# Vibration Signal Analysis for Machinery Condition Monitoring

Imre Kocsis
Krisztián Deák

University of Debrecen Faculty of Engineering

2022

# Part I

(Imre Kocsis)



# Part II

(Krisztián Deák)

# Content

# Introduction

The word ‘diagnostics’ originally referred to the identification of human health problems resulting the ‘diagnosis’, the evaluation of the actual health status of a human. Technical diagnostics aims the diagnosis of technical objects (machines, parts) to support engineering decisions (reparation or change of a part, process improvement).

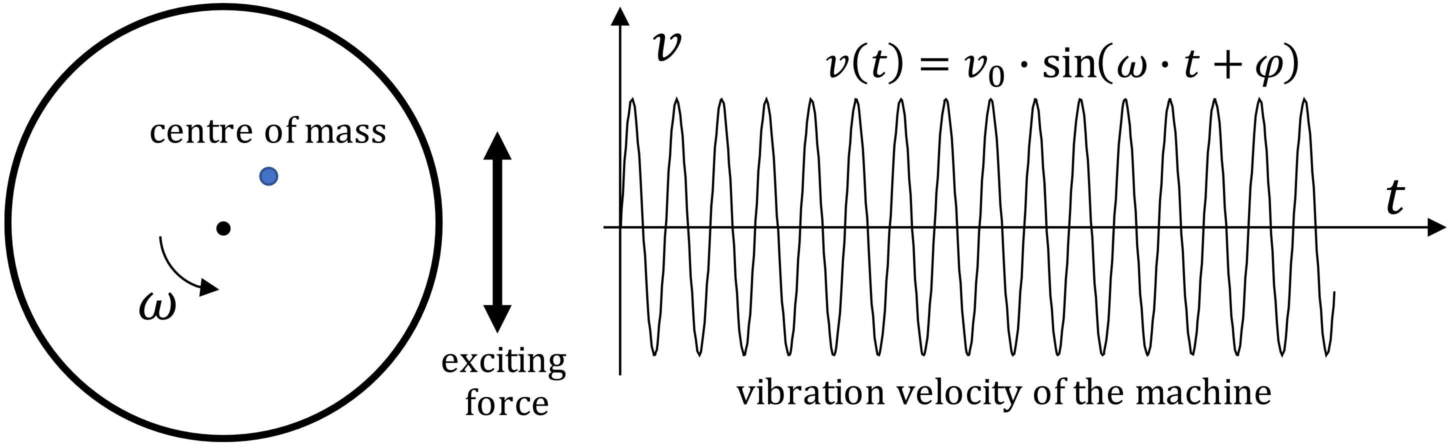
*Fault* is the abnormal behaviour of a machine (state), *failure* is the termination of the ability to perform a required function (event), *failure mode* is an observable phenomenon caused by a failure. *Root cause failure analysis* is an examination aiming to identify the causes of a failure mode.

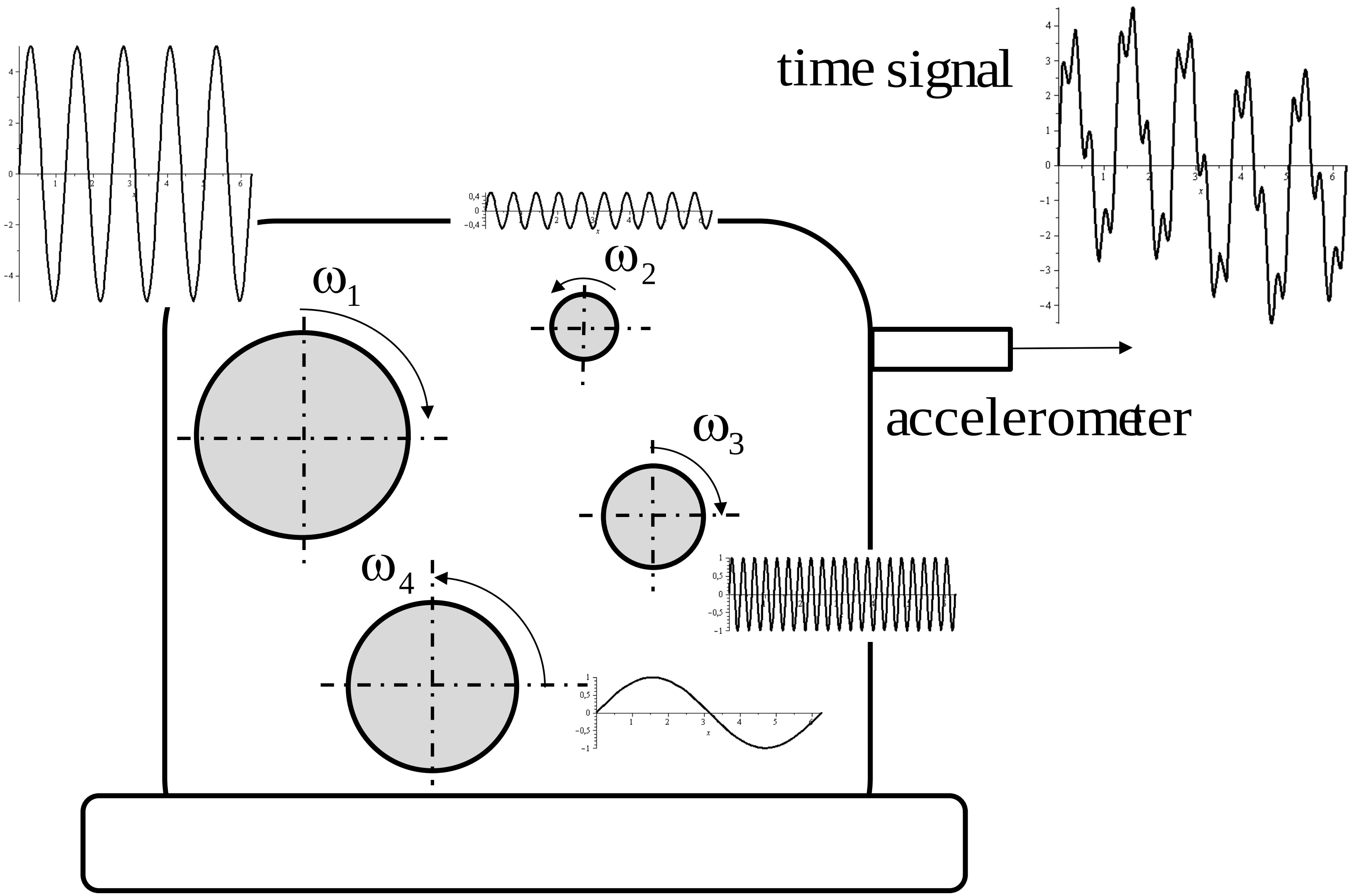
The first step of condition monitoring is to find connection between failures and measurable symptoms generated by the failures of interest. Since the symptoms are very different in appearance (vibration, sound, heat, wear particles in oil), technical diagnostics has several fields using special measuring systems as vibration monitoring, acoustics, thermography, oil analysis, etc.

In the field of vibration monitoring symptoms can be detected with the analysis of the sampled vibration signal. (In simple everyday situations the diagnosis can be based on the human senses – vision, hearing and touch first of all – but their accuracy and range is not enough for diagnosis of modern machineries.) Some symptoms appear in the so-called time-domain (e.g. vibration velocity vs. time function) others can be revealed from the frequency spectrum (frequency-domain analysis).

The basic tool for vibration analysis, from the beginning, is the Fourier analysis. Several mechanical failures of rotating parts generate periodic, nearly harmonic vibrations. Such failures, for example, are unbalance, angle or shift problems of couplings, looseness, misalignment of shafts, bended shafts.

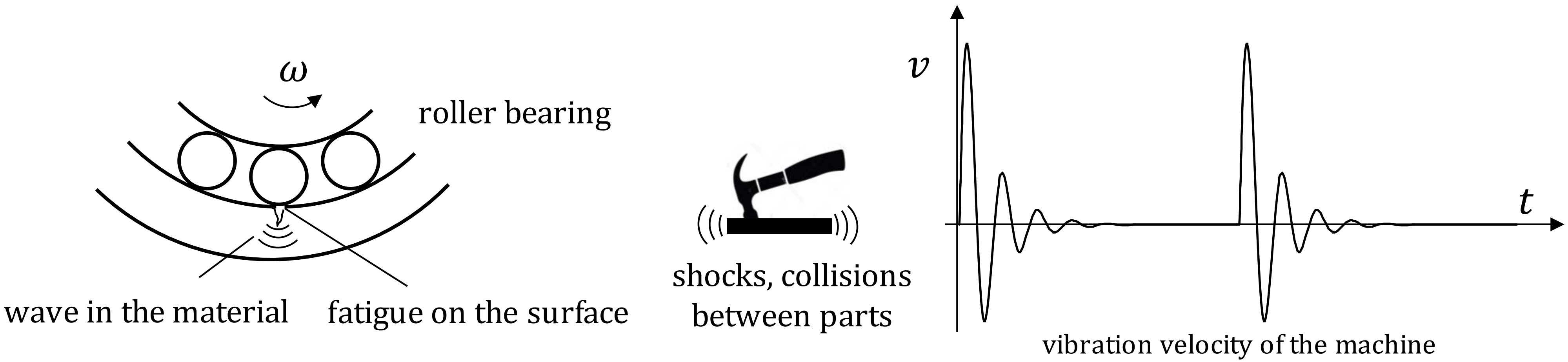
The generated vibrations have special frequencies depending on the rotational speed and the geometry of the rotating component. Special patterns containing a group of lines in the frequency spectrum belong to the majority of failures. Thus, in many cases, pattern recognition is required rather than the detection of a certain frequency value.






Connecting an accelerometer the superposition of approximately harmonic vibrations generated by rotating parts of the machine and other vibrations (beats, noise) can be measured. The vibration spectrum provided by the Fourier analysis shows the frequencies appearing in the vibration signal and the magnitudes belonging to them. Based on these data the problematic components and the severity of the failures can be identified.

Another type of failures causes so-called shock pulses (transient vibrations) rather than periodic vibrations. The most important examples are bearing and gear failures. Since these components are crucial in mechanical engineering, monitoring these parts are important. Shock pulses are non-periodic transient waves in the time signal, so Fourier analysis is not effective in detection of these parts of the signal. Transient components can be revealed using so-called simultaneous time-frequency methods, for instance short time Fourier transform or wavelet transform. Shock pulses repeat periodically with a certain magnitude and frequency which depend on the geometry and the rotational speed of the machine element and are characteristic to the failure.




The aim of this problem book is to introduce some selected topics from the field of vibration analysis using a “practical approach”. The level of problems is varying from the basics (e.g. properties of sine and cosine functions) to advanced investigations in abstract spaces.

The modern methods of technical diagnostics are based on digital measuring and data acquisition systems, this is why discrete transformations are in the focus when practical industrial problems are to be solved.

As an example, some functions of the SPM Condition Monitoring System related to the frequency spectrum are mentioned. (https://www.spminstrument.com)

Questions at the end of the chapters are related to the theoretical part and the exercises are connected to the practical usage of the methods. From chapter 7 to 12, exercises are related to Matlab applications and examples of the official Mathworks homepage (https://www.mathworks.com/help/examples.html) are used to present the possible methods and solutions for the actual problem.

# 1st week

# 1 Trigonometric and Exponential Functions

# Trigonometric and Exponential Functions

Signal processing is based on different decompositions of functions. In the classic Fourier theory the decomposition is about trigonometric (sin, cos) or complex exponential functions. For further use some properties of these functions are overviewed.

Trigonometric (sin, cos) and exponential functions are defined as power series:

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k+1)!} \cdot z^{2k+1}, \quad z \in \mathbb{C}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k)!} \cdot z^{2k}, \quad z \in \mathbb{C}$$

$$EXP(z) = e^z = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^k, \quad z \in \mathbb{C}$$

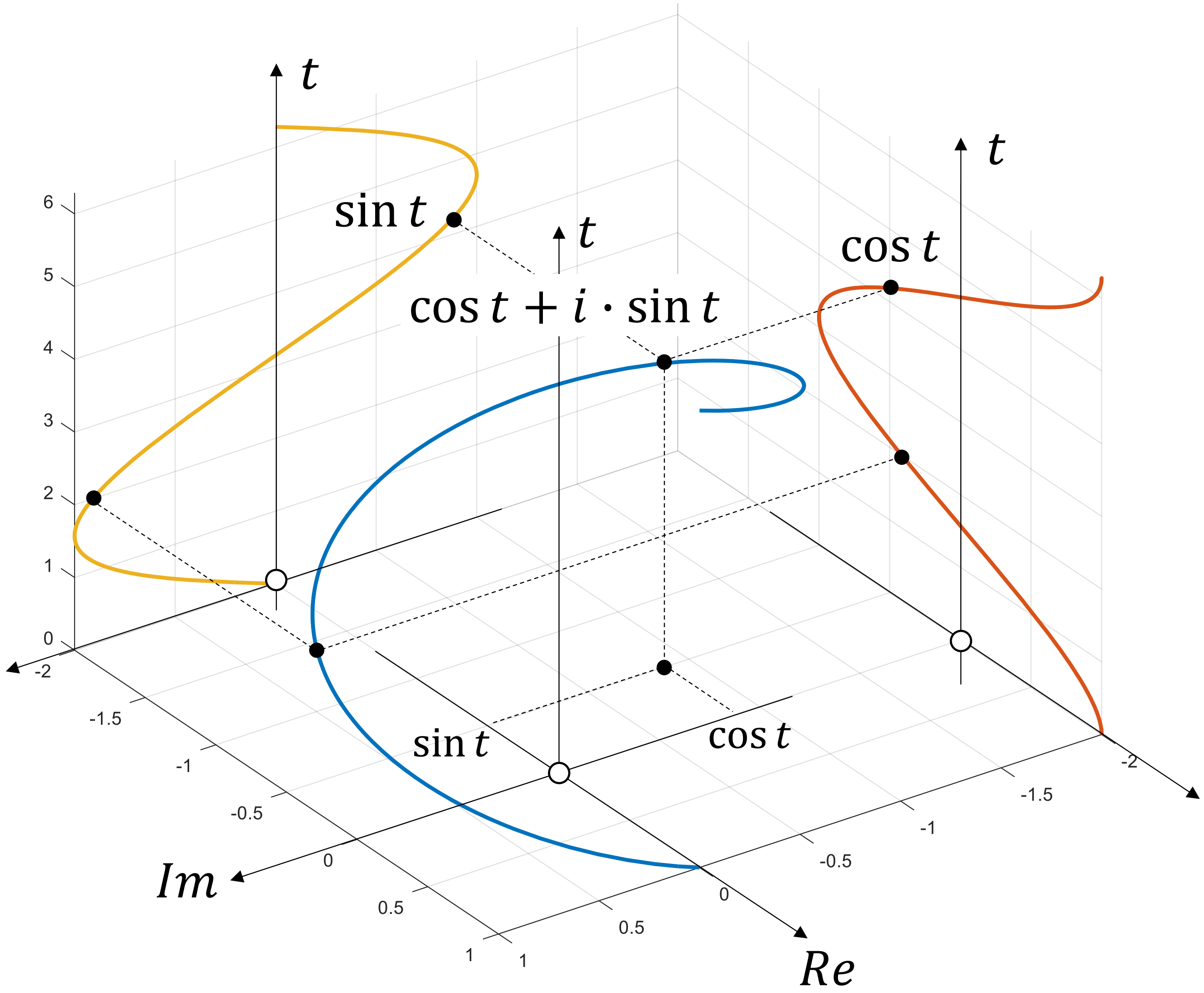
***Remark***

The real trigonometric and exponential functions are obtained as restrictions to $\mathbb{R}$.

The so-called Euler formula

$$e^{i \cdot \varphi} = \cos \varphi + i \cdot \sin \varphi, \quad \varphi \in \mathbb{R}$$

comes directly from the definitions.


t
sin t
t
cos t
cos t + i · sin t
t
6
5
4
3
2
1
0
-2
-1.5
-1
-0.5
0
0.5
1
sin t
cos t
Im
Re
-2
-1.5
-1
-0.5
0
0.5
1

Using the properties of the exponential functions we have that for an arbitrary complex number $z = \sigma + s \cdot i$, $(\sigma, s \in \mathbb{R})$

$$e^z = e^{\sigma + s \cdot i} = e^\sigma \cdot e^{s \cdot i} = e^\sigma \cdot (\cos s + i \cdot \sin s)$$

holds.

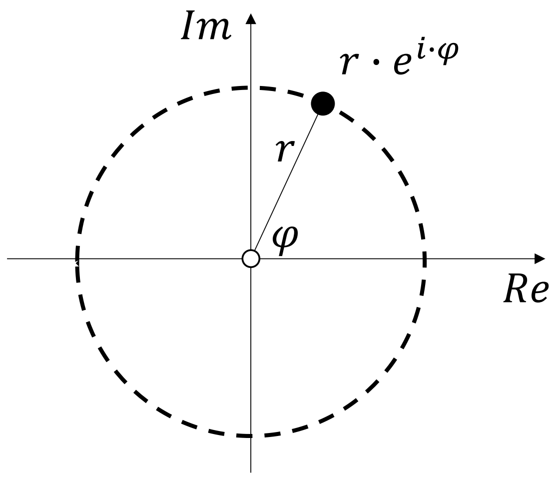
***Corollary***

Values of the complex exponential function can be calculated from values of real trigonometric and exponential functions.

Since $e^\sigma$ is a positive real number and $\left|e^{s \cdot i}\right| = |\cos s + i \cdot \sin s| = \sqrt{\cos^2 s + \sin^2 s} = 1$, in formula

$$e^z = e^\sigma \cdot e^{s \cdot i}$$

$r = e^\sigma$ is the norm and $s$ is the argument ('angle') of $e^z$.

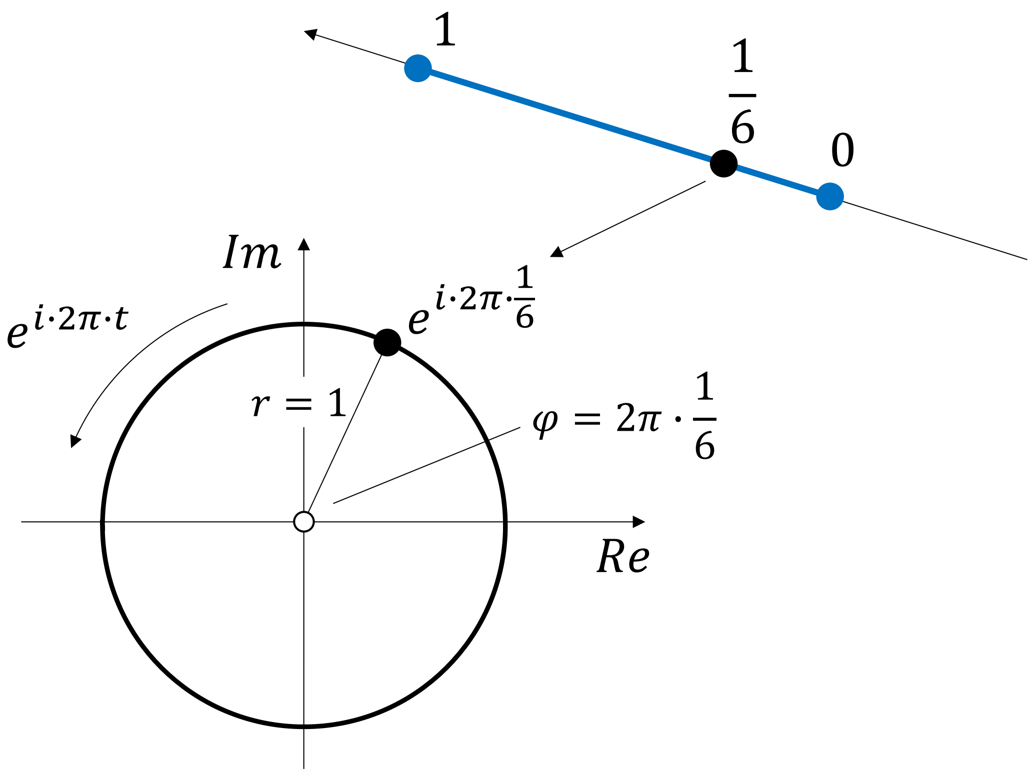



Function

$$t \to e^{i \cdot 2\pi \cdot t}, \quad t \in \mathbb{R}$$

has an important role in Fourier series and Fourier transforms.

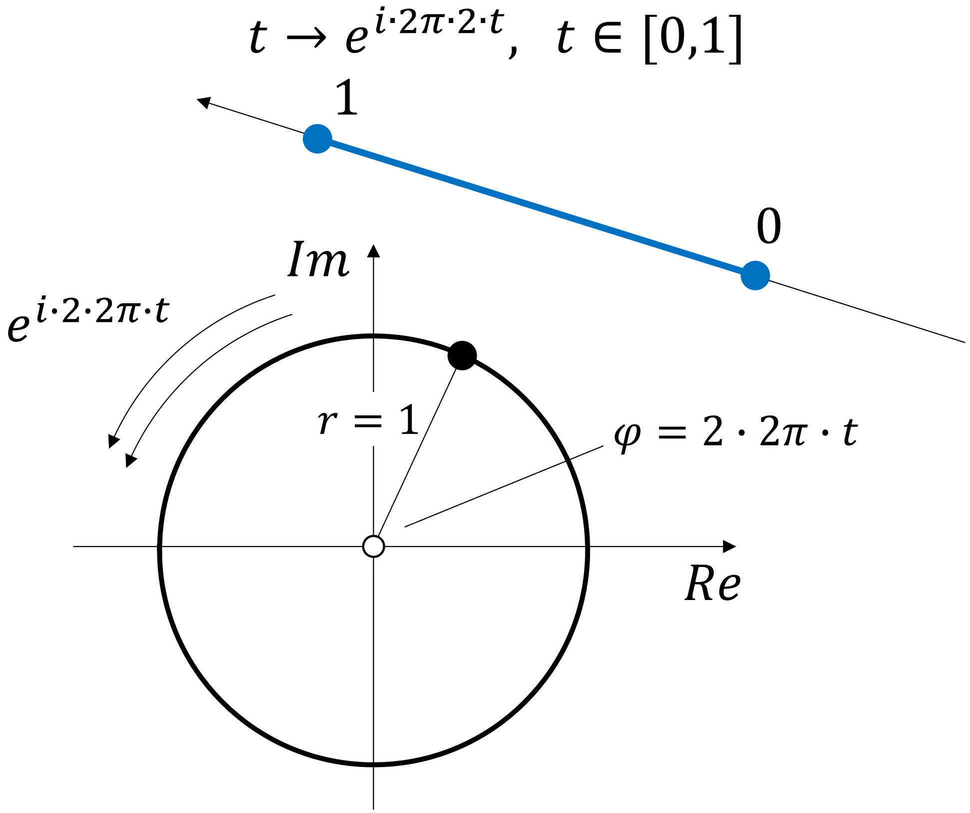
Function $t \to e^{i \cdot 2\pi \cdot t}, t \in \mathbb{R}$ is 1-periodic and the range of $t \to e^{i \cdot 2\pi \cdot t}, t \in [0,1]$ is the unit circle of the complex plane.

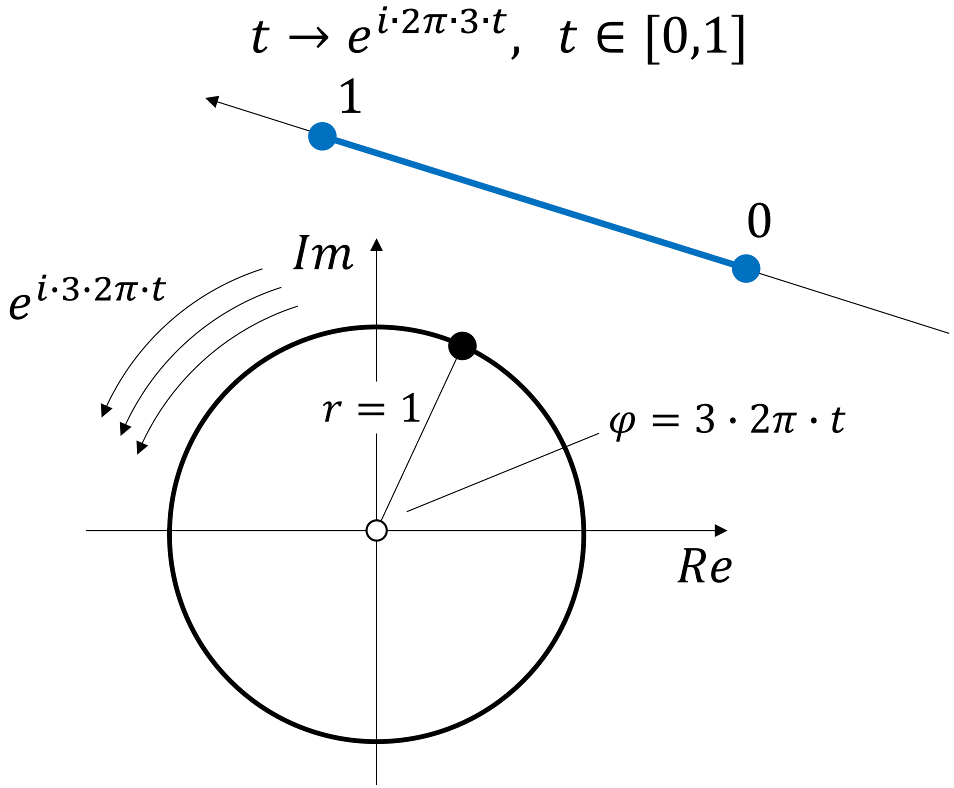



The period of function $t \to e^{i\cdot f\cdot 2\pi\cdot t}$ is $T = \frac{1}{f}$.

Considering $t \to e^{i\cdot f\cdot 2\pi\cdot t}$ as a "position-time function" in the complex plane and using the SI units we have that $T = [s]$, $f = \left[\frac{1}{s}\right] = [Hz]$.

Quantity $f = \frac{1}{T}$ can be called 'rotational frequency' which gives the number of rotations per second.




***Remark***

$f$ in formula

$$e^{i\cdot f\cdot 2\pi\cdot t} = \cos(f \cdot 2\pi \cdot t) + i \cdot \sin(f \cdot 2\pi \cdot t)$$

can be called 'frequency' (the frequency of the harmonic vibrations descripted by the trigonometric functions.).

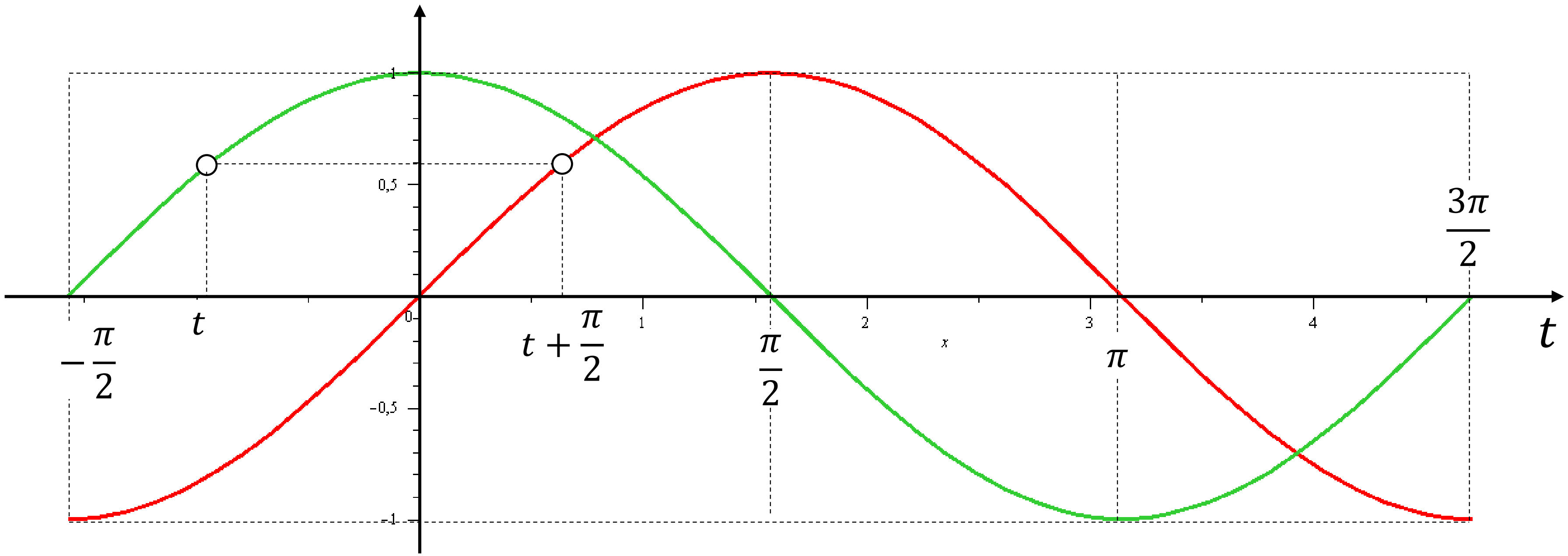
## *Remark*

Using the well-known equality $\omega = 2\pi \cdot f$ we can write

$$e^{i \cdot f \cdot 2\pi \cdot t} = e^{i \cdot f \omega \cdot t}$$

as well.

Since $\cos t = \sin\left(t + \frac{\pi}{2}\right)$, it is generally enough to give the properties of the sin function, these are also valid for the cos function.




In the Fourier theory the following three equivalent formulas are used to describe harmonic vibrations

$$A \cdot \sin(\omega \cdot t + \varphi) = A \cdot \sin(2\pi f \cdot t + \varphi) = A \cdot \sin\left(\frac{2\pi}{T} \cdot t + \varphi\right),$$

where the physical quantities are

- $\omega$ is the angular frequency $\omega = \left[\frac{rad}{s}\right]$
- $f$ is the frequency $f = \left[\frac{1}{s}\right] = \left[\frac{rad}{s}\right] = [Hz]$
- $T$ is the period $T = [s]$
- $\varphi$ is the phase $\varphi = [rad]$.

If

$$u(t) = A_u \cdot \sin(\omega \cdot t + \varphi_u)$$

is the input and

$$y(t) = A_y(\omega) \cdot \sin\big(\omega \cdot t + \varphi_y(\omega)\big)$$

is the steady state output of a linear system, then the ratio $\frac{A_y(\omega)}{A_u}$ (gain) the difference $\varphi_y(\omega) - \varphi_u$ (phase shift) can characterize the system.

Giving a certain type of decompositions we will use that a linear combination of sin and cos functions of the same frequency can be written as a “shifted” sin function with the same frequency as follows

$$A \cdot \sin t + B \cdot \cos t = \sqrt{A^2 + B^2} \cdot \sin(t + \varphi),$$

where

$$\varphi = \begin{cases} \operatorname{arctg}\dfrac{B}{A}, & \text{ha } A \geq 0 \\ \operatorname{arctg}\dfrac{B}{A} \pm \pi, & \text{ha } A < 0 \end{cases}$$

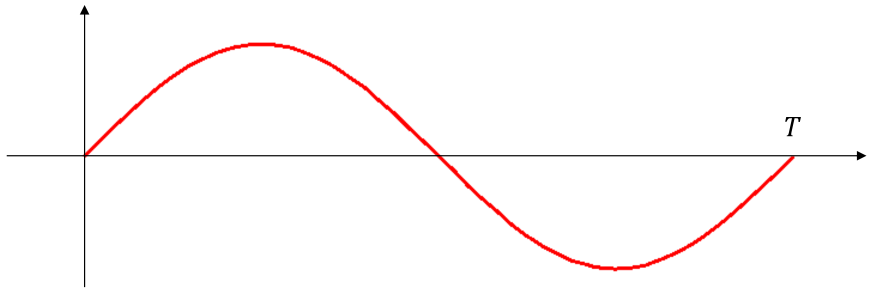
Giving a decomposition of a function according to the trigonometric or exponential system we identify the ’frequencies’ and the related amplitudes (or energies) of components contained by the signal.

In vibration diagnostics the most frequently used symptoms of some mechanical and electrical faults are special combinations of frequency values appearing in the frequency spectrum.

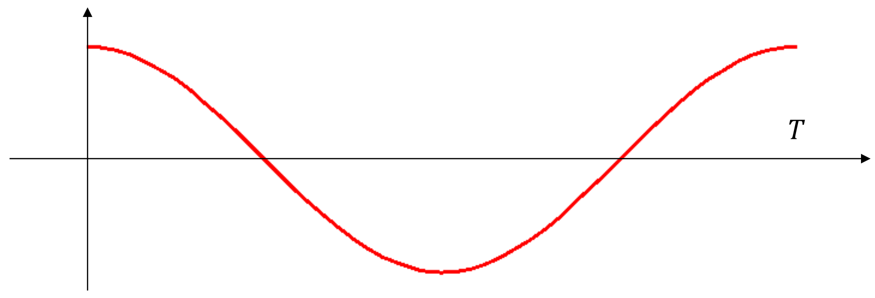
A trigonometric system contains sin and cos functions of frequencies $f_0, 2f_0, 3f_0, \dots$ where $f_0$ is the basic frequency:

| | period | frequency |
|---|---|---|
| $\cos\left(\frac{2\pi}{T}\cdot t\right) = \cos(2\pi f_0 \cdot t)$ | $T = \frac{1}{f_0}$ | $f_0 = \frac{1}{T}$ |
| $\sin\left(\frac{2\pi}{T}\cdot t\right) = \sin(2\pi f_0 \cdot t)$ | $T = \frac{1}{f_0}$ | $f_0 = \frac{1}{T}$ |
| $\cos\left(2\cdot\frac{2\pi}{T}\cdot t\right) = \cos(2\cdot 2\pi f_0 \cdot t)$ | $T/2$ | $2f_0$ |
| $\sin\left(2\cdot\frac{2\pi}{T}\cdot t\right) = \sin(2\cdot 2\pi f_0 \cdot t)$ | $T/2$ | $2f_0$ |
| $\cos\left(3\cdot\frac{2\pi}{T}\cdot t\right) = \cos(3\cdot 2\pi f_0 \cdot t)$ | $T/3$ | $3f_0$ |
| $\sin\left(3\cdot\frac{2\pi}{T}\cdot t\right) = \sin(3\cdot 2\pi f_0 \cdot t)$ | $T/3$ | $3f_0$ |
| $\cos\left(4\cdot\frac{2\pi}{T}\cdot t\right) = \cos(4\cdot 2\pi f_0 \cdot t)$ | $T/4$ | $4f_0$ |
| $\sin\left(4\cdot\frac{2\pi}{T}\cdot t\right) = \sin(4\cdot 2\pi f_0 \cdot t)$ | $T/4$ | $4f_0$ |

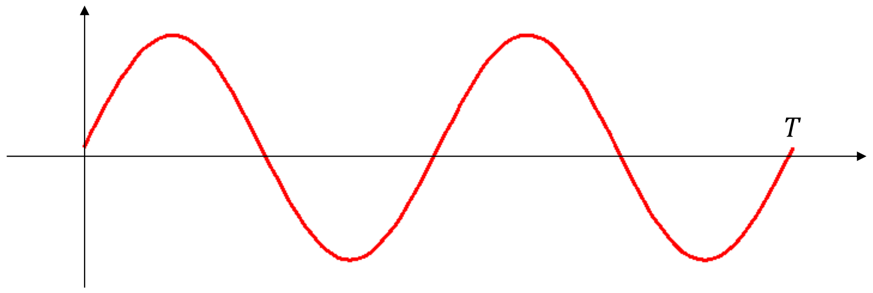
$$t \to \sin\left(\frac{2\pi}{T} \cdot t\right) = \sin(2\pi f_0 \cdot t)$$



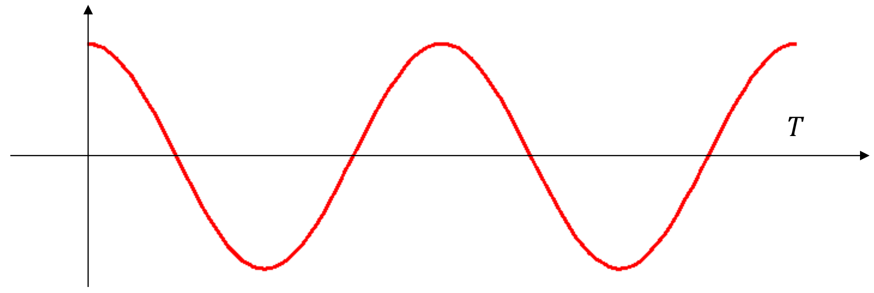

$$t \to \cos\left(\frac{2\pi}{T} \cdot t\right) = \cos(2\pi f_0 \cdot t)$$



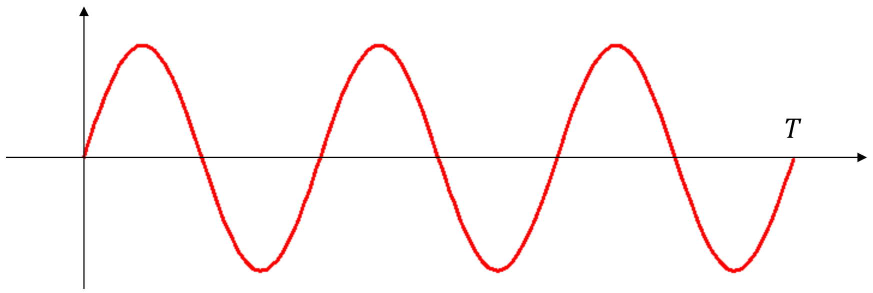

$$t \to \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(2 \cdot 2\pi f_0 \cdot t)$$



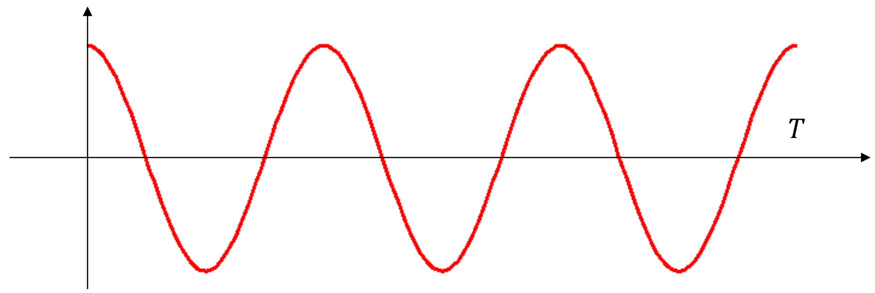

$$t \to \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



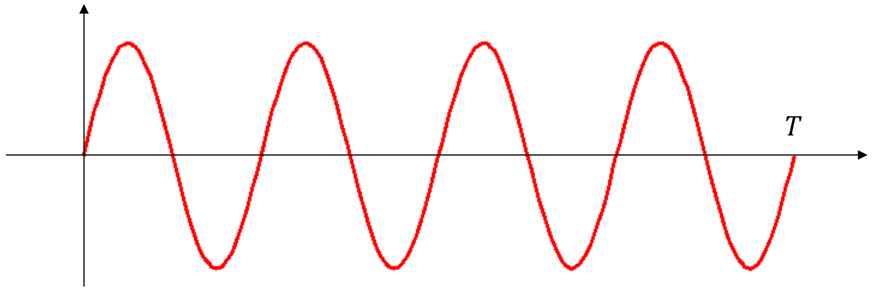

$$t \to \sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(3 \cdot 2\pi f_0 \cdot t)$$



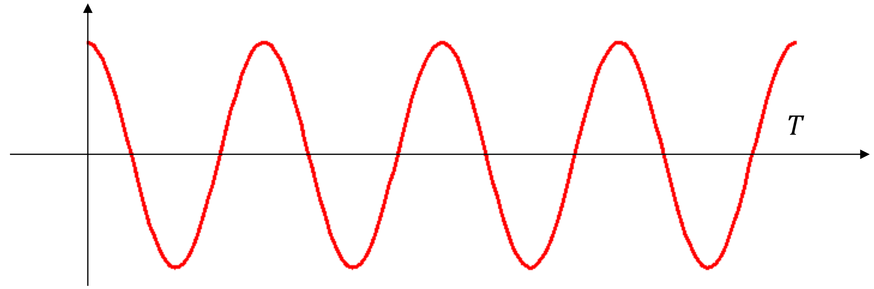

$$t \to \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(3 \cdot 2\pi f_0 \cdot t)$$



$$t \to \sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(4 \cdot 2\pi f_0 \cdot t)$$

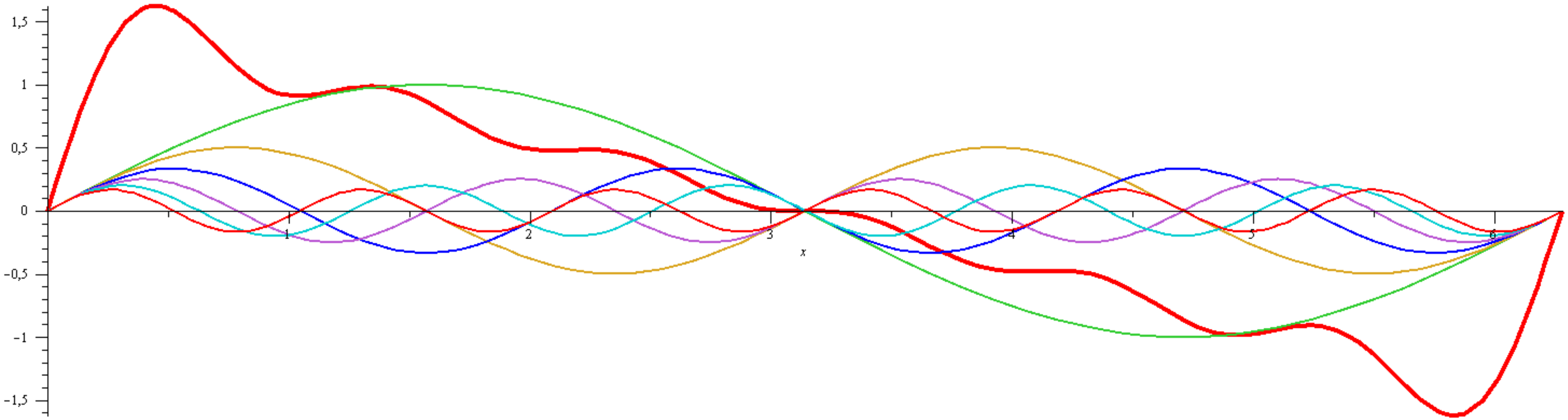



$$t \to \cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(4 \cdot 2\pi f_0 \cdot t)$$




When a T-periodic function is analysed, then $f_0 = \frac{1}{T}$.

The figure shows a periodic function (red thick line) and its six harmonic components.




In practice, when processing a vibration signal, sampled signals are available. The ratio of sampling frequency and the maximum frequency of the signal components determines the "quality" of signal processing, a too low value of the sampling frequency leads to the appearance of fake frequencies in the frequency spectrum (aliasing).

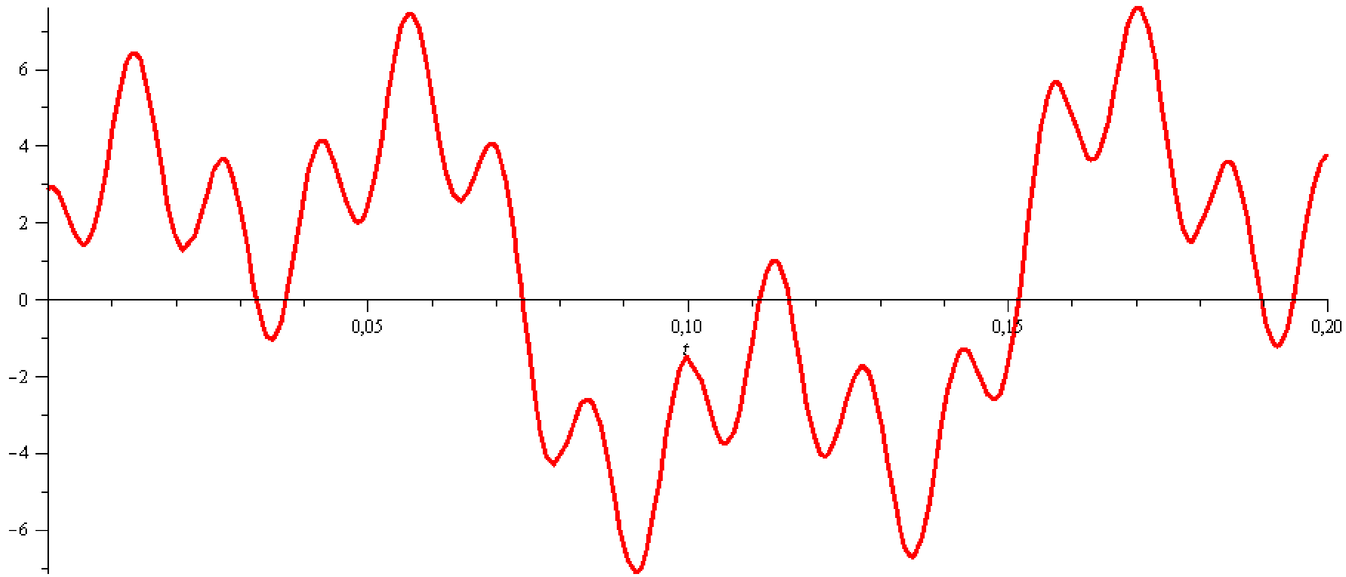
***Example***

Suppose that there are three rotating parts in a machine generating harmonic vibrations:

| | rotational speed (RPM) | harmonics (order) | amplitude $\left[A, \frac{mm}{s}\right]$ | phase $[\varphi, rad]$ |
|---|---|---|---|---|
| Part 1 | 416 | 1 | 4 | $-0.11$ |
| Part 2 | 580 | 2 | 3 | 0.47 |
| Part 3 | 1050 | 4 | 2 | 1.86 |

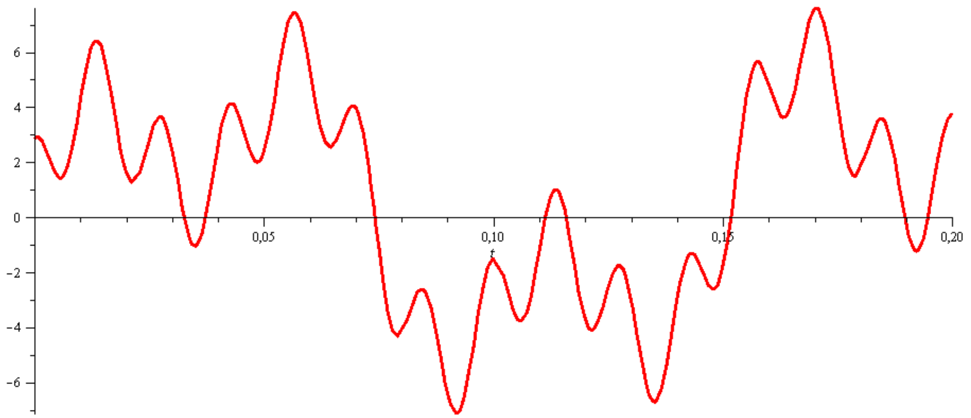
Then the vibration signal is the sum of three sin functions

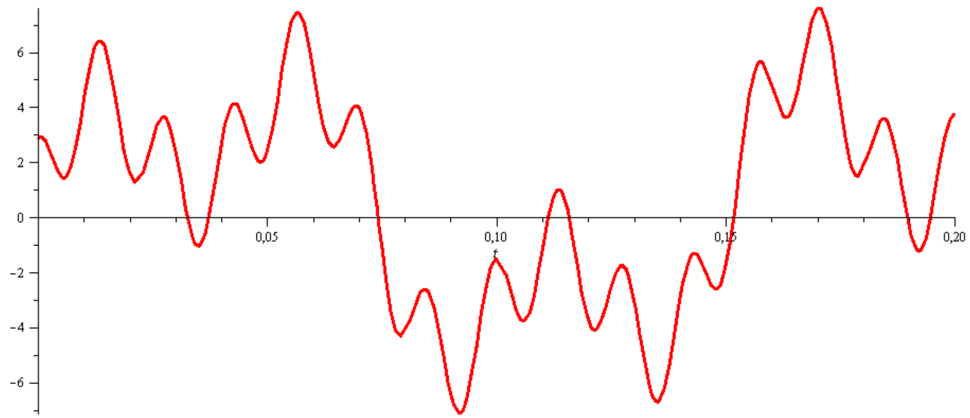
$$v(t) = 4 \cdot \sin(43.56 \cdot t - 0.11) + 3 \cdot \sin(121.47 \cdot t + 0.47) + 2 \cdot \sin(439.81 \cdot t + 1.86)$$

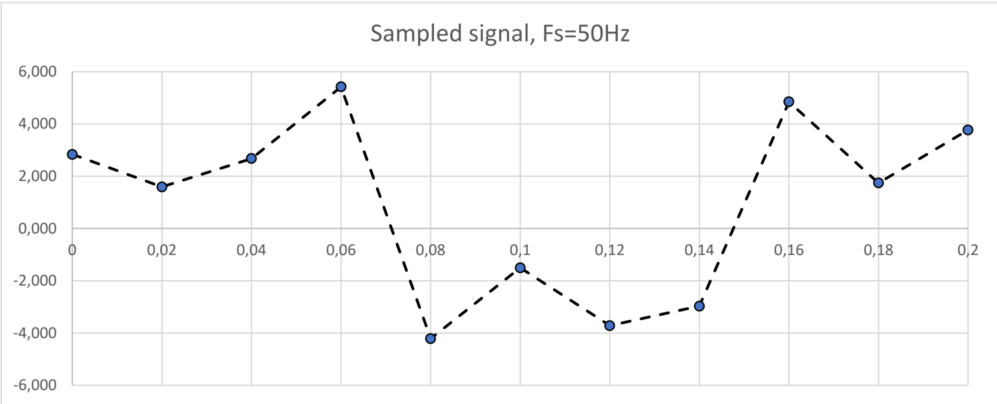


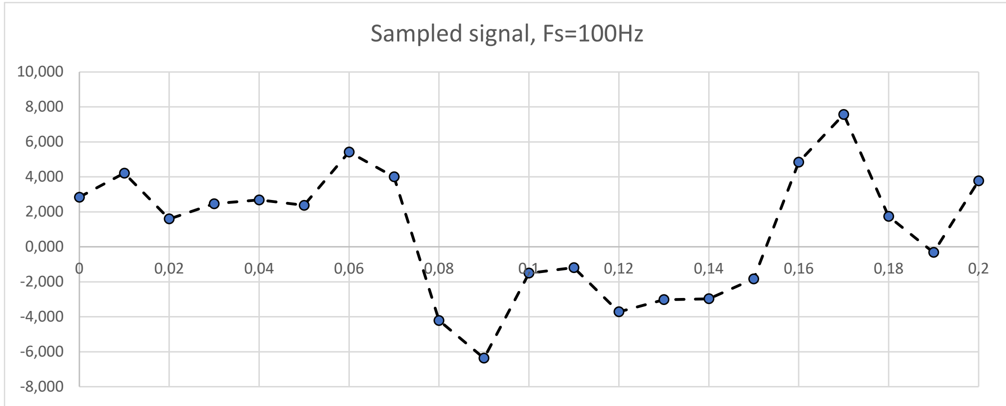

Sample the vibration velocity for $T = 0.2$ seconds with sampling frequencies $f_{s1} = 50[Hz]$, $f_{s2} = 100[Hz]$, $f_{s3} = 200[Hz]$, respectively.

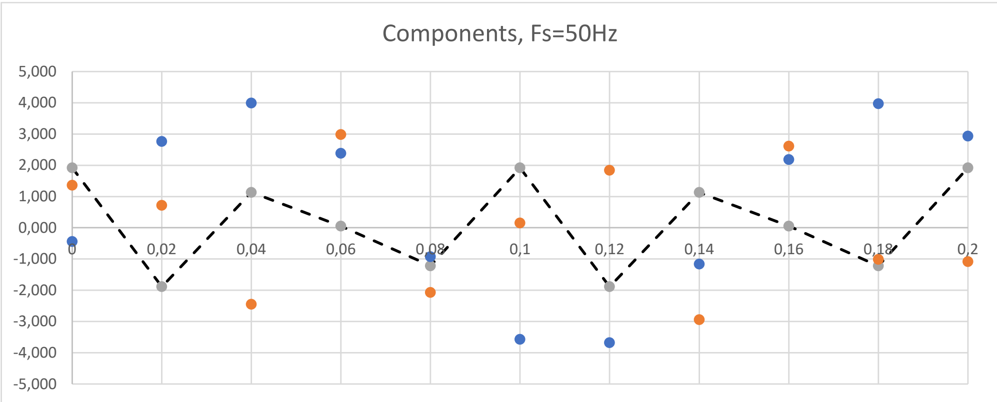
The following diagrams show the sampled signals obtained with the different sampling frequencies.

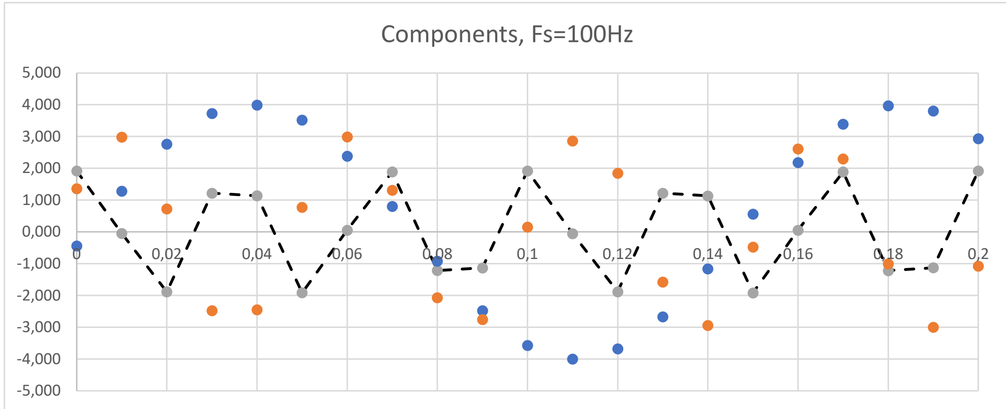

6
4
2
0
−2
−4
−6
0,05
0,10
0,15
0,20
t

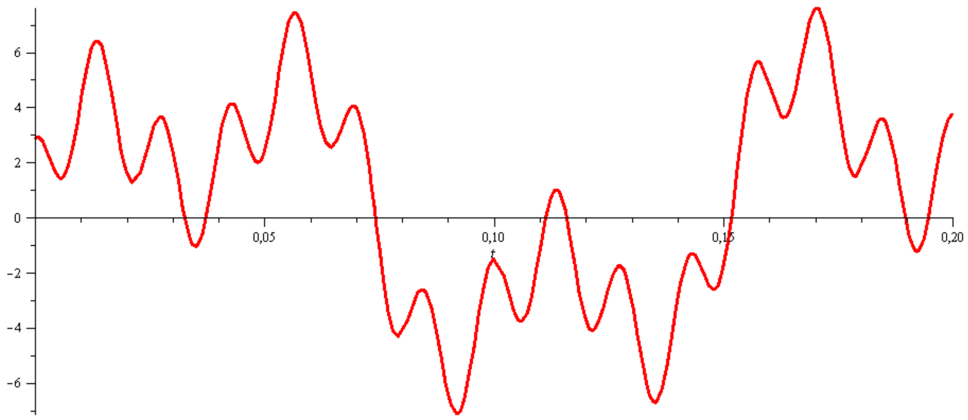

6
4
2
0
−2
−4
−6
0,05
0,10
0,15
0,20

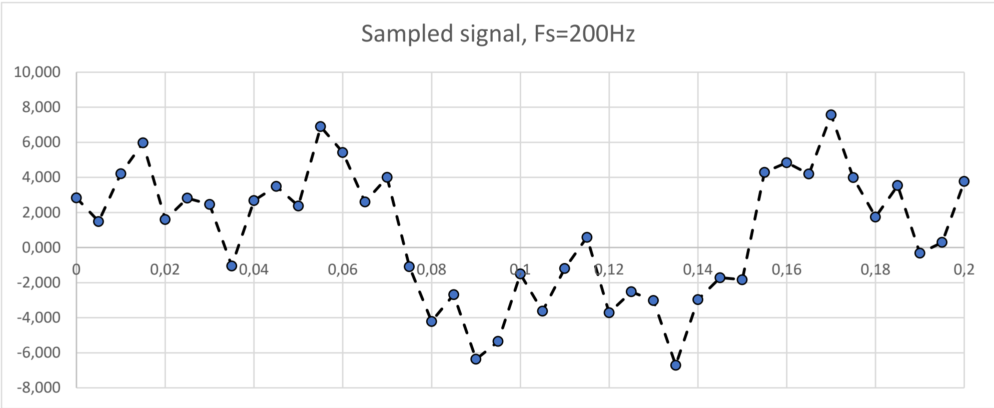

Sampled signal, Fs=50Hz
6,000
4,000
2,000
0,000
-2,000
-4,000
-6,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2

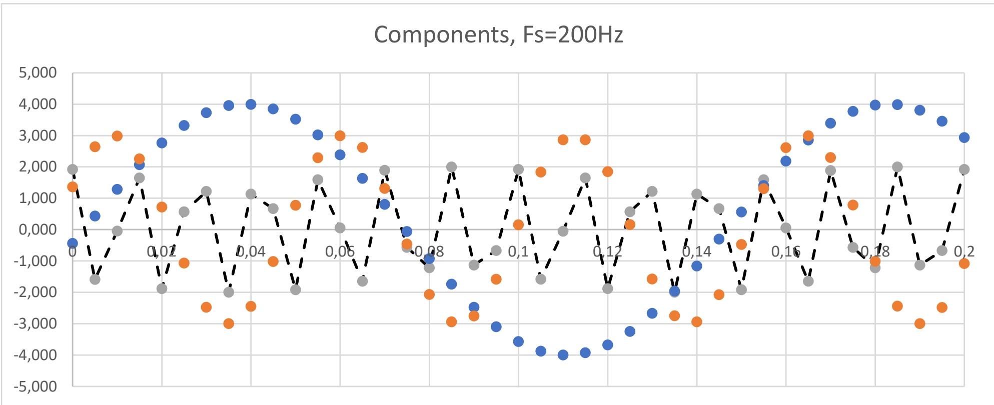

Sampled signal, Fs=100Hz
10,000
8,000
6,000
4,000
2,000
0,000
-2,000
-4,000
-6,000
-8,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2


Components, Fs=50Hz
5,000
4,000
3,000
2,000
1,000
0,000
-1,000
-2,000
-3,000
-4,000
-5,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2


Components, Fs=100Hz
5,000
4,000
3,000
2,000
1,000
0,000
-1,000
-2,000
-3,000
-4,000
-5,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2


6
4
2
0
−2
−4
−6
0,05
0,10
0,15
0,20


Sampled signal, Fs=200Hz
10,000
8,000
6,000
4,000
2,000
0,000
-2,000
-4,000
-6,000
-8,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2


Components, Fs=200Hz
5,000
4,000
3,000
2,000
1,000
0,000
-1,000
-2,000
-3,000
-4,000
-5,000
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2

In the theory of Fourier series functions

$$a_0 + \sum_{k=1}^{n} \left( a_k \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right),$$

the so-called trigonometric polynomials have important role.

Dealing with linear combination of periodic functions it is a question whether a linear combination of them is also a periodic function, since it can be shown that sum of periodic functions is not necessarily periodic.

Let $x_1, \ldots, x_n$ be periodic functions with period $T_1 > 0, \ldots, T_n > 0$. Function

$$x_1 + \cdots + x_n$$

is periodic if and only if there exists $T > 0$ such that for some positive integers $k_1, \ldots, k_n$

$$T = k_1 \cdot T_1 = k_2 \cdot T_2 = \ldots = k_n \cdot T_n$$

holds, that is, $T$ is a common multiple of periods. The smallest positive $T$ satisfying the system of equations above is the period of $x_1 + \cdots + x_n$.

Equalities $T = k_1 \cdot T_1 = \ldots = k_n \cdot T_n$ imply that ratios

$$\frac{T_2}{T_1} = \frac{k_1}{k_2}, \ldots, \frac{T_n}{T_1} = \frac{k_1}{k_n},$$

are rational numbers.

For example, function $\sin t + \sin(\pi \cdot t)$ is non-periodic, since the ratio of the two periods is $\pi$ which is not rational.

Period of the sum can be obtained by multiplying period $T_1$ with the least common multiple of denominators in the simplest forms of fractions $\frac{T_2}{T_1}, \ldots, \frac{T_n}{T_1}$.

***Remark***

The considerations about periodicity above are valid for linear combinations

$$\alpha_1 \cdot x_1 + \cdots + \alpha_n \cdot x_n, \quad \alpha_i \neq 0$$

as well.

***Example***

Determine the period of function

$$10.2 \cdot \sin\left(\frac{2\pi}{40} \cdot t\right) - 3.3 \cdot \sin\left(\frac{2\pi}{60} \cdot t\right) + 0.8 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$$

Periods of the three functions in the linear combination are

$$T_1 = \frac{2\pi}{\frac{2\pi}{40}} = 40,\;\; T_2 = \frac{2\pi}{\frac{2\pi}{60}} = 60,\;\; T_3 = \frac{2\pi}{\frac{2\pi}{30}} = 30$$

The ratios of periods in the simplest form

$$\frac{T_1}{T_2} = \frac{2}{\mathbf{3}}, \quad \frac{T_1}{T_3} = \frac{4}{\mathbf{3}}$$

The least common multiple of denominators: $m = \text{lcm}\{3,3\} = 3$

Thus, the period is

$$T = T_1 \cdot m = 40 \cdot 3 = 120$$

A similar result can be obtained for frequency $f = \frac{1}{T}$.

Let $x_1, \ldots, x_n$ be periodic functions with frequency $f_1 > 0, \ldots, f_n > 0$, respectively.

Function

$$x_1 + \cdots + x_n$$

is periodic if and only if there exists $f > 0$ such that for some positive integers $k_1, \ldots, k_n$

$$f_1 = k_1 \cdot f, \ldots, f_n = k_n \cdot f$$

holds, that is, all frequencies are multiples of frequency $f$.

The largest $f$ satisfying the system of equations above is the frequency of $x_1 + \cdots + x_n$.

Equalities $f_1 = k_1 \cdot f, f_2 = k_2 \cdot f, \ldots, f_n = k_n \cdot f$ imply that ratios frequencies are rational numbers.

Period of the elements of the trigonometric system

$$\left\{\cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$$

are

$$T, \frac{T}{2}, \dots, \frac{T}{k}, \dots$$

while the frequencies are

$$f, 2 \cdot f, \dots, k \cdot f, \dots$$

respectively.

The period and frequency of linear combinations of elements of the system are $T$ and $f$, respectively, supposing that the coefficient of $\cos\left(\frac{2\pi}{T}\right)$ or the coefficient of $\sin\left(\frac{2\pi}{T} \cdot t\right)$ is different from zero.

# 1st week – Questions

***Question 1***

Give the complex sin, cos, and exp functions as power series, and the Euler formula.

***Answer***

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k+1)!} \cdot z^{2k+1}, \quad z \in \mathbb{C}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k)!} \cdot z^{2k}, \quad z \in \mathbb{C}$$

$$EXP(z) = e^z = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^k, \quad z \in \mathbb{C}$$

$$e^{i\cdot\varphi} = \cos\varphi + i \cdot \sin\varphi, \quad \varphi \in \mathbb{R}$$

## Question 2

Give the formula of harmonic vibrations using the angular frequency, the frequency, and the period, respectively.

## Answer

$$A \cdot \sin(\omega \cdot t + \varphi)$$

$$A \cdot \sin(2\pi f \cdot t + \varphi)$$

$$A \cdot \sin\left(\frac{2\pi}{T} \cdot t + \varphi\right)$$

### *Question 3*

Give a necessary and sufficient condition for the periodicity of the sum of periodic functions. Give the period of the sum.

### *Answer*

If functions $x_1, \dots, x_n$ are periodic with period $T_1 > 0, \dots, T_n > 0$, then function

$$x_1 + \cdots + x_n$$

is periodic if and only if there exists $T > 0$ such that for some positive integers $k_1, \dots, k_n$

$$T = k_1 \cdot T_1 = k_2 \cdot T_2 = \dots = k_n \cdot T_n$$

holds, that is, $T$ is a common multiple of periods.

The period of

$$x_1 + \cdots + x_n$$

is the smallest positive $T$ satisfying the system of equations

$$T = k_1 \cdot T_1 = k_2 \cdot T_2 = \dots = k_n \cdot T_n.$$

# 1st week – Exercises

Calculating the discrete Fourier transform

$$e^{i\cdot 2\pi \cdot k \cdot \frac{\mathrm{n}}{N}}, \quad k = 0,1, \dots , N - 1$$

values of the complex exponential function are used

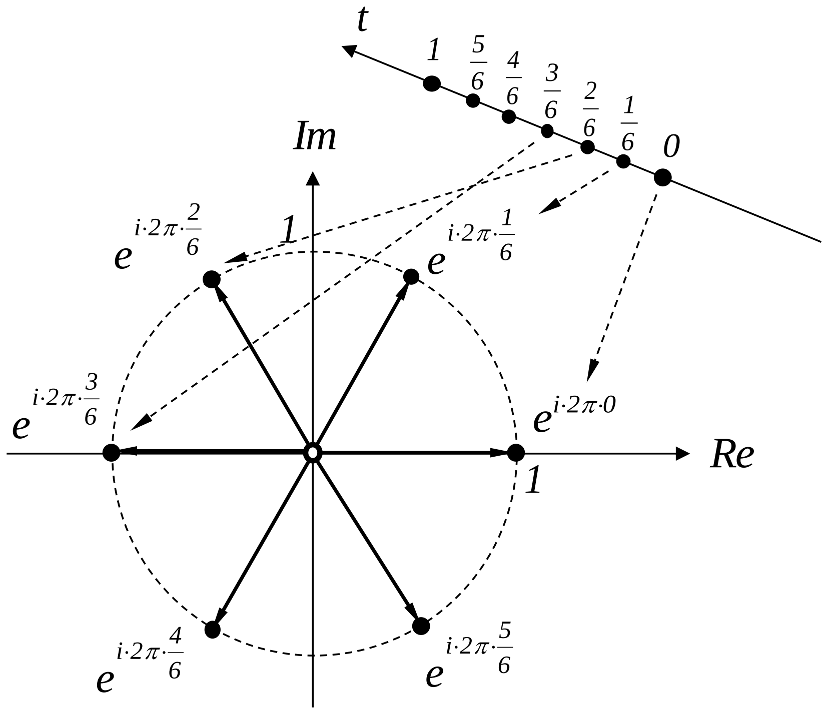
***Exercise***

Plot values

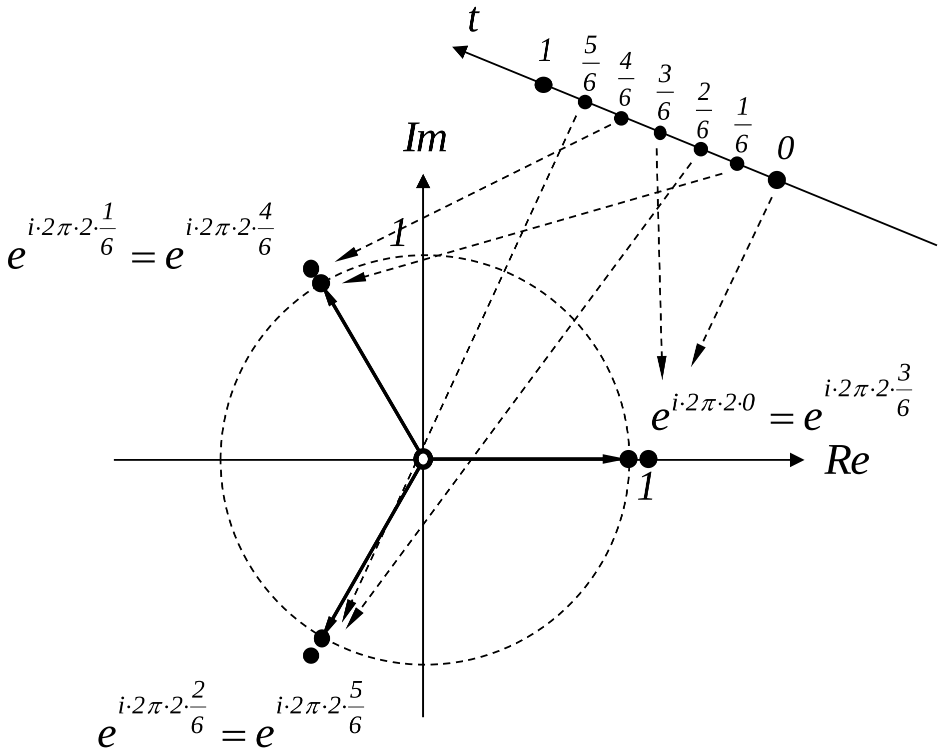
$$e^{i\cdot 2\pi \cdot k \cdot \frac{n}{6}}, \quad k = 1, \dots ,3, n = 0, \dots ,5$$

on the complex plane.

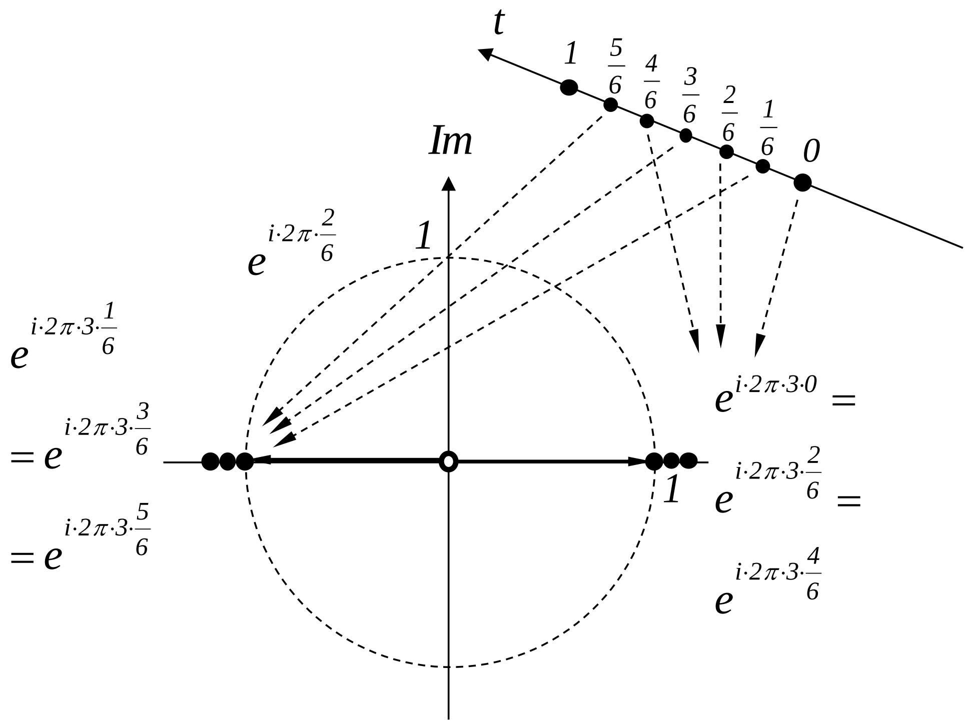
## Case $k = 1, n = 0, \ldots, 5$




## Case $k = 2, n = 0, \ldots, 5$



Case $k = 3, n = 0, \dots, 5$



## *Exercise*

Show that

$$\sum_{n=0}^{5} e^{i\cdot 2\pi\cdot\frac{n}{6}} = 0$$

$$\sum_{n=0}^{5} e^{i\cdot 2\pi\cdot\frac{n}{6}} = e^0 + e^{i\cdot\frac{2\pi}{6}} + e^{i\cdot\frac{4\pi}{6}} + e^{i\cdot\frac{6\pi}{6}} + e^{i\cdot\frac{8\pi}{6}} + e^{i\cdot\frac{10\pi}{6}} =$$

$$= e^0 + e^{i\cdot\frac{\pi}{3}} + e^{i\cdot\frac{2\pi}{3}} + e^{i\cdot\pi} + e^{i\cdot\frac{4\pi}{3}} + e^{i\cdot\frac{5\pi}{3}} =$$

$$= 1 + \left(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3}\right) + \left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right) +$$

$$+(\cos\pi + i\cdot\sin\pi) + \left(\cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3}\right) + \left(\cos\frac{5\pi}{3} + i\cdot\sin\frac{5\pi}{3}\right) =$$

$$= 1 + \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} - \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} - 1 - \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2} + \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2} = 0$$

## ***Remark***

It can be proven that

$$\sum_{n=0}^{N-1} e^{i\cdot 2\pi\cdot\frac{n}{N}} = 0$$

holds for all positive integers $N$.

## *Exercise*

Plot the given functions and characterize them in term of properties given in the table

| | |
|---|---|
| amplitude | |
| maximum | |
| minimum | |
| period | |
| frequency | |
| smallest positive zero place | |

$$t \to \sin t\,,\quad t \to \sin(2\cdot t)\,,\quad t \to \sin(3\cdot t)\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \sin\left(\frac{\pi}{2}\cdot t\right),\quad x \to \sin(\pi\cdot t)\,,\quad t \to \sin(2\pi\cdot t)\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \cos t\,,\quad t \to \cos(2\cdot t)\,,\quad t \to \cos(3\cdot t)\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \cos\left(\frac{\pi}{2}\cdot t\right),\quad x \to \cos(\pi\cdot t)\,,\quad t \to \cos(2\pi\cdot t)\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \sin t\,,\quad t \to \frac{1}{3}\cdot \sin t\,,\quad t \to 3\cdot \sin t\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \cos t\,,\quad t \to \frac{1}{3}\cdot \sin t\,,\quad t \to 3\cdot \sin t\,,\quad t \in [-2\pi, 2\pi]$$

$$t \to \sin t\,, \quad t \to \sin t - 1, \quad t \to \sin t + 2, \quad t \in [-2\pi, 2\pi]$$

---

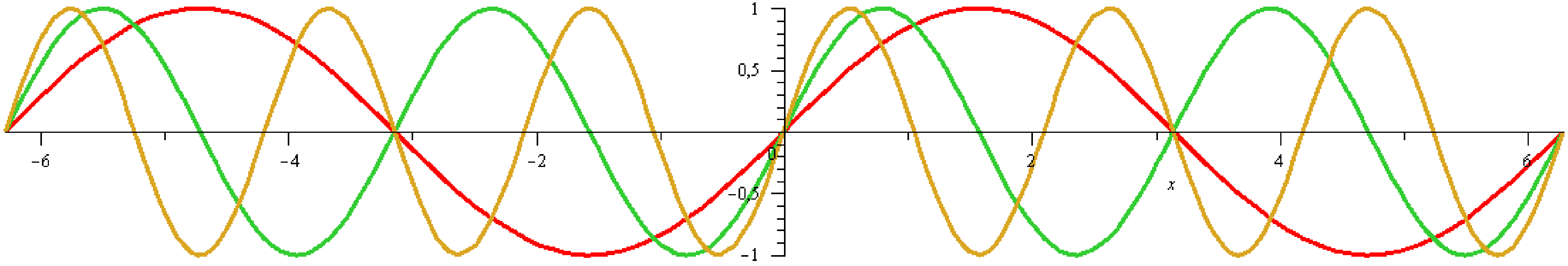
$$t \to \cos t\,, \quad t \to \cos t - 2, \quad t \to \cos t + 1, \quad t \in [-2\pi, 2\pi]$$

---

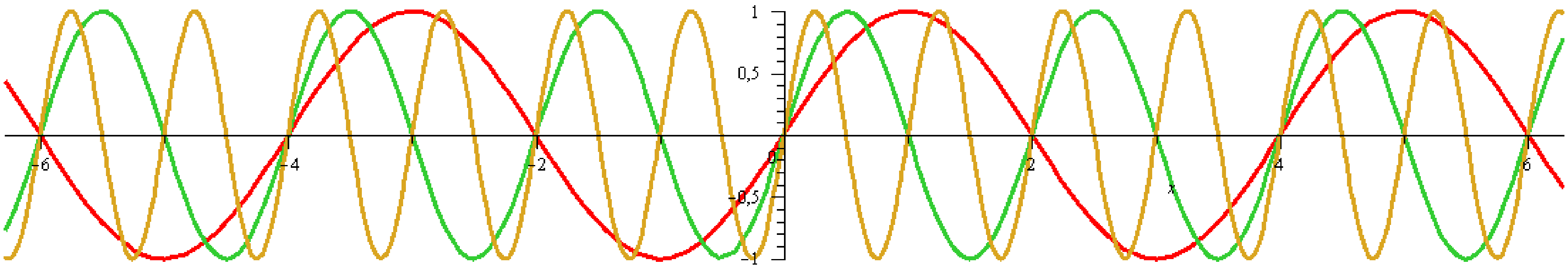
$$t \to \sin t\,, \quad t \to \sin\left(t - \frac{\pi}{2}\right), \quad t \to \sin\left(t + \frac{\pi}{2}\right), \quad t \in [-2\pi, 2\pi]$$

---

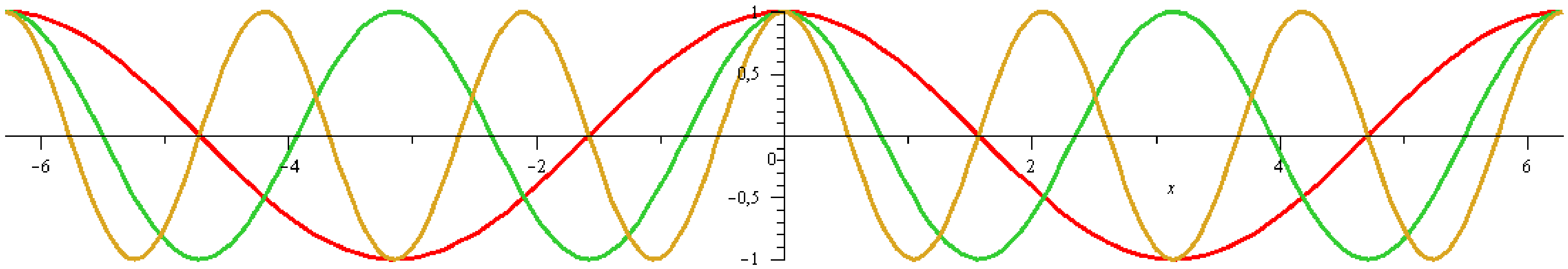
$$t \to \cos t\,, \quad t \to \cos\left(t - \frac{\pi}{2}\right), \quad t \to \cos\left(t + \frac{\pi}{2}\right), \quad t \in [-2\pi, 2\pi]$$



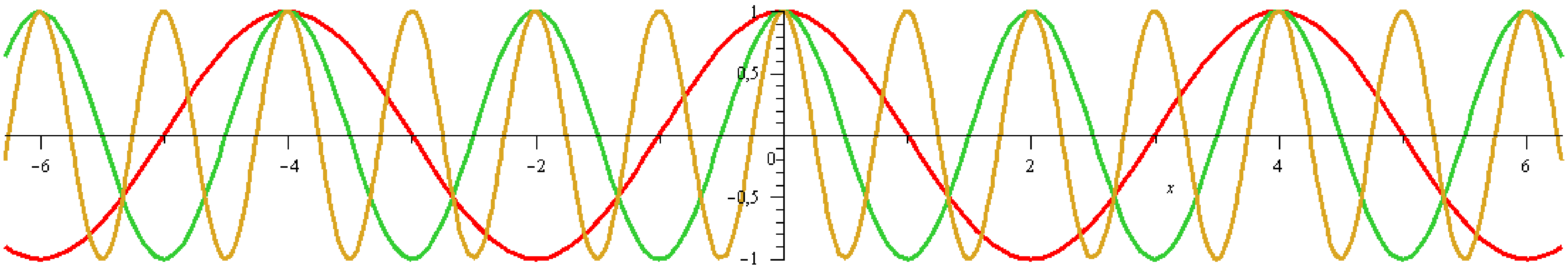

| | $t \to \sin t$ | $t \to \sin 2t$ | $t \to \sin 3t$ |
|---|---|---|---|
| amplitude | $1$ | $1$ | $1$ |
| maximum | $1$ | $1$ | $1$ |
| minimum | $-1$ | $-1$ | $-1$ |
| period | $2\pi$ | $\pi$ | $2\pi/3$ |
| frequency | $1/2\pi$ | $1/\pi$ | $3/2\pi$ |
| smallest positive zero place | $\pi$ | $\frac{\pi}{2}$ | $\frac{\pi}{3}$ |



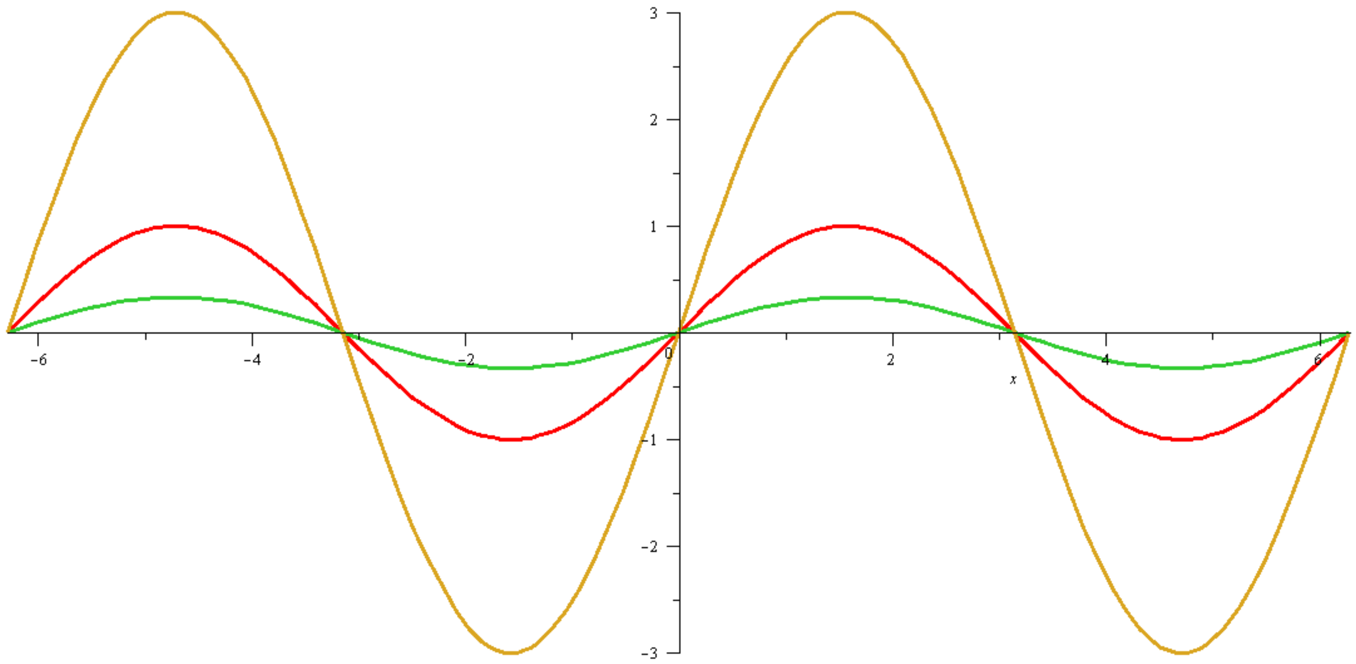

| | $t \to \sin \frac{\pi}{2} t$ | $t \to \sin \pi t$ | $t \to \sin 2\pi t$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 1 | 1 |
| minimum | $-1$ | $-1$ | $-1$ |
| period | 4 | 2 | 1 |
| frequency | 1/4 | 1/2 | 1 |
| smallest positive zero place | 2 | 1 | 1/2 |



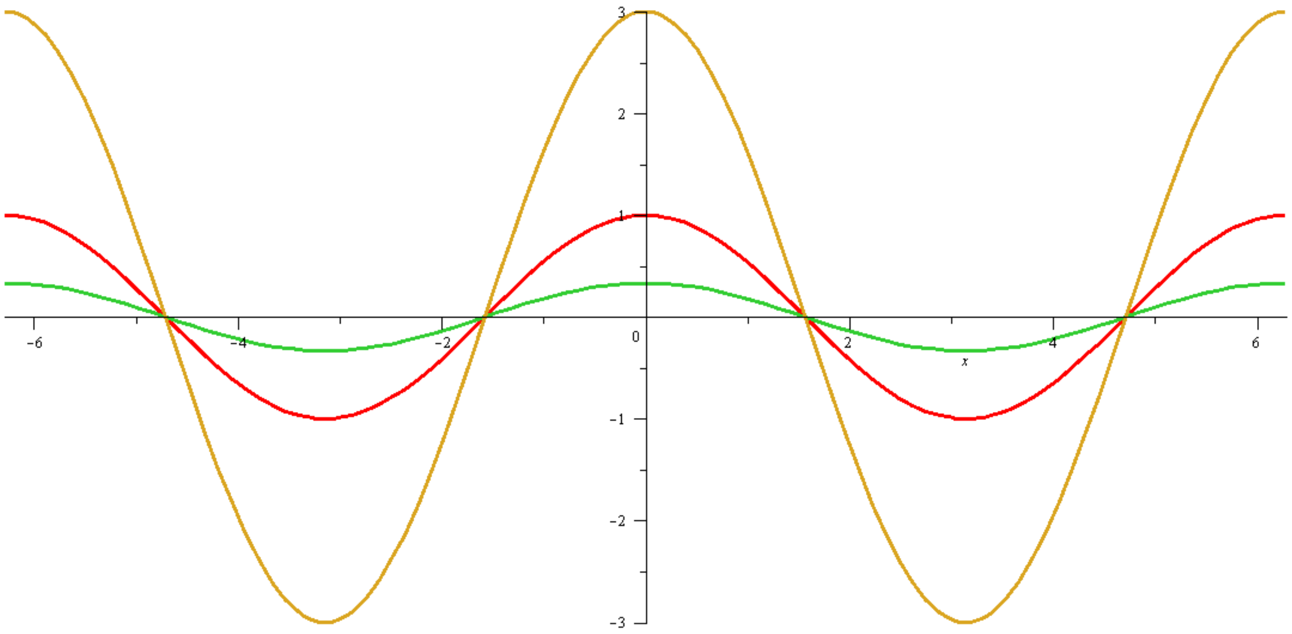

| | $t \to \cos t$ | $t \to \cos 2t$ | $t \to \cos 3t$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 1 | 1 |
| minimum | $-1$ | $-1$ | $-1$ |
| period | $2\pi$ | $\pi$ | $2\pi/3$ |
| frequency | $1/2\pi$ | $1/\pi$ | $3/2\pi$ |
| smallest positive zero place | $\pi/2$ | $\pi/4$ | $\pi/6$ |



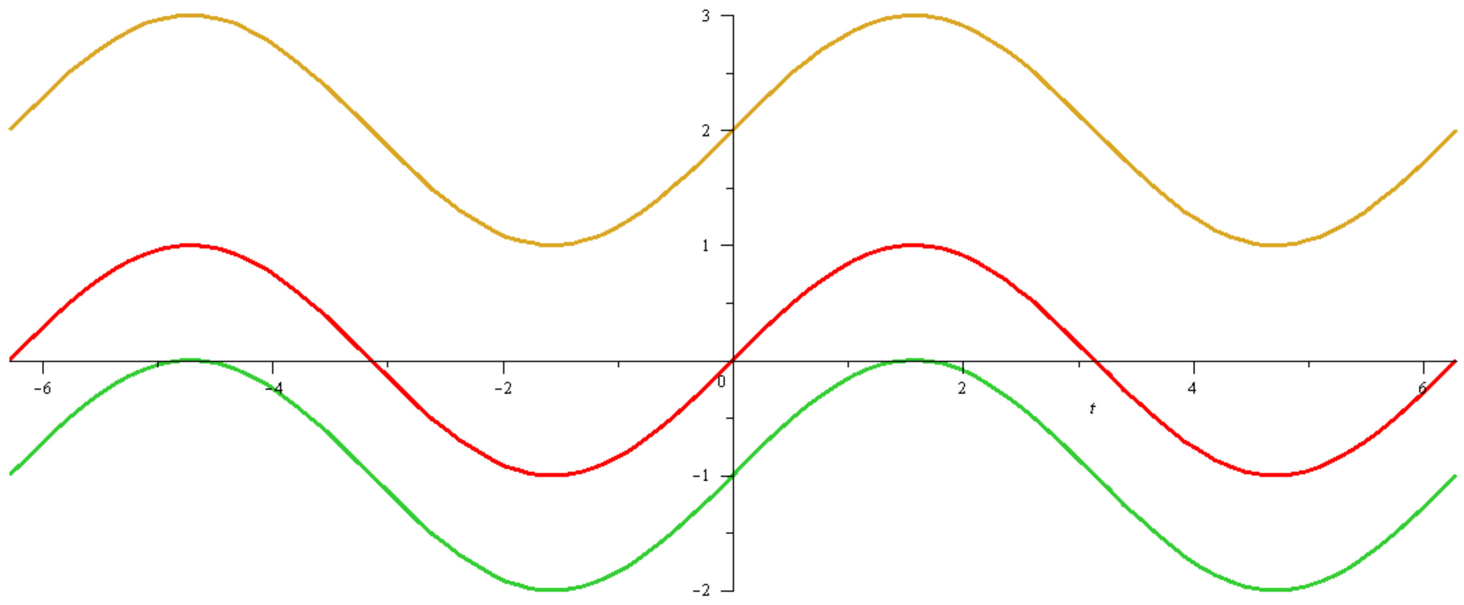

| | $t \to \cos\frac{\pi}{2}t$ | $x \to \cos \pi t$ | $t \to \cos 2\pi t$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 1 | 1 |
| minimum | $-1$ | $-1$ | $-1$ |
| period | 4 | 2 | 1 |
| frequency | 1/4 | 1/2 | 1 |
| smallest positive zero place | 1 | 1/2 | 1/4 |



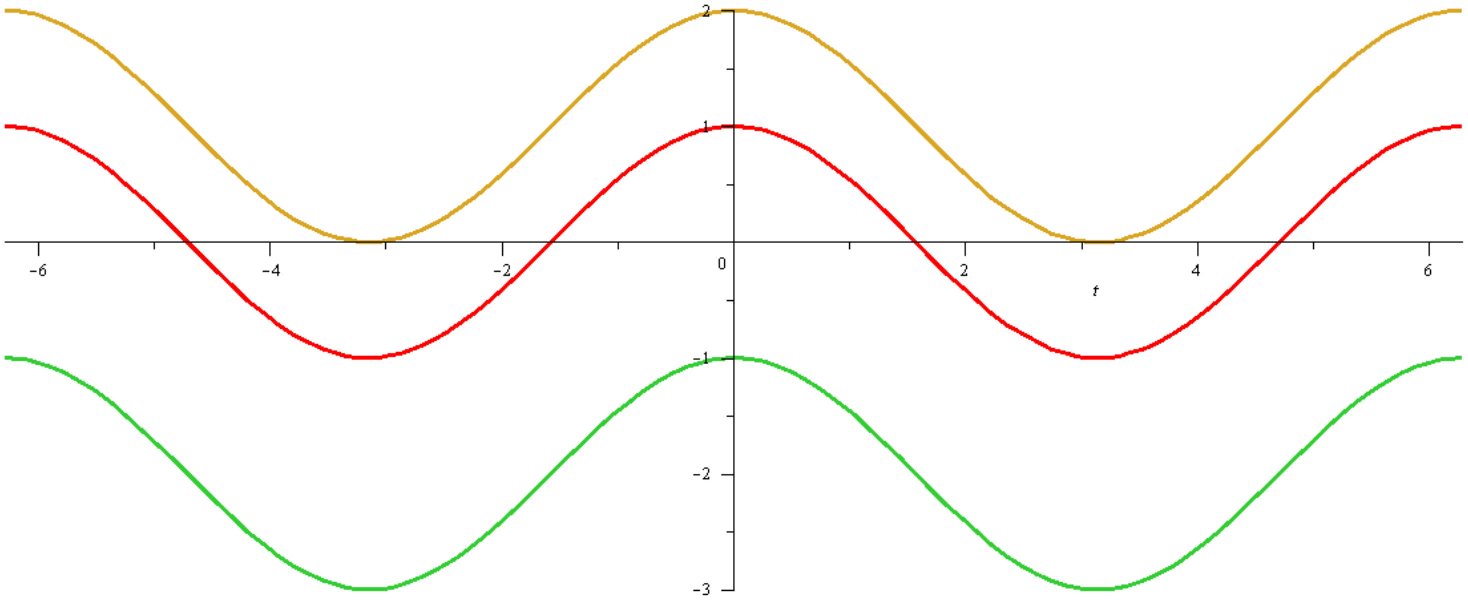

| | $t \to \sin t$ | $t \to \frac{1}{3}\sin t$ | $t \to 3\sin t$ |
|---|---|---|---|
| amplitude | 1 | 1/3 | 3 |
| maximum | 1 | 1/3 | 3 |
| minimum | −1 | −1/3 | −3 |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |
| smallest positive zero place | $\pi$ | $\pi$ | $\pi$ |



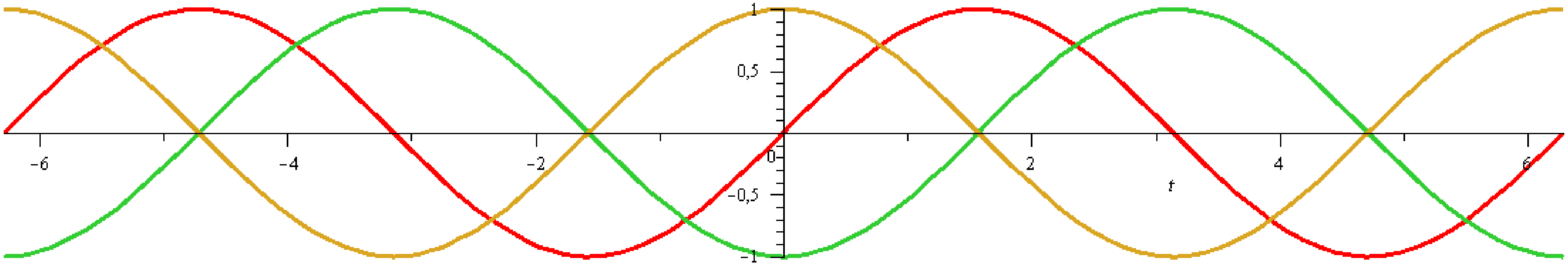

| | $t \to \cos t$ | $t \to \frac{1}{3}\cos t$ | $t \to 3\cos t$ |
|---|---|---|---|
| amplitude | $1$ | $1/3$ | $3$ |
| maximum | $1$ | $1/3$ | $3$ |
| minimum | $-1$ | $-1/3$ | $-3$ |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |
| smallest positive zero place | $\pi/2$ | $\pi/2$ | $\pi/2$ |



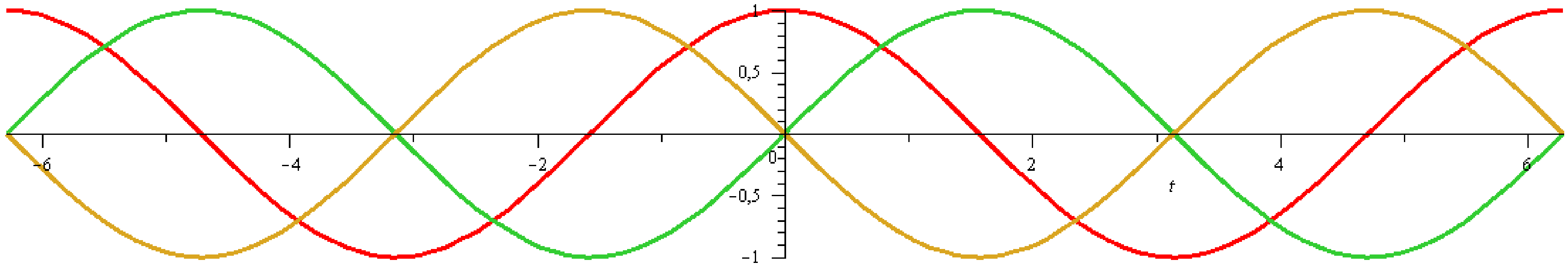

| | $t \to \sin t$ | $t \to \sin t - 1$ | $t \to \sin t + 2$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 0 | 3 |
| minimum | $-1$ | $-2$ | 1 |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |




| | $t \to \cos t$ | $t \to \cos t - 2$ | $t \to \cos t + 1$ |
|---|---|---|---|
| amplitude | $1$ | $1$ | $1$ |
| maximum | $1$ | $-1$ | $2$ |
| minimum | $-1$ | $-3$ | $0$ |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |




| | $t \to \sin t$ | $t \to \sin\left(t - \frac{\pi}{2}\right)$ | $t \to \sin\left(t + \frac{\pi}{2}\right)$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 1 | 1 |
| minimum | $-1$ | $-1$ | $-1$ |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |
| smallest positive zero place | $\pi$ | $\pi/2$ | $\pi/2$ |



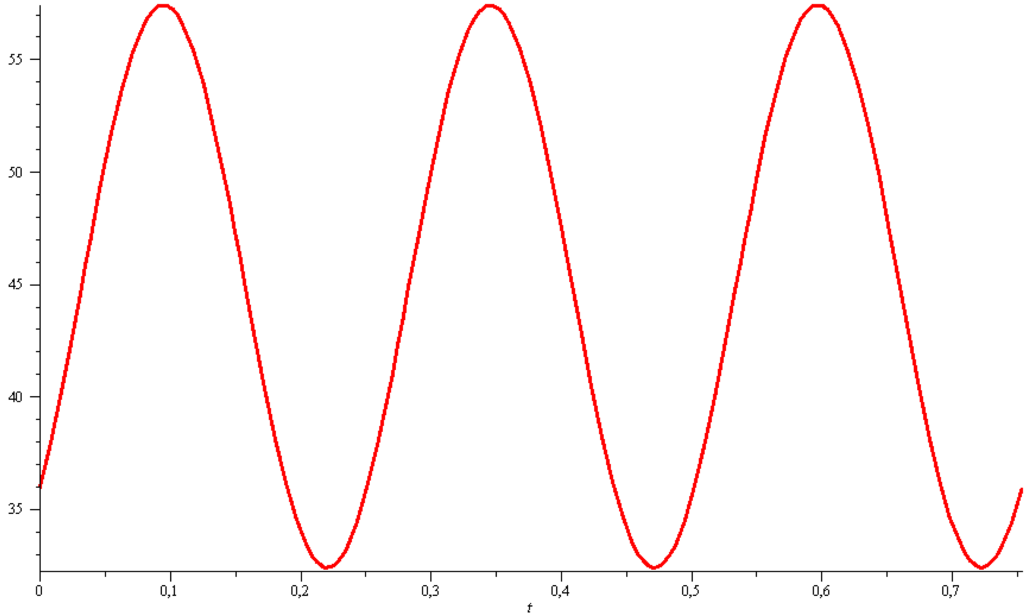

| | $t \to \cos t$ | $t \to \cos\left(t - \frac{\pi}{2}\right)$ | $t \to \cos\left(t + \frac{\pi}{2}\right)$ |
|---|---|---|---|
| amplitude | 1 | 1 | 1 |
| maximum | 1 | 1 | 1 |
| minimum | $-1$ | $-1$ | $-1$ |
| period | $2\pi$ | $2\pi$ | $2\pi$ |
| frequency | $1/2\pi$ | $1/2\pi$ | $1/2\pi$ |
| smallest positive zero place | $\pi$ | $\pi/2$ | $\pi/2$ |

## *Exercise*

Plot function

$$x(t) = 12.5 \cdot \sin(25t - 0{,}8) + 44.9$$

on interval $[0{,}3T]$ and characterize it.



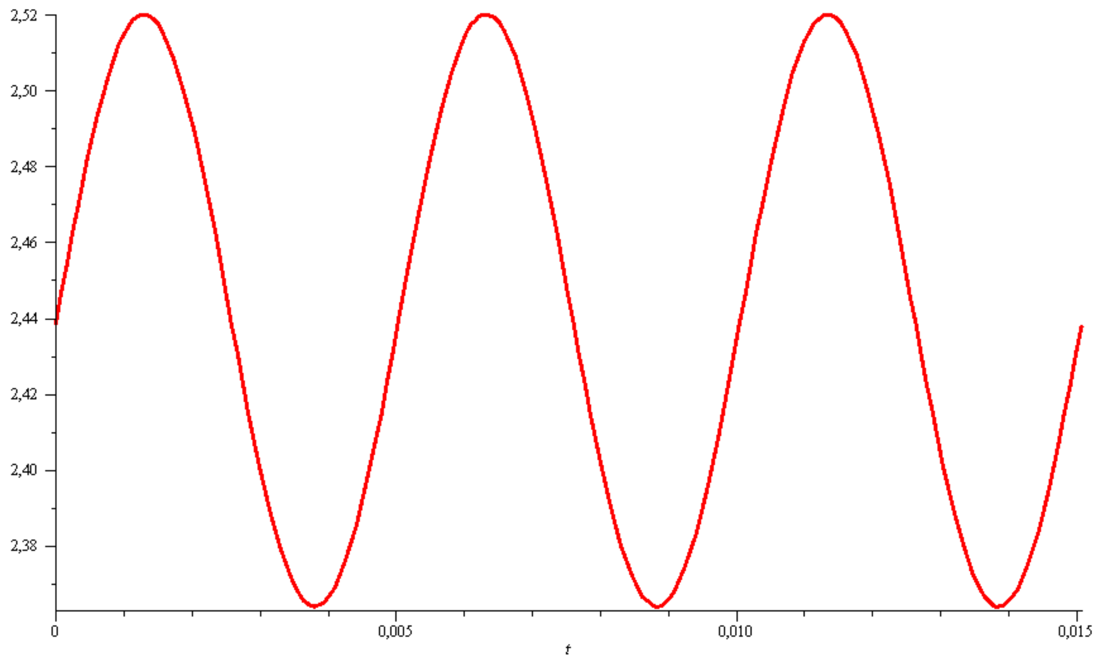

| | |
|---|---|
| amplitude | 12.5 |
| maximum | 57.4 |
| minimum | 32.4 |
| period | $\frac{2\pi}{25} \approx 0.251$ |
| frequency | $\frac{25}{2\pi} \approx 3.979$ |
| angular frequency | 25 |
| phase | $-0.8$ |

## ***Exercise***

Plot function

$$x(t) = 0.078 \cdot \sin(1250t - 0.05) + 2.442$$

on interval $[0,3T]$ and characterize it.




| | |
|---|---|
| amplitude | 0.078 |
| maximum | 2.520 |
| minimum | 2.364 |
| period | $\frac{2\pi}{1250} \approx 0.005$ |
| frequency | $\frac{1250}{2\pi} \approx 200$ |
| angular frequency | 1250 |
| phase | $-0.05$ |

***Exercise:*** Determine the period of function

$$10.2 \cdot \sin(0.12 \cdot x) - 3.3 \cdot \sin(5.5 \cdot x) + 0.8 \cdot \cos(1.28 \cdot x)$$

Periods of the three functions in the linear combination are

$$T_1 = \frac{2\pi}{0.12}, \quad T_2 = \frac{2\pi}{5.5}, \quad T_3 = \frac{2\pi}{1.28}$$

The ratios of periods in the simplest form

$$\frac{T_1}{T_2} = \frac{5.5}{0.12} = \frac{\mathbf{275}}{\mathbf{6}}, \qquad \frac{T_1}{T_3} = \frac{1.28}{0.12} = \frac{\mathbf{32}}{\mathbf{3}}$$

The least common multiple of denominators: $m = lkkt\{6,3\} = 6$

Thus, the period is

$$T = T_1 \cdot m = \frac{2\pi}{0.12} \cdot 6 = 100\pi$$

# 2nd week

## 2 Statistical Analysis of Vibration Signals

Certain types of mechanical damage of rotating parts imply the change of some statistical parameters in time, such as mean, standard deviation, RMS, peak value, skewness and kurtosis of the vibration velocity or acceleration data in the sampled signal.

The changed shape of the probability density function of the vibration velocity or acceleration data can be an indicator of failures. The level of shock pulses generated by a healthy ball bearing follows normal distribution, the appearance of damage in the bearing results in the change of probability density function.

Some of the abovementioned parameters are related to the statistical moments of the probability density function.

The $n$-th raw moment (i.e., moment about zero) of a distribution is defined by

$$M_n = \int_{-\infty}^{\infty} x^n \cdot f(x)\,dx, \quad n = 1,2,\ldots,$$

while the $n$-th moment about $c$ is

$$M_{n,c} = \int_{-\infty}^{\infty} (x-c)^n \cdot f(x)\,dx, \quad n = 0,1,2,\ldots,$$

where $f$ is the probability density function.

If $c$ is the expected value, then $M_{n,c}$ is called the $n$-th central moment.

The normalised $n$-th central moment (or standardised moment) is the $n$-th central moment divided by the $n$-th power of the standard deviation:

$$\frac{\int_{-\infty}^{\infty}\left(x-E(X)\right)^n \cdot f(x)\,dx}{\left(\sqrt{\int_{-\infty}^{\infty}\left(x-E(X)\right)^2 \cdot f(x)\,dx}\right)^n} = \frac{E\left((X-\mu)^n\right)}{\sigma^n},$$

where

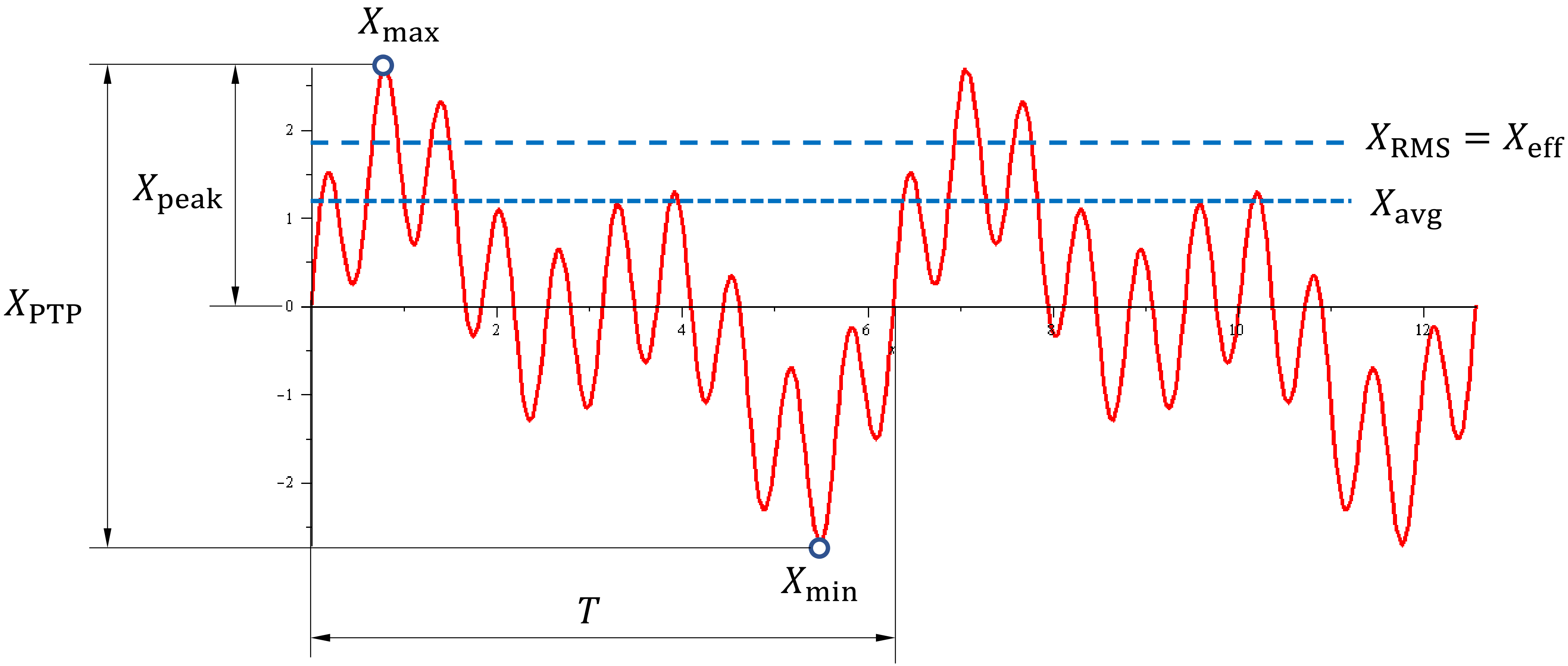
$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

is the mean and

$$\sigma = \sqrt{Var(X)} = \sqrt{\int_{-\infty}^{\infty}\left(x-E(X)\right)^2 \cdot f(x)\,dx}$$

is the standard deviation.

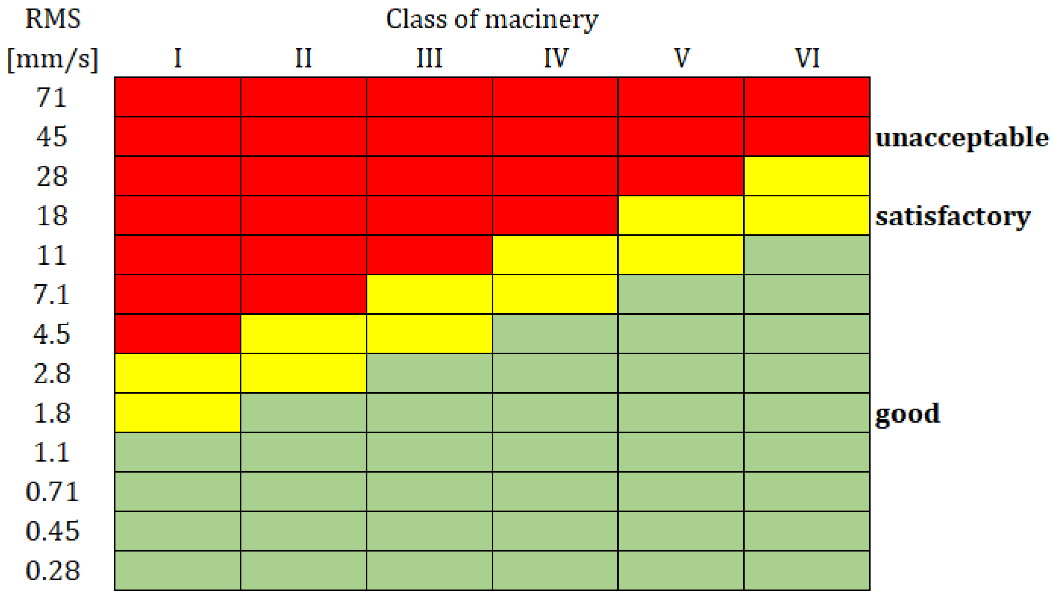
| moment ordinal | raw moment | central moment | standardised moment |
|---|---|---|---|
| 1 | mean | | |
| 2 | | variance | |
| 3 | | | skewness |
| 4 | | | kurtosis |



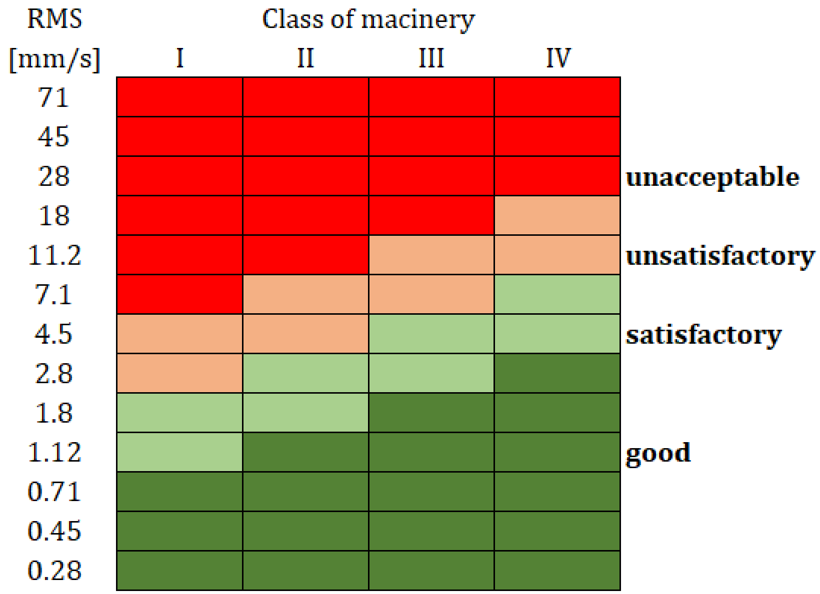
# Mean values

Mean values of the vibration velocity signal inform us about the severity of the vibration. In vibration standards there are limits for the RMS of the vibration velocity belonging to the different classes of machines (classified according to size, power and function).

**ISO 2372**




**ISO 10816**

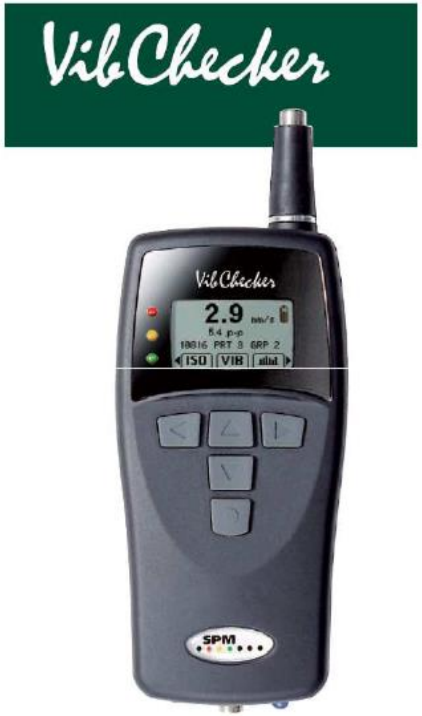


The **Root Mean Square** (***RMS*** or effective value) of quantity $X$ and its estimation from a sample are

| definition | estimation |
|---|---|
| $X_{\mathrm{RMS}} = X_{\mathrm{eff}} = \sqrt{\frac{1}{T} \cdot \int_0^T x^2(t)\,dt}$ <br> where $T$ is the period of $x(t)$. | $\sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} x_i^2}$ |

***Remark:***

Simple measuring equipment generally provide the RMS of the vibration velocity



RMS does not increase (significantly) with the isolated peaks in the signal – while a periodic series of high energy events will increase the overall level of vibration (value of RMS). Thus, RMS is not sensitive to incipient failures and starts indicating a fault only after the damage crossed a certain level of severity. The main usage of this parameter is to monitor the overall vibration level and is used in conjunction with other parameters.

The **mean of the absolute value** of quantity $X$ and its estimation from a sample are

| definition | estimation |
|---|---|
| $$X_{avg} = \frac{1}{T} \cdot \int_0^T \lvert x(t) \rvert \, dt$$ where $T$ is the period of $x(t)$. | $$\frac{1}{n} \cdot \sum_{i=1}^{n} \lvert x_i \rvert$$ |

# Peak values

**Peak to Peak value** of quantity $X$ and its estimation

| definition | estimation |
|---|---|
| $X_{\mathrm{PTP}} = x_{\max} - x_{\min}$ | $X_{\mathrm{PTP}} = x_{\max} - x_{\min}$ |

**Peak value** of quantity $X$ and its estimation

| definition | estimation |
|---|---|
| $X_{\mathrm{peak}} = max\{\lvert x_{\max}\rvert, \lvert x_{\min}\rvert\}$ | $X_{\mathrm{peak}} = max\{\lvert x_{\max}\rvert, \lvert x_{\min}\rvert\}$ |

# Moments

## ***Skewness***

Skewness characterises the symmetry of the distribution around its mean. The skewness for symmetric distributions is zero. Its negative / positive value means that the tail of the probability density function in the left/right side is longer than that in the opposite side.

The mean of positively / negatively skewed data will be greater / less than the median.

| definition | estimation |
| --- | --- |
| $\frac{E((X-\mu)^3)}{\sigma^3}$ | a biased estimation<br>$$\frac{\frac{1}{n}\cdot\sum_{i=1}^{n}(x_i-\overline{x})^3}{\left(\sqrt{\frac{1}{n}\cdot\sum_{i=1}^{n}(x_i-\overline{x})^2}\right)^3}$$<br>an unbiased estimation<br>$$\frac{n}{(n-1)\cdot(n-2)}\cdot\frac{\sum_{i=1}^{n}(x_i-\overline{x})^3}{s^3}$$ |

### ***Kurtosis***

Kurtosis describes the shape of the distribution. In signal processing, kurtosis can reveal the flatness or the spikiness of the signal. Its value is low for good bearing and high for bearings due to spiky nature of the signal.

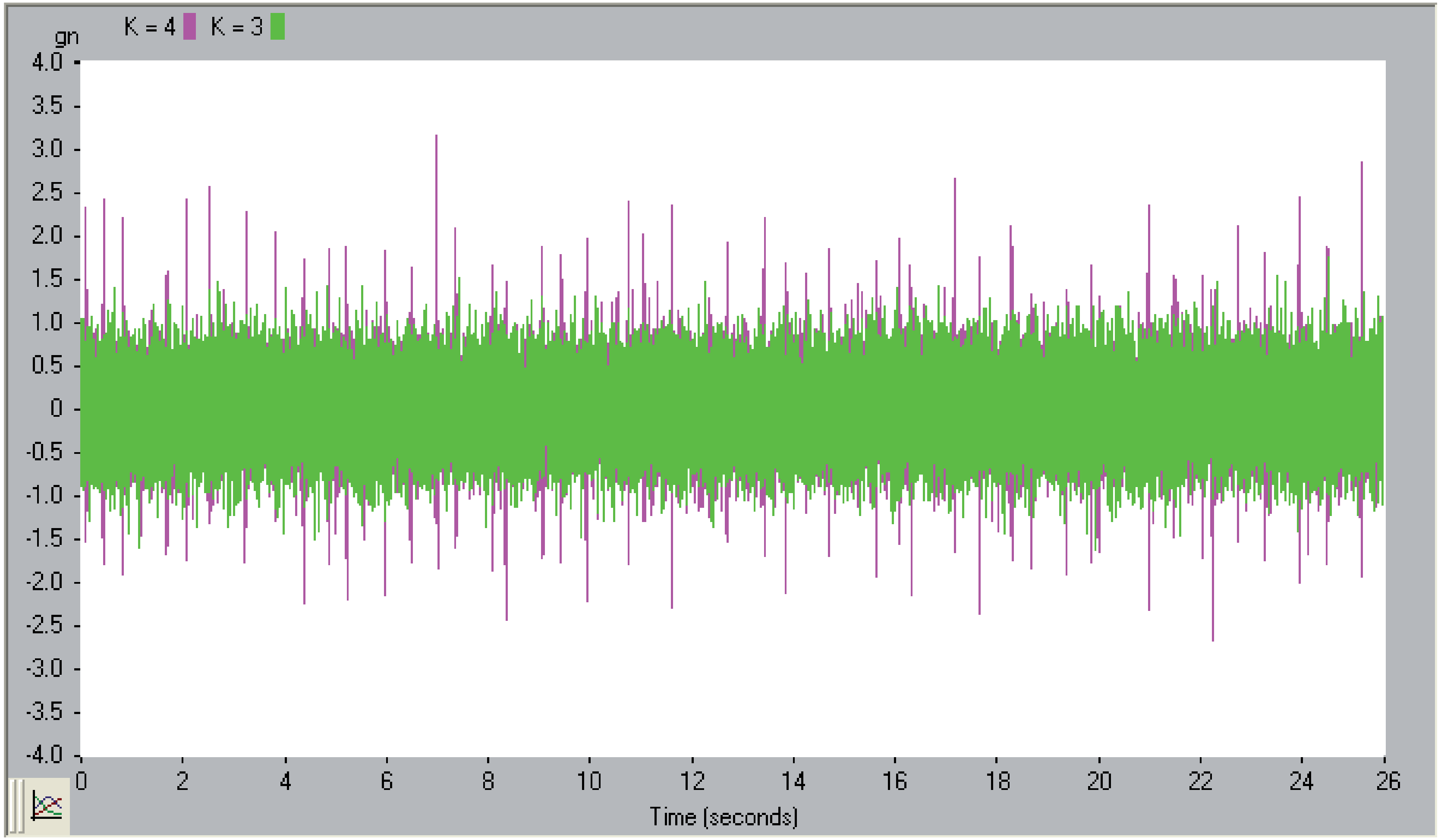
The kurtosis for a signal of Gaussian distribution is around 3. As faults appear on the ball bearing and the signal becomes noisy kurtosis will higher than 3.

| definition | estimation |
|---|---|
| $\frac{E((X-\mu)^4)}{\sigma^4}-3$ | a biased estimation<br>$\frac{\frac{1}{n}\cdot\sum_{i=1}^{n}(x_i-\overline{x})^4}{\left(\sqrt{\frac{1}{n}\cdot\sum_{i=1}^{n}(x_i-\overline{x})^2}\right)^4}-3$<br>an unbiased estimation<br>$\frac{n\cdot(n-1)}{(n-1)\cdot(n-2)\cdot(n-3)}\cdot\frac{\sum_{i=1}^{n}(x_i-\overline{x})^4}{s^4}-3\cdot\frac{(n-1)^2}{(n-2)\cdot(n-3)}$ |

**Kurtosis** is a parameter defined as the fourth centralized moment of the signal normalized by square of the variance of the signal.

If a vibration signal contains events which are impulsive in nature, then its overall amplitude distribution function is sharper, leading to higher kurtosis values. However, as the gear fault develops from being localised to more widely distributed, the generated vibration acceleration signal becomes less impulsive and transforms into more complex signal containing high energy, more widely distributed components, which reduces the peakedness of the amplitude distribution and causes the kurtosis to drop. [4]

Signals that have a higher kurtosis value have more peaks that are greater than three-sigma. In the real world many kinds of vibration environments are characterized by signals that have high kurtosis value, the fatigue and damage potential for these vibrations are higher.




Comparison of the control signals for a Gaussian signal (kurtosis=3) and a non-Gaussian signal with kurtosis=4.

## *Crest factor*

Crest factor was designed to detect early impulses appearing in the signal that are characteristic for an incipient gear fault:

$$\frac{X_{peak}}{X_{RMS}}$$

For example in gear monitoring, as the gear tooth condition deteriorates the impulsive content within the signal ($X_{peak}$) increases, while the energy within the impulses is not big enough to cause noticeable changes in $X_{RMS}$. This results in increase of the crest factor. However, as the damage progress the RMS values start to increase quicker than the maximal absolute amplitude which causes the overall crest factor value to decrease. Thus crest factor can be useful in indicating the early stages of gear fault development.

# Reliability

Reliability is a characteristic of the item / system, expressed by the probability that

- it will perform its required function
- under given conditions
- for a stated time interval.

For a given mission time $T$

$$R = P(\tau > T)$$

is a simple number ($\tau$ is the time to failure or the failure-free time).

$R$ is the probability that no failure will occur in the interval $[0, T]$.

If $n$ statistically identical and independent items are put into operation at time $t = 0$ to perform a given mission and $\bar{\nu}$ of them accomplish it successfully, then the ratio

$$\frac{\bar{\nu}}{n}$$

is a random variable which converges for increasing $n$ to the true value of the reliability.

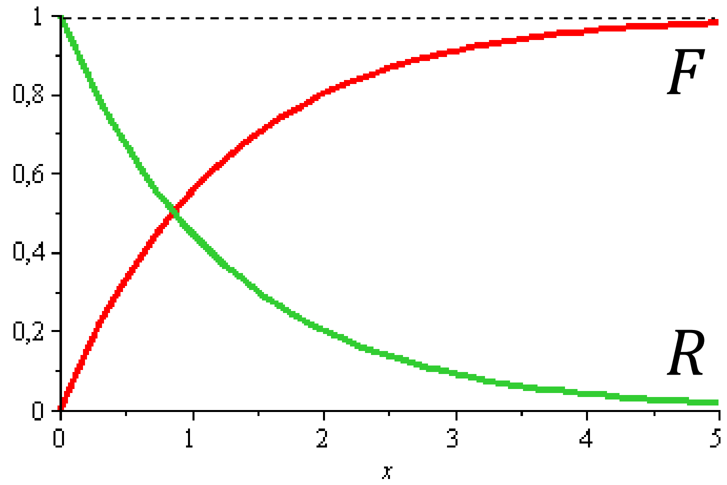
## Reliability function

$$R(t) = P(\tau > t), \quad t > 0$$

If $F$ is the cumulative distribution function of the failure-free time, then

$$R(t) = P(\tau > t) = 1 - F(t), \quad t > 0$$

The figure shows the cumulative distribution function and the reliability function of exponentially distributed time to failure.



# Failure Rate, Mean Time to Failure, Mean Time Between Failures

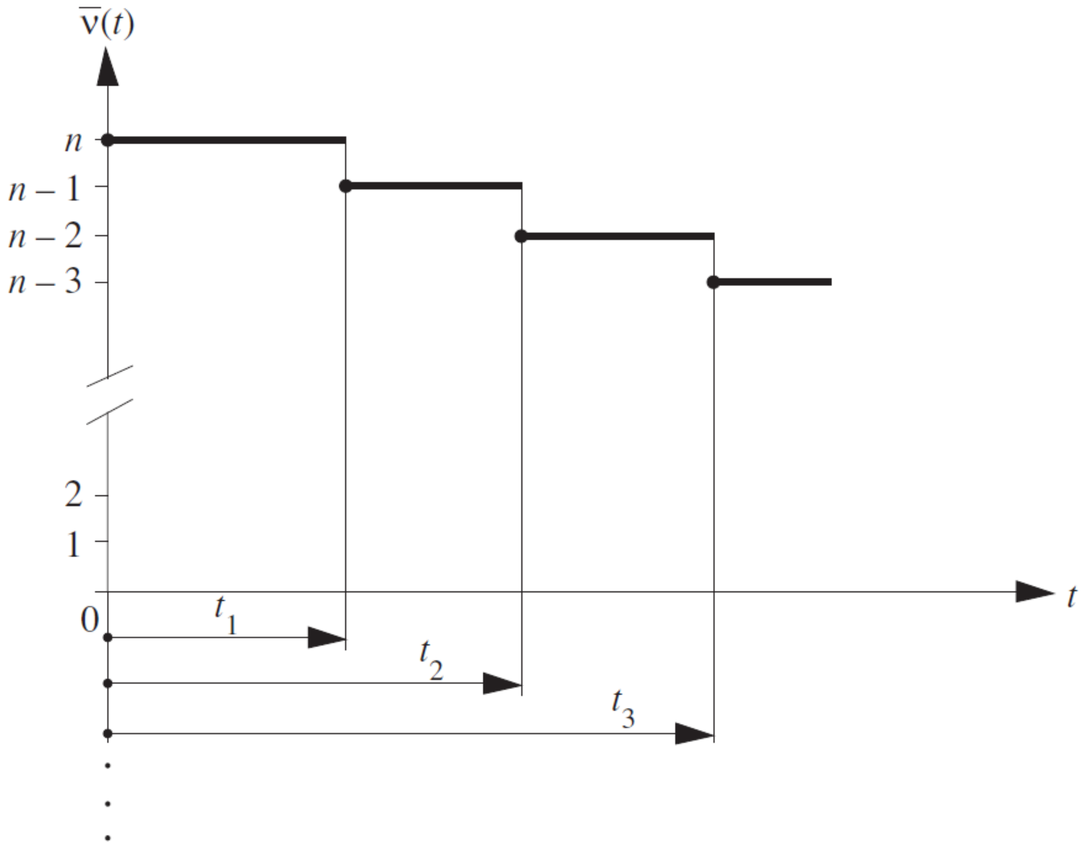
Let us assume that $n$ statistically identical, new, and independent items are put into operation at time $t = 0$, under the same conditions.

$$\bar{v}(t)$$

is the number of items that have not yet failed at the time $t$.

$$t_1, \ldots t_n$$

are the observed failure-free times (operating times to failure).



Empirical reliability function

$$\hat{R}(t) = \frac{\bar{v}(t)}{n}$$

***Remark***

$\hat{R}(t)$ converges to $R(t)$ for $n \to \infty$.

Empirical failure rate for an interval $[t, t + \delta t]$

$$\hat{\lambda}(t) = \frac{\bar{v}(t) - \bar{v}(t + \delta t)}{\bar{v}(t)} \cdot \frac{1}{\delta t} = \frac{\hat{R}(t) - \hat{R}(t + \delta t)}{\hat{R}(t)} \cdot \frac{1}{\delta t}$$

***Remark***

$\hat{\lambda}(t) \cdot \delta t$ is the ratio of the items failed in the interval $[t, t + \delta t]$ to the number of items still operating (or surviving) at time $t$.

If $R(t)$ is derivable

$$\lambda(t) = -\frac{1}{R(t)} \cdot \frac{dR(t)}{dt}$$

Considering $R(0) = 1$ we have

$$R(t) = e^{-\int_0^t \lambda(x) dx}$$

In many practical applications, $\lambda(t) = \lambda$ can be assumed. Then

- $R(t) = e^{-\lambda t}$
- the failure-free time $\tau$ is exponentially distributed ($F(t) = 1 - e^{-\lambda t}$, t $> 0$).
- $\hat{\lambda} = \frac{k}{T}$ ($k$ is the number of failures during $T$)

Mean time to failure

$$MTTF = E[\tau] = \int_0^\infty t \cdot f(t)\,dt$$

***Remark***

If $\lim_{t\to\infty} t \cdot R(t) = 0$, then

$$MTBF = E(T) = \int_0^\infty t \cdot f(t)\,dt = \int_0^\infty R(t)\,dt$$

For $\lambda(t) = \lambda$ it follows $E[\tau] = \frac{1}{\lambda}$.

***Example***

1,500 pieces of an electronic component are installed. The table shows the failure rate values for this type of components. Let's calculate the number of operating (surviving) components month-by-month using the given failure rates.

| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| | | **1500** |
| 1 | 0.05 | |
| 2 | 0.015 | |
| 3 | 0.008 | |
| 4 | 0.005 | |
| 5 | 0.005 | |

*Solution:*

| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| | | **1500** |
| 1 | 0.05 | **1425** |
| 2 | 0.015 | **1404** |
| 3 | 0.008 | **1392** |
| 4 | 0.005 | **1385** |
| 5 | 0.005 | **1379** |

# WEIBULL DISTRIBUTION

Probability density function of the Weibull distribution is

$$f(t) = \frac{\alpha}{\beta} \cdot \left(\frac{t}{\beta}\right)^{\alpha-1} \cdot e^{-\left(\frac{t}{\beta}\right)^{\alpha}}, \quad t > 0$$

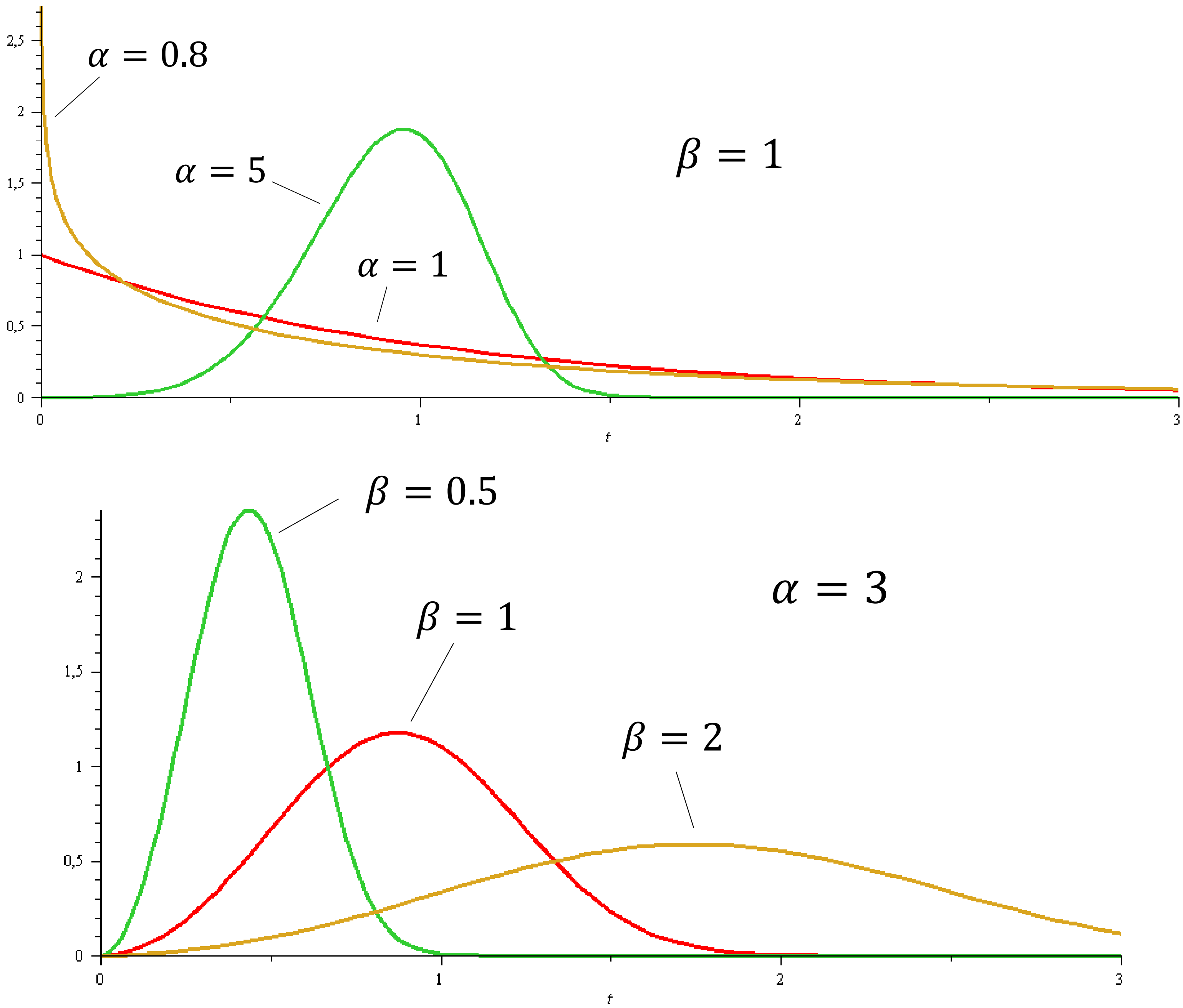
$\beta > 0$ is the scale parameter

$\alpha > 0$ is the shape parameter.

When $\beta = 1$, the Weibull distribution is identical to the exponential distribution.

Cumulative distribution function of the Weibull distribution is

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^{\alpha}}$$



$\alpha = 0.8$
$\alpha = 5$
$\beta = 1$
$\alpha = 1$
2,5
2
1,5
1
0,5
0
0
1
2
3
t
$\beta = 0.5$
$\alpha = 3$
$\beta = 1$
$\beta = 2$
2
1,5
1
0,5
0
0
1
2
3
t

# Linearization of the Weibull model

$$F = 1 - e^{-\left(\frac{x}{\boldsymbol{\beta}}\right)^{\boldsymbol{\alpha}}}$$

$$\ln\left(\ln\frac{1}{1-F}\right) = \boldsymbol{\alpha} \cdot \ln x - \alpha \cdot \ln \beta$$

$\alpha$: slope $\Rightarrow \boldsymbol{\alpha}$

$-\alpha \cdot \ln \beta$: intercept $\Rightarrow \boldsymbol{\beta}$

$$F = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

$$1 - F = e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

$$\ln(1-F) = -\left(\frac{x}{\beta}\right)^{\alpha}$$

$$\ln\frac{1}{1-F} = \left(\frac{x}{\beta}\right)^{\alpha}$$

$$\ln\left(\ln\frac{1}{1-F}\right) = \alpha \cdot \ln\frac{x}{\beta}$$

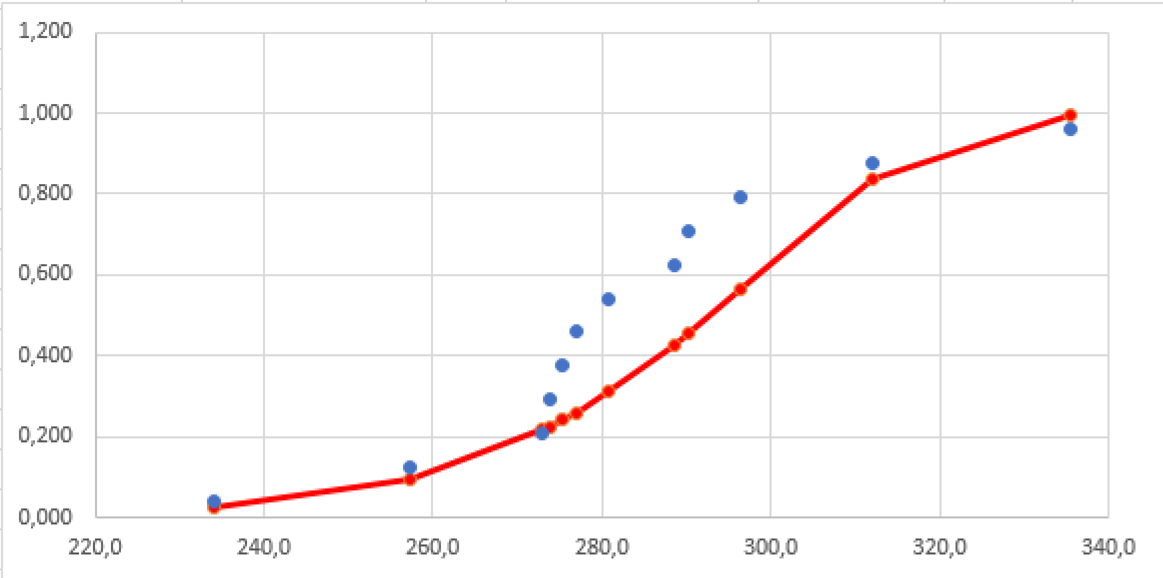
$$\ln\left(\ln\frac{1}{1-F}\right) = \alpha \cdot \ln x - \alpha \cdot \ln \beta$$

# Weibull model fitting with linearization

| $n$ | $x$ (ordered sample) | $ln\, x$ | $F = \frac{n-0.5}{N}$ (cumulative probability) | $ln\left(ln\frac{1}{1-F}\right)$ |
|---|---|---|---|---|
| 1 | $x_1$ | $ln\, x_1$ | $0.5/N$ | |
| 2 | $x_2$ | $ln\, x_2$ | $1.5/N$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $N$ | $x_N$ | $ln\, x_N$ | $(N-0.5)/N$ | |

# Weibull model fitting with Excel Solver

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| alpha: | Weibull modulus | | 15,00 | | | |
| beta: | scale factor | | 300,00 | | | |
| | | | | | | |
| **failure stress (sample)** | **failure stress (S) (sorted sample)** | | **Cumulative probability (F)** | **Weibull** | **SS** | |
| 257,4 | 234,0 | 1 | 0,042 | 0,023779541 | 0,273618 | |
| 290,2 | 257,4 | 2 | 0,125 | 0,095644873 | | |
| 335,4 | 273,0 | 3 | 0,208 | 0,215734898 | | |
| 273,8 | 273,8 | 4 | 0,292 | 0,224239757 | | |
| 276,9 | 275,3 | 5 | 0,375 | 0,240881141 | | |
| 312,0 | 276,9 | 6 | 0,458 | 0,259644862 | | |
| 234,0 | 280,8 | 7 | 0,542 | 0,309816861 | | |
| 273,0 | 288,6 | 8 | 0,625 | 0,428377156 | | |
| 280,8 | 290,2 | 9 | 0,708 | 0,455362219 | | |
| 275,3 | 296,4 | 10 | 0,792 | 0,565848328 | | |
| 296,4 | 312,0 | 11 | 0,875 | 0,834856999 | | |
| 288,6 | 335,4 | 12 | 0,958 | 0,995149973 | | |



# 2nd week – Questions

### Question 1

Give the definition of RMS

### Answer

$$X_{\mathrm{RMS}} = X_{\mathrm{eff}} = \sqrt{\frac{1}{T} \cdot \int_{0}^{T} x^2(t)\, dt}$$

## Question 2

Give the definition of skewness and kurtosis

## Answer

$$\frac{E((X-\mu)^3)}{\sigma^3}$$

$$\frac{E((X-\mu)^4)}{\sigma^4}-3$$

## *Question 3*

Give the Cumulative distribution function of the Weibull distribution and its linearization.

## *Answer*

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^{\alpha}}$$

$$\ln\left(\ln\frac{1}{1-F}\right) = \alpha \cdot \ln x - \alpha \cdot \ln \beta$$

# 2nd week – Exercises

### *Exercise*

In cell A1:A1000 of an Excel worksheet a 1000-element sample is available.

Calculate the following values

| | |
|---|---|
| $X_{RMS}$, | kurtosis |
| $X_{PTP}$, | skewness |
| $X_{peak}$, | crest factor |

☑

| | |
|---|---|
| $X_{RMS}$ | SQRT(SUMSQ(A1:A1000)/1000) |
| $X_{PTP}$ | MAX(A1:A1000)-MIN(A1:A1000) |
| $X_{peak}$ | MAX(ABS(MAX(A1:A1000));ABS(MIN(A1:A1000))) |
| kurtosis | KURT(A1:A100) |
| skewness | SKEW(A1:A100) |
| crest factor | $X_{peak}/X_{RMS}$ |

# Exercise

1500 electronic parts have been built into devices. The table below contains the observed failure rate values for every month in two years.

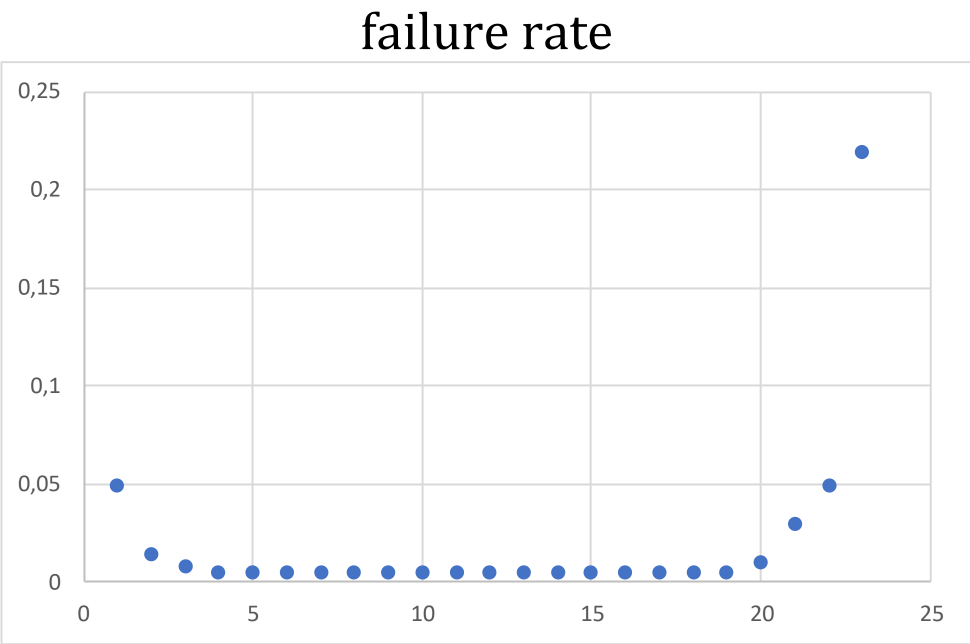
Calculate the number of operating components at the end of the months supposing that real number of failures per month is equal that follows from the failure rate.

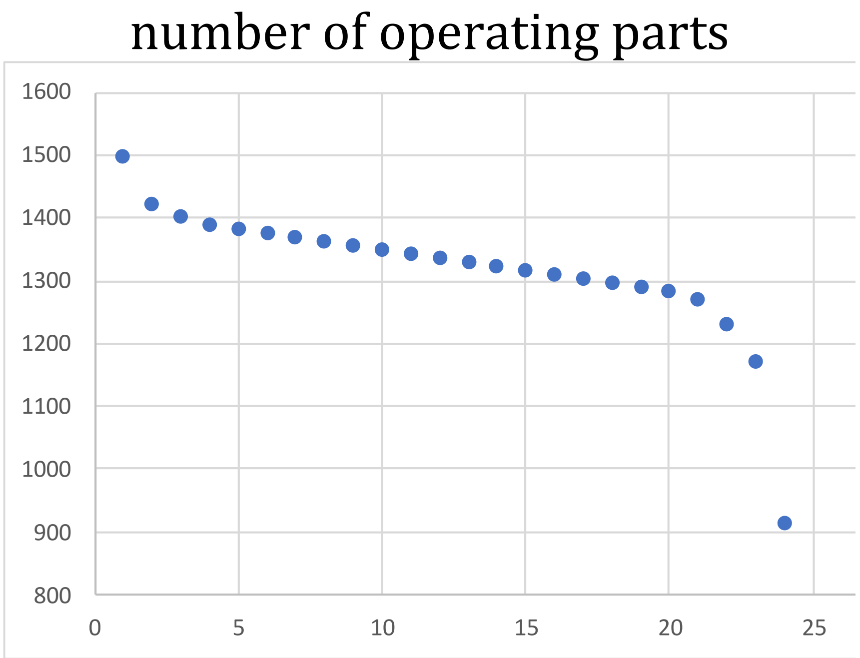
| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| 1 | 0.05 | |
| 2 | 0.015 | |
| 3 | 0.008 | |
| 4 | 0.005 | |
| 5 | 0.005 | |
| 6 | 0.005 | |
| 7 | 0.005 | |
| 8 | 0.005 | |
| 9 | 0.005 | |
| 10 | 0.005 | |
| 11 | 0.005 | |
| 12 | 0.005 | |

| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| 13 | 0.005 | |
| 14 | 0.005 | |
| 15 | 0.005 | |
| 16 | 0.005 | |
| 17 | 0.005 | |
| 18 | 0.005 | |
| 19 | 0.005 | |
| 20 | 0.01 | |
| 21 | 0.03 | |
| 22 | 0.05 | |
| 23 | 0.22 | |
| 24 | | |

| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| | | 1500 |
| 1 | 0.05 | 1425 |
| 2 | 0.015 | 1404 |
| 3 | 0.008 | 1392 |
| 4 | 0.005 | 1385 |
| 5 | 0.005 | 1379 |
| 6 | 0.005 | 1372 |
| 7 | 0.005 | 1365 |
| 8 | 0.005 | 1358 |
| 9 | 0.005 | 1351 |
| 10 | 0.005 | 1344 |
| 11 | 0.005 | 1338 |
| 12 | 0.005 | 1331 |

| month | failure rate (1/month) | number of operating parts |
|---|---|---|
| | | |
| 13 | 0.005 | 1324 |
| 14 | 0.005 | 1318 |
| 15 | 0.005 | 1311 |
| 16 | 0.005 | 1305 |
| 17 | 0.005 | 1298 |
| 18 | 0.005 | 1292 |
| 19 | 0.005 | 1285 |
| 20 | 0.01 | 1272 |
| 21 | 0.03 | 1234 |
| 22 | 0.05 | 1172 |
| 23 | 0.22 | 914 |
| 24 | | 804 |


failure rate
0,25
0,2
0,15
0,1
0,05
0
0
5
10
15
20
25


number of operating parts
1600
1500
1400
1300
1200
1100
1000
900
800
0
5
10
15
20
25

# 3rd week

# 3 Hilbert Spaces, Orthogonality, Similarity of Functions

# The Concept of Hilbert Spaces

Let $X$ be a real or complex linear space. A function $\langle\quad\rangle: X \times X \to \mathbb{C}$ is called *inner product* (or *scalar product*) if

$$\mathbb{R} \ni \langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0$$

$$\langle x, y \rangle = \langle y, x \rangle^*$$

$$\langle \lambda \cdot x, y \rangle = \lambda \cdot \langle x, y \rangle$$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

hold for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$.

$x^*$ denotes the complex conjugate of $x \in \mathbb{C}$.

***Remark***

$$\langle x, \lambda \cdot y \rangle = \langle \lambda \cdot y, x \rangle^* = (\lambda \cdot \langle y, x \rangle)^* = \lambda^* \cdot \langle y, x \rangle^* = \lambda^* \cdot \langle x, y \rangle$$

***Remark***

$$\langle x, y + z \rangle = \langle y + z, x \rangle^* = \langle y, x \rangle^* + \langle z, x \rangle^* = \langle x, y \rangle + \langle x, z \rangle$$

***Remark***

In the special case when the inner product is a real-valued function $\langle\quad\rangle: X \times X \to \mathbb{R}$ the second property can be simply written as $\langle x, y \rangle = \langle y, x \rangle$.

The pair $(X, \langle\quad\rangle)$ is called *inner product space*.

An inner product space $(X, \langle \quad \rangle)$ is called *Hilbert space* if $X$ is a Banach space with the norm function $\| \quad \|: X \to \mathbb{R}$ defined as $\|x\| = \sqrt{\langle x, x \rangle}$.

***Remark***

Each Hilbert space $(X, \langle \quad \rangle)$ is a normed space with the norm

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in X.$$

The value of inner product characterizes the 'similarity' of elements in a Hilbert space. The higher the value of the inner product is, the more 'similar' the two elements are.

***Remark***

In the space of spatial vectors the inner product of two vectors of given length is zero if they are orthogonal ('not similar'), and it reaches its maximum if they are parallel ('similar').

# Finite Dimensional Hilbert spaces

A Hilbert space $(X, \langle \quad \rangle)$ is finite dimensional if $X$ is a finite dimensional linear space.

Let $n$ be a positive integer and suppose that $X$ is an $n$-dimensional Hilbert space.

A system of vectors $\{b_1, \dots, b_k\} \subset X, k \in \mathbb{N}$ is *orthogonal* if its elements are pairwise orthogonal.

***Remark***

It is easy to see that each orthogonal system of non-zero vectors is linearly independent.

The system is *orthonormal* if orthogonal and normed, that is, the elements are unit vectors (the norm of each vector in the system is equal to 1).

If an orthogonal (orthonormal) system $\{b_1, \dots, b_n\} \subset X$ is a basis of $X$, it is called *orthogonal (orthonormal) basis* of $X$.

If $\{b_1, \dots, b_n\} \subset X$ is an orthonormal basis of $X$ and $x \in X$ then

$$x = \sum_{i=1}^{n} \langle x, b_i \rangle \cdot b_i.$$

This sum is also called the *decomposition* $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$. The coefficients

$$\langle x, b_i \rangle, \quad i = 1, .., n$$

are the *coordinates* of $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$.

### *Remark*

Coordinates of a vector $x \in X$ with respect to an orthonormal basis $\{b_1, \dots, b_n\}$ can be calculated as the inner product of $x$ and the basis vectors $b_i$. We will use the generalization of this statement in function spaces when we are talking about the decomposition of functions with respect to an orthonormal system of functions.

## *The space of spatial (planar) vectors*

The space of *spatial (planar) vectors* is a 3-dimensional (2-dimensional) Hilbert space with the inner product

$$\langle \bar{u}, \bar{v} \rangle = \|\bar{u}\| \cdot \|\bar{v}\| \cdot \cos \sphericalangle(\bar{u}, \bar{v})$$

### *Remark*

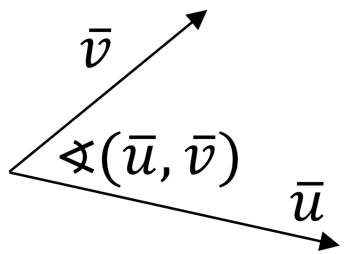
The inner product of spatial (planar) vectors is generally called scalar product and denoted simply by $\bar{u} \cdot \bar{v}$ or $\bar{u}\bar{v}$.

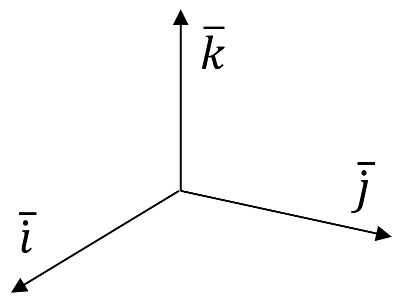
### *Remark*

Using the definition we have

$$\sphericalangle(\bar{u}, \bar{v}) = \arccos \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|},$$

that is, the angle of spatial (planar) vectors can be calculated from the scalar product and the magnitude (norm) of the vectors.




Spatial (planar) vectors $\bar{u}$ and $\bar{v}$ are *perpendicular* in geometry iff $\bar{u} \cdot \bar{v} = 0$.

Perpendicular vectors are also called *orthogonal*.

In applications (for instance in mechanics) the orthonormal basis is generally denoted by $\{\bar{i},\bar{j},\bar{k}\}$. Coordinates of a spatial vector $\bar{v}$ with respect to the orthonormal basis $\{\bar{i},\bar{j},\bar{k}\}$ are

$$v_x = \langle \bar{v},\bar{i}\,\rangle,\;\; v_y = \langle \bar{v},\bar{j}\,\rangle,\;\; v_z = \langle \bar{v},\bar{k}\,\rangle$$

and the decomposition $\bar{v}$ with respect to the orthonormal basis $\{\bar{i},\bar{j},\bar{k}\}$ is

$$\bar{v} = v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k} = \langle \bar{v},\bar{i}\,\rangle \cdot \bar{i} + \langle \bar{v},\bar{j}\,\rangle \cdot \bar{j} + \langle \bar{v},\bar{k}\,\rangle \cdot \bar{k}$$

### ***The space of real n-tuples***

$\mathbb{R}^n$ is a $n$-dimensional Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i\,,\quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, y = (y_1, \dots, y_n) \in \mathbb{R}^n\,.$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are called *orthogonal* if

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i = 0.$$

The 'natural' orthonormal basis is in $\mathbb{R}^n$ is $\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\right\}$ which is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$ and $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^2$.

## *The space of complex n-tuples*

$\mathbb{C}^n$ (which is an $n$-dimensional linear space over $\mathbb{C}$) is a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot y_i^*, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n, y = (y_1, \dots, y_n) \in \mathbb{C}^n \,.$$

$x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are called *orthogonal* if

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot y_i^* = 0.$$

# Orthogonality and Similarity of Functions

Let $I$ be an interval. A function $x: I \rightarrow \mathbb{C}$ is *square integrable* if

$$\int_I |x(t)|^2 < \infty.$$

$|\quad|$ denotes the magnitude (norm) of a complex number. The space of the square integrable functions defined on $I$ is denoted by $L_2(I)$.

***Remark***

A real valued function $x: I \rightarrow \mathbb{R}$ is square integrable if $\int_I \; x^2 < \infty$.

***Remark***

Square integrable functions are mathematical representations of finite energy signals.

The *inner product* of functions $x \in L_2(I)$ and $\psi \in L_2(I)$ is

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = \int_I x(t) \cdot \psi^*(t)\, dt.$$

***Remark***

If $x \in L_2(I)$ and $\psi \in L_2(I)$ are real-valued functions then $\psi^* = \psi$, and we can write

$$\langle x, \psi \rangle = \int_X (x \cdot \psi) = \int_X x(t) \cdot \psi(t)\, dt.$$

***Remark***

In the definition of the inner product in $L_2(I)$ the so-called Lebesgue integral is used, which is more general than the Riemann integral and more suitable for the general Fourier theory. If a function is Riemann integrable then it is also Lebesgue integrable, and the two integrals are equal. In engineering models, principally, the so-called piecewise continuous functions appear which are Riemann integrable. In our examples we have piecewise continuous functions, thus we have to calculate Riemann integrals.

***Remark***

In the definition the inner product the Lebesgue integrability of functions is supposed. If we want to introduce the concept of inner product $\langle x, \psi \rangle = \int_I (x \cdot \psi^*)$ in the class of Riemann integrable functions we also have to suppose the Riemann integrability of $x \cdot \psi^*$.

The *norm* of function $x \in L_2(I)$ is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_I |x|^2}.$$

***Remark***

If $x \in L_2(I)$ is a real-valued function we can write $\|x\| = \sqrt{\int_I \; x^2}$.

***Remark***

In engineering textbooks $\int_I \; |x|^2$ is frequently referred as the "energy content" of signal $x \in L_2(I)$.

Functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are *orthogonal* if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = 0.$$

***Remark***

The real-valued functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are orthogonal if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi) = 0.$$

***Remark***

The value of the inner product characterizes the "similarity" of the functions.

***Example***

Consider the following functions in $L_2([0,2\pi])$:

$$x_1(t) = \sin t, \qquad x_2(t) = \cos t,$$
$$x_3(t) = \sin 2t, \qquad x_4(t) = \cos 2t, \qquad t \in [0,2\pi]$$

$\langle x_i, x_j \rangle = 0$ if $i \neq j$, that is, functions $x_i$ are pairwise orthogonal.

For example, the calculation of $\langle x_1, x_4 \rangle$ and $\langle x_3, x_4 \rangle$ is as follows

$$\langle x_1, x_4 \rangle = \int_0^{2\pi} \sin t \cdot \cos 2t \, dt = \left[-\tfrac{2}{3} \cdot \cos^3 t - \cos t\right]_0^{2\pi} = 0$$

*Details of the integration:*

$$\int \sin t \cdot \cos 2t \, dt = \int \sin t \cdot (\cos^2 t - \sin^2 t) \, dt = \int \sin t \cdot (2\cos^2 t - 1) \, dt =$$
$$= -2 \int -\sin t \cdot \cos^2 t \, dt - \int \sin t \, dt = -\tfrac{2}{3} \cdot \cos^3 t - \cos t$$

$$\langle x_3, x_4 \rangle = \int_0^{2\pi} \sin 2t \cdot \cos 2t \, dt = \left[-\frac{1}{8} \cdot \cos 4t\right]_0^{2\pi} = 0$$

*Details of the integration:*

$$\int \sin 2t \cdot \cos 2t \, dt = \frac{1}{2} \int \sin 4t \, dt = -\frac{1}{8} \cdot \cos 4t$$

The norm of all the four functions is $\sqrt{\pi}$.

To get the norm it is more convenient to calculate $\|x_i\|^2, i = 1,2,3,4$ as follows

$$\|x_1\|^2 = \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 - \cos 2t) \, dt = \frac{1}{2} \cdot \left[t - \frac{1}{2} \cdot \sin 2t\right]_0^{2\pi} = \pi$$

$$\|x_2\|^2 = \int_0^{2\pi} \cos^2 t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 + \cos 2t) \, dt = \frac{1}{2} \cdot \left[t + \frac{1}{2} \cdot \sin 2t\right]_0^{2\pi} = \pi$$

$$\|x_3\|^2 = \int_0^{2\pi} \sin^2 2t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 - \cos 4t) \, dt = \frac{1}{2} \cdot \left[t - \frac{1}{4} \cdot \sin 4t\right]_0^{2\pi} = \pi$$

$$\|x_4\|^2 = \int_0^{2\pi} \cos^2 2t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 + \cos 4t) \, dt = \frac{1}{2} \cdot \left[t + \frac{1}{4} \cdot \sin 4t\right]_0^{2\pi} = \pi$$

That is, $\|x_i\| = \sqrt{\pi}, i = 1,2,3,4$.

***Example***

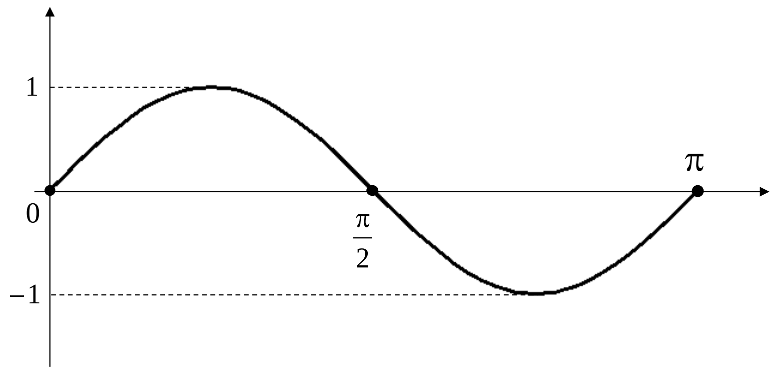
Consider the following functions in $L_2([0,\pi])$:

$$\psi(t) = \sin 2t\,, t \in [0,\pi] \qquad x_1(t) = \begin{cases} 1 & \text{if } t \in \left[0,\frac{\pi}{2}\right[ \\ -1 & \text{if } t \in \left[\frac{\pi}{2},\pi\right] \end{cases}$$

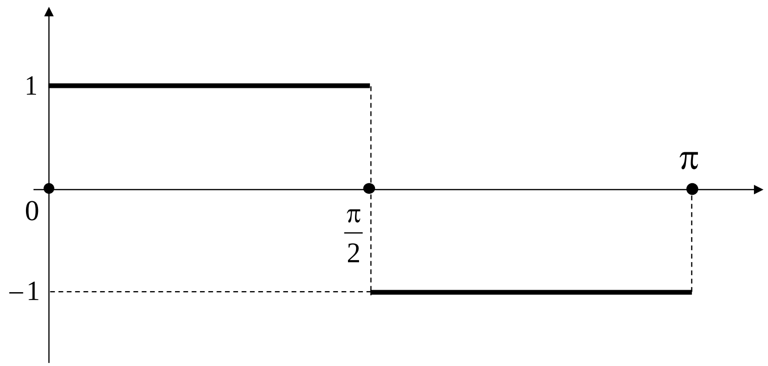
$$x_2(t) = \begin{cases} 1 & \text{if } t \in \left[0,\frac{\pi}{4}\right[ \text{ or } t \in \left[\frac{\pi}{2},\frac{3\pi}{4}\right[ \\ -1 & \text{if } t \in \left[\frac{\pi}{4},\frac{\pi}{2}\right[ \text{ or } t \in \left[\frac{3\pi}{4},\pi\right] \end{cases} \qquad x_3(t) = \begin{cases} 1 & \text{if } t \in \left[0,\frac{\pi}{3}\right[ \text{ or } t \in \left[\frac{2\pi}{3},\pi\right] \\ -1 & \text{if } t \in \left[\frac{\pi}{3},\frac{2\pi}{3}\right[ \end{cases}$$

Calculate the inner product of $\psi$ with $x_1$, $x_2$ and $x_3$, respectively, and compare the similarity of $\psi$ with the three functions.

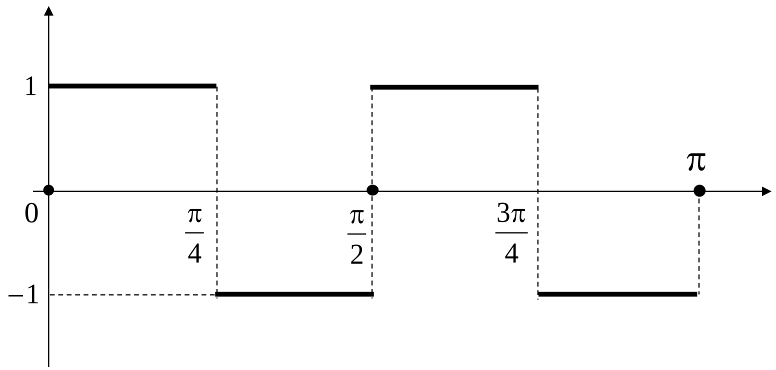
$\psi$



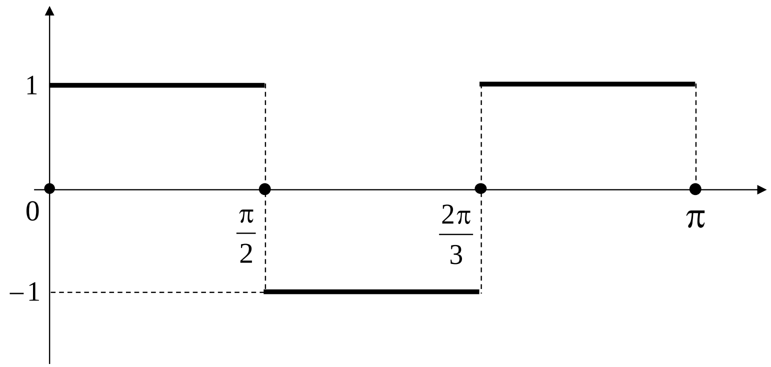

$x_1$



$x_2$




$x_3$




The inner product of $\psi$ with $x_1, x_2$ and $x_3$ are

$$\langle x_1, \psi \rangle = \int_0^{\pi} x_1(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{2}} \sin 2t\, dt - \int_{\frac{\pi}{2}}^{\pi} \sin 2t\, dt = -\frac{1}{2} \cdot [\cos 2t]_0^{\frac{\pi}{2}} + \frac{1}{2} \cdot [\cos 2t]_{\frac{\pi}{2}}^{\pi} = 2$$

$$\langle x_2, \psi \rangle = \int_0^{\pi} x_2(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{4}} \sin 2t\, dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2t\, dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2t\, dt - \int_{\frac{3\pi}{4}}^{\pi} \sin 2t\, dt = 0$$

$$\langle x_3, \psi \rangle = \int_0^{\pi} x_3(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{3}} \sin 2t\, dt - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin 2t\, dt + \int_{\frac{2\pi}{3}}^{\pi} \sin 2t\, dt = 1$$

That is, $0 = \langle x_2, \psi \rangle < \langle x_3, \psi \rangle < \langle x_1, \psi \rangle$.

This result implies that the similarity is the highest between $x_1$ and $\psi$, while $x_2$ and $\psi$ are not similar (actually, they are orthogonal).

# 3rd week – Questions

***Question 1***

Give the *inner product* of functions in $L_2(I)$

***Answer***

$$\langle f, g \rangle = \int_I (f \cdot g^*)$$

## Question 2

Give the norm in $L_2(I)$

## Answer

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_I \; |f|^2}.$$

## *Question 3*

Give the concept of orthogonality in $L_2(I)$

### *Answer*

Functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are *orthogonal* if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = 0.$$

# 3rd week – Exercises

## *Exercise*

Give the decomposition of vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to the orthonormal system

$$\left\{ a_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, a_2 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, a_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} \right\}$$

## *Solution*

$$v_1 = \langle v, a_1 \rangle = -\frac{1}{3}$$

$$v_2 = \langle v, a_2 \rangle = \frac{5}{3} \qquad v = -\frac{1}{3} \cdot a_1 + \frac{5}{3} \cdot a_2 + \frac{10}{3} \cdot a_3$$

$$v_3 = \langle v, a_3 \rangle = \frac{10}{3}$$

### Exercise

Let

$$x = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \quad y = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

Is the system $\{b_1, b_2, b_3\}$ orthonormal?

If yes, give the decomposition of $x$ with respect to system $\{b_1, b_2, b_3\}$.

### Solution

System $\{b_1, b_2, b_3\}$ is orthonormal, since on the one hand

$$\|b_1\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1, \quad \|b_2\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} = 1, \quad \|b_3\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1,$$

on the other hand

$$\langle b_1, b_2 \rangle = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \left(-\frac{2}{\sqrt{6}}\right) = 0.$$

$$\langle b_1, b_3 \rangle = \frac{1}{\sqrt{3}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot 0 = 0.$$

$$\langle b_2, b_3 \rangle = \frac{1}{\sqrt{6}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{2}{\sqrt{6}}\right) \cdot 0 = 0.$$

that is, vectors $b_i$ are pairwise orthogonal.

The inner products of $x$ and the elements of the system are

$$\langle x, b_1 \rangle = \frac{1}{\sqrt{3}}, \quad \langle x, b_2 \rangle = \frac{13}{\sqrt{6}}, \quad \langle x, b_3 \rangle = \frac{1}{\sqrt{2}}$$

so the decomposition of $x$ with respect to the orthonormal system $\{b_1, b_2, b_3\}$ is

$$x = \frac{1}{\sqrt{3}} \cdot b_1 + \frac{13}{\sqrt{6}} \cdot b_2 + \frac{1}{\sqrt{2}} \cdot b_3$$

### Exercise

Calculate the inner product of the given pairs of functions

| | | | |
|---|---|---|---|
| ***a*** | $f(t) = 200$ | $g(t) = \cos(0.001 \cdot t)$ | $t \in [0,1]$ |
| ***b*** | $f(t) = e^{10 \cdot t}$ | $g(t) = \sin(100\pi \cdot t)$ | $t \in [0,1]$ |
| ***c*** | $f(t) = (15 + i) \cdot t$ | $g(t) = e^{4\pi \cdot i \cdot t}$ | $t \in [0,1]$ |

### Solution a

$$\langle f, g \rangle = \int_0^1 200 \cdot \cos(0.001 \cdot t)\, dt = 200000 \cdot [\sin(0.001 \cdot t)]_0^1 = 200000 \cdot \sin(0.001) \approx 100$$

### Solution b

$$\langle f, g \rangle = \int_0^1 e^{10 \cdot t} \cdot \sin(100\pi \cdot t)\, dt =$$

$$= \frac{1}{1 + 100\pi^2} \cdot \left[\frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t)\right]_0^1 = \frac{\pi \cdot (1 - e^{10})}{1 + 100\pi^2}$$

*Details of the integration:*

$$\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt=\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-10\pi\cdot\int e^{10\cdot t}\cdot\cos(100\pi\cdot t)\,dt=$$

$$\begin{bmatrix} g(t)=\sin(100\pi\cdot t) & \Longrightarrow & g'(t)=100\pi\cdot\cos(100\pi\cdot t) \\ f'(t)=e^{10\cdot t} & \Longrightarrow & f(t)=\frac{1}{10}\cdot e^{10\cdot t}\end{bmatrix}$$

$$\begin{bmatrix} g(t)=\cos(100\pi\cdot t) & \Longrightarrow & g'(t)=-100\pi\cdot\sin(100\pi\cdot t) \\ f'(t)=e^{10\cdot t} & \Longrightarrow & f(t)=\frac{1}{10}\cdot e^{10\cdot t}\end{bmatrix}$$

$$=\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-10\pi\cdot\left(\frac{1}{10}\cdot e^{10\cdot t}\cdot\cos(100\pi\cdot t)+10\pi\cdot\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt\right)=$$

$$=\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-\pi\cdot e^{10\cdot t}\cdot\cos(100\pi\cdot t)-100\pi^2\cdot\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt$$

$$\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt=$$

$$=\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-\pi\cdot e^{10\cdot t}\cdot\cos(100\pi\cdot t)-100\pi^2\cdot\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt$$

$$(1+100\pi^2)\cdot\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt=\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-\pi\cdot e^{10\cdot t}\cdot\cos(100\pi\cdot t)$$

$$\int e^{10\cdot t}\cdot\sin(100\pi\cdot t)\,dt=\frac{1}{1+100\pi^2}\cdot\left(\frac{1}{10}\cdot e^{10\cdot t}\cdot\sin(100\pi\cdot t)-\pi\cdot e^{10\cdot t}\cdot\cos(100\pi\cdot t)\right)$$

☑ ***Solution c***

$$\langle f,g\rangle = \int_0^1 (15+i)\cdot t\cdot e^{4\pi\cdot i\cdot t}\,dt = \left[\left(\frac{1-15\cdot i}{4\pi}\cdot t + \frac{15+i}{16\pi^2}\right)\cdot e^{4\pi\cdot i\cdot t}\right]_0^1 =$$

$$= \left(\frac{1-15\cdot i}{4\pi} + \frac{15+i}{16\pi^2}\right)\cdot e^{4\pi\cdot i} - \frac{15+i}{16\pi^2} = \frac{1-15\cdot i}{4\pi}$$

*Details of the integration:*

$$\int (15+i)\cdot t\cdot e^{4\pi\cdot i\cdot t}\,dt = (15+i)\cdot\int t\cdot e^{4\pi\cdot i\cdot t}\,dt = (15+i)\cdot\left(\frac{1}{4\pi\cdot i}\cdot t\cdot e^{4\pi\cdot i\cdot t} - \frac{1}{4\pi\cdot i}\cdot\int e^{4\pi\cdot i\cdot t}\,dt\right)$$

$$=$$

$$\begin{bmatrix} g(t)=t & \Longrightarrow & g'(t)=1 \\ f'(t)=e^{4\pi\cdot i\cdot t} & \Longrightarrow & f(t)=\dfrac{1}{4\pi\cdot i}\cdot e^{4\pi\cdot i\cdot t} \end{bmatrix}$$

$$= (15+i)\cdot\left(\frac{1}{4\pi\cdot i}\cdot t\cdot e^{4\pi\cdot i\cdot t} - \frac{1}{(4\pi\cdot i)^2}\cdot e^{4\pi\cdot i\cdot t}\right) = \left(\frac{1-15\cdot i}{4\pi}\cdot t + \frac{15+i}{16\pi^2}\right)\cdot e^{4\pi\cdot i\cdot t}$$

### Exercise

Show that functions

$$x_1(t) = \sin\left(\frac{6\pi}{T} \cdot t\right) \quad \text{and} \quad x_2(t) = \cos\left(\frac{6\pi}{T} \cdot t\right)$$

are orthogonal in $L_2([0,T])$ space. Give the norm of $x_2$.

### Solution

$$\langle x_1, x_2 \rangle = \int_0^T \left(\sin\left(\frac{6\pi}{T} \cdot t\right) \cdot \cos\left(\frac{6\pi}{T} \cdot t\right)\right) dt = \frac{1}{2} \cdot \int_0^T \sin\left(\frac{12\pi}{T} \cdot t\right) dt$$

$$= -\frac{1}{2} \cdot \frac{T}{12\pi} \cdot \left[\cos\left(\frac{12\pi}{T} \cdot t\right)\right]_0^T = -\frac{T}{24\pi} \cdot (1 - 1) = 0$$

$$\|x_2\|^2 = \int_0^T \cos^2\left(\frac{6\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \int_0^T 1 + \cos\left(\frac{12\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \left[t + \frac{T}{12\pi} \cdot \sin\left(\frac{12\pi}{T} \cdot t\right)\right]_0^T = \frac{T}{2}$$

$$\|x_2\| = \sqrt{\frac{T}{2}}$$

## Exercise

Show that functions

$$x_1(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{6\pi}{T} \cdot t} \quad \text{and} \quad x_2(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{10\pi}{T} \cdot t}$$

are orthogonal in $L_2([0, T])$ space. Give the norm of $x_2$.

## Solution

$$\langle x_1, x_2 \rangle = \int_0^T \left( \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{6\pi}{T} \cdot t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i \cdot \frac{10\pi}{T} \cdot t} \right) dt = \frac{1}{T} \cdot \int_0^T e^{i \cdot \frac{-4\pi}{T} \cdot t} \, dt =$$

$$= \frac{1}{T} \cdot \frac{1}{i \cdot \frac{-4\pi}{T}} \cdot \left[ e^{i \cdot \frac{-4\pi}{T} \cdot t} \right]_0^T = \frac{-1}{4\pi \cdot i} \cdot \left( e^{-4\pi \cdot i} - 1 \right) = 0$$

$$\|x_2\|^2 = \int_0^T \left( \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{10\pi}{T} \cdot t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i \cdot \frac{10\pi}{T} \cdot t} \right) dt = \int_0^T \frac{1}{T} dt = 1$$

# 4th week

# 4 Orthonormal Systems, Fourier Series, Trigonometric System

# Orthonormal Systems

Let $I$ be an interval. A *sequence* of functions $\{\varphi_j\}_{j\in\mathbb{N}} \subset L_2(I)$ is *orthonormal* if its elements are pairwise orthogonal and the norm of each element is 1.

***Remark***

An orthonormal sequence is also called *orthonormal system*.

***Remark***

In the definition of orthonormal sequences (systems) we can write $\mathbb{Z}$ or $\mathbb{N} \cup \{0\}$ instead of $\mathbb{N}$. For example, in the case of the exponential system we have $j \in \mathbb{Z}$, while in the case of the trigonometric system we write $j \in \mathbb{N}$ or $j \in \mathbb{N} \cup \{0\}$.

The *Fourier coefficients* of a function $x \in L_2(I)$ with respect to the orthonormal system $\{\varphi_j\}_{j\in\mathbb{N}} \subset L_2(I)$ are

$$\hat{x}_k = \langle x, \varphi_k \rangle = \int_I (x \cdot \varphi_k^*)\,, \quad k \in \mathbb{N}$$

The series of functions

$$\mathcal{FS}(x) = \sum_{k=1}^{\infty} (\hat{x}_k \cdot \varphi_k) = \sum_{k=1}^{\infty} (\langle x, \varphi_k \rangle \cdot \varphi_k)$$

is called the *Fourier series* of $x$ with respect to the orthonormal system $\{\varphi_j\}_{j\in\mathbb{N}}$.

The connection between $x \in L_2(I)$ and $\mathcal{FS}(x)$ is important in the Fourier theory. In $L_2$ we say that $x = \sum_{k=1}^{\infty}(\hat{x}_k \cdot \varphi_k)$ if $\lim_{n\to\infty} \|x - \sum_{k=1}^{n}(\hat{x}_k \cdot \varphi_k)\| = 0$. An orthonormal system $\{\varphi_j\}_{j\in\mathbb{N}} \subset L_2(I)$ is called *complete* if $x = \mathcal{FS}(x)$ for all $x \in L_2(I)$.

From the point of view of engineering practice, it is generally enough to know that the Fourier series of a piecewise continuous function converges to the value of the function at every point $t$ where the function is continuous ($\lim_{n\to\infty} \sum_{k=1}^{n} \hat{x}_k \cdot \varphi_k(t) = x(t)$) and converges to the midpoint of the discontinuity (the average of the left- and right-hand limits) wherever the function is discontinuous.

The *Parseval equality*

$$\|x\|^2 = \sum_{k=-\infty}^{\infty} |\hat{x}_k|^2$$

states that the square norm of a function (energy content of a signal) can be calculated directly from its Fourier coefficients.

# The Trigonometric System

## ***The Orthonormal Trigonometric System***

Let $T > 0$. System of functions

$$\left\{\mathrm{CONST(t)} = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

$T$-periodic functions

$$t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(\frac{2\pi}{T} \cdot t\right) \quad \text{and} \quad t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(\frac{2\pi}{T} \cdot t\right)$$

of 'frequency' $f_0 = \frac{1}{T}$ are called the *basic functions* of the system, while $T/k$-periodic functions

$$t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad k = 2,3, \ldots$$

of frequency $k \cdot f_0 = k/T$ are the *harmonics*.

The *Fourier series* of a function $x \in L_2([0,T])$ with respect to the orthonormal trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{A}_0 \cdot \mathrm{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \mathrm{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \mathrm{SIN}_k(t),$$

where

$$\hat{A}_0 = \langle x, \mathrm{CONST} \rangle = \int_0^T x(t) \cdot \mathrm{CONST(t)}\, dt = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} dt$$

$$\hat{A}_k = \langle x, \mathrm{COS}_k \rangle = \int_0^T x(t) \cdot \mathrm{COS}_k(t)\, dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \;\; k = 1,2,\ldots$$

$$\hat{B}_k = \langle x, \mathrm{SIN}_k \rangle = \int_0^T x(t) \cdot \mathrm{SIN}_k(t)\, dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \;\; k = 1,2,\ldots$$

$\hat{A}_0, \hat{A}_k$ and $\hat{B}_k, k = 1,2,\ldots$ are the *Fourier coefficients* of $x$ with respect to the orthonormal trigonometric system.

In the special case $T = 2\pi$ the orthonormal trigonometric system is

$$\left\{\mathrm{CONST}(t) = \frac{1}{\sqrt{2\pi}}, \mathrm{COS}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t), \mathrm{SIN}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right\}_{k \in \mathbb{N}}$$

and the Fourier coefficients of $x$ are

$$\hat{A}_0 = \int_0^{2\pi} x(t) \cdot \frac{1}{\sqrt{2\pi}} dt$$

$$\hat{A}_k = \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t)\right) dt, \quad k = 1,2, \ldots$$

$$\hat{B}_k = \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right) dt, \quad k = 1,2, \ldots$$

If function $x$ is odd, then $\hat{A}_k = 0, k = 0,1,2, \ldots$ , if $x$ is even, then $\hat{B}_k = 0, k = 1,2, \ldots$

The Parseval's equality in the case of the orthonormal trigonometric system is

$$\|x\|^2 = \int_0^T x^2 = \hat{A}_0^2 + \sum_{k=1}^{\infty} \hat{A}_k^2 + \sum_{k=1}^{\infty} \hat{B}_k^2$$

***Remark:*** When calculating the Fourier coefficients of the $T$-periodic functions with respect to the orthonormal trigonometric system we can take the integrals on any interval of length $T$. E.g. we often do the calculations on interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

***Example***

Let $T > 0$. Show that the system of functions

$$\left\{\mathrm{CONST(t)} = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

***Solution***

$$\int_0^T \mathrm{CONST}^2(t)\,\mathrm{dt} = \int_0^T \frac{1}{T} dt = 1$$

$$\int_0^T \mathrm{COS}_k^2(t)\,\mathrm{dt} = \int_0^T \frac{2}{T} \cdot \cos^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) \mathrm{dt} = \frac{1}{T} \cdot \int_0^T \left(1 + \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) \mathrm{dt} =$$

$$= \frac{1}{T} \cdot \left[t + \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right]_0^T = 1$$

$$\int_0^T \mathrm{SINC}_k^2(t)\,\mathrm{dt} = \int_0^T \frac{2}{T} \cdot \sin^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) \mathrm{dt} = \frac{1}{T} \cdot \int_0^T \left(1 - \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) \mathrm{dt} =$$

$$= \frac{1}{T} \cdot \left[ t - \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right) \right]_0^T = 1$$

If $k \neq n$

$$\int_0^T \mathrm{COS}_k(t) \cdot \mathrm{SIN}_n(t)\, dt = \frac{2}{T} \cdot \int_0^T \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \left[ \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) \right]_0^T +$$

$$+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \left[ \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \right]_0^T =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot (\sin(k \cdot 2\pi) \cdot \sin(n \cdot 2\pi) - \sin 0 \cdot \sin 0) +$$

$$+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot (\cos(k \cdot 2\pi) \cdot \cos(n \cdot 2\pi) - \cos 0 \cdot \cos 0) = 0$$

*Details of the integration:*

$$\int \cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$\begin{bmatrix} g(t)=\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & g'(t)=n\cdot\frac{2\pi}{T}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right) \\ f'(t)=\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & f(t)=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{bmatrix}$$

$$=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)-\frac{n}{k}\cdot\int \sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$\begin{bmatrix} g(t)=\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & g'(t)=-n\cdot\frac{2\pi}{T}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right) \\ f'(t)=\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & f(t)=-\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{bmatrix}$$

$$=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)-$$

$$-\frac{n}{k}\cdot\left(-\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)-\frac{n}{k}\cdot\int \cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt\right)=$$

$$= \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) +$$

$$+ \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n^2}{k^2} \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt$$

$$\left(1 - \frac{n^2}{k^2}\right) \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)$$

$$\int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)$$

We can show similarly that

$$\int_0^T \mathrm{SIN}_k(t) \cdot \mathrm{SIN}_n(t)\, dt = 0 \quad \text{and} \quad \int_0^T \mathrm{COS}_k(t) \cdot \mathrm{COS}_n(t)\, dt = 0$$

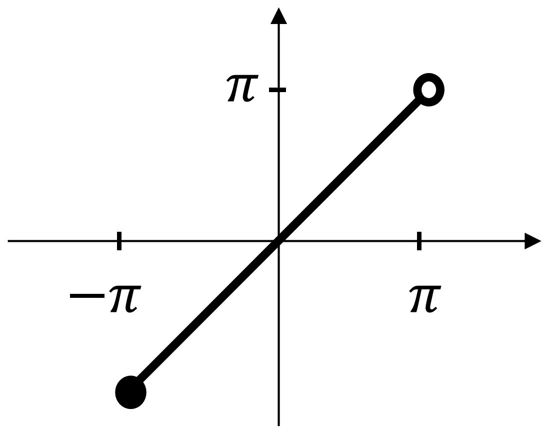
## *Example*

Calculate the Fourier coefficients of the $2\pi$-periodic function $x$ defined as

$$x(t) = t, \quad -\pi \leq t < \pi$$

with respect to the orthonormal trigonometric system.

Use the Parseval's equality to give the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$.




## *Solution*

Since function $x$ is odd, $\hat{A}_k = 0, k = 0,1,2, \ldots$

$$\hat{B}_k = \int_{-\pi}^{\pi} t \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right) dt = \frac{1}{\sqrt{\pi}} \cdot \left[-\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t)\right]_{-\pi}^{\pi} =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(\left(-\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) + \frac{1}{k^2} \cdot \sin(k \cdot \pi)\right) - \left(\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) - \frac{1}{k^2} \cdot \sin(k \cdot \pi)\right)\right) =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(-\frac{2}{k} \cdot \pi \cdot \cos(k \cdot \pi)\right) = 2 \cdot \sqrt{\pi} \cdot (-1)^{k+1} \cdot \frac{1}{k}$$

*Details of the calculation (integration by parts):*

$$\int t \cdot \sin(k \cdot t)\, dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k} \cdot \int \cos(k \cdot t)\, dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t)$$

$$\begin{bmatrix} g(t) = t & \Longrightarrow & g'(t) = 1 \\ f'(t) = \sin(k \cdot t) & \Longrightarrow & f(t) = -\frac{1}{k} \cdot \cos(k \cdot t) \end{bmatrix}$$

According to the Parseval's equality

$$\|x\|^2 = \int_{-\pi}^{\pi} t^2\, dt = \sum_{k=1}^{\infty} \hat{B}_k^2 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Since $\int_{-\pi}^{\pi} t^2\, dt = \frac{1}{3} \cdot [t^3]_{-\pi}^{\pi} = \frac{2}{3} \cdot \pi^3$ we have

$$\frac{2}{3} \cdot \pi^3 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

that is

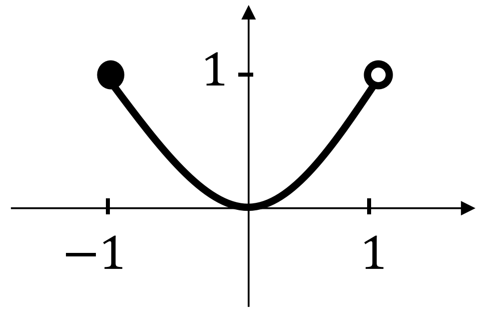
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

***Example***

Calculate the Fourier coefficients of the 2-periodic function $x$ defined as

$$x(t) = t^2, \quad -1 \le t < 1$$

with respect to the orthonormal trigonometric system.




***Solution***

Since function $x$ is even, $\hat{B}_k = 0, k = 1,2, \ldots$

$$\hat{A}_0 = \langle x, \mathrm{CONST} \rangle = \int_{-1}^{1} t^2 \cdot \frac{1}{\sqrt{2}} dt = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot [t^3]_{-1}^{1} = \frac{\sqrt{2}}{3}$$

$$\hat{A}_k = \langle x, \mathrm{COS}_k \rangle = \int_{-1}^{1} t^2 \cdot (\cos(k \cdot \pi \cdot t))\, dt =$$

$$= \left[\frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) + \frac{2}{k^2 \cdot \pi^2} \cdot t \cdot \cos(k \cdot \pi \cdot t) - \frac{2}{k^3 \pi^3} \cdot \sin(k \cdot \pi \cdot t)\right]_{-1}^{1} =$$

$$= \left(\frac{1}{k \cdot \pi} \cdot \sin(k \cdot \pi) + \frac{2}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi)\right) -$$

$$-\left(-\frac{1}{k\cdot\pi}\cdot\sin(k\cdot\pi)-\frac{2}{k^2\cdot\pi^2}\cdot\cos(k\cdot\pi)+\frac{2}{k^3\cdot\pi^3}\cdot\sin(k\cdot\pi)\right)=$$

$$=\frac{2}{k\cdot\pi}\cdot\sin(k\cdot\pi)+\frac{4}{k^2\cdot\pi^2}\cdot\cos(k\cdot\pi)-\frac{4}{k^3\cdot\pi^3}\cdot\sin(k\cdot\pi)=\frac{4}{k^2\cdot\pi^2}\cdot\cos(k\cdot\pi)$$

$$\hat{A}_k=\begin{cases}\dfrac{4}{k^2\cdot\pi^2} & \text{if } k \text{ is even}\\ -\dfrac{4}{k^2\cdot\pi^2} & \text{if } k \text{ is odd}\end{cases}$$

*Details of the calculation (integration by parts):*

$$\int t^2\cdot(\cos(k\cdot\pi\cdot t))\,dt=\frac{1}{k\pi}\cdot t^2\cdot\sin(k\cdot\pi\cdot t)-\frac{2}{k\pi}\cdot\int t\cdot\sin(k\cdot\pi\cdot t)\,dt=$$

$$\begin{bmatrix} g(t)=t^2 & \Longrightarrow & g'(t)=2t\\ f'(t)=\cos(k\cdot\pi\cdot t) & \Longrightarrow & f(t)=\dfrac{1}{k\pi}\cdot\sin(k\cdot\pi\cdot t)\end{bmatrix}$$

$$\begin{bmatrix} g(t)=t & \Longrightarrow & g'(t)=1\\ f'(t)=\sin(k\cdot\pi\cdot t) & \Longrightarrow & f(t)=-\dfrac{1}{k\pi}\cdot\cos(k\cdot\pi\cdot t)\end{bmatrix}$$

$$=\frac{1}{k\cdot\pi}\cdot t^2\cdot\sin(k\cdot\pi\cdot t)-\frac{2}{k\cdot\pi}\cdot\left(-\frac{1}{k\cdot\pi}\cdot t\cdot\cos(k\cdot\pi\cdot t)+\frac{1}{k\cdot\pi}\cdot\int\cos(k\cdot\pi\cdot t)\,dt\right)=$$

$$=\frac{1}{k\cdot\pi}\cdot t^2\cdot\sin(k\cdot\pi\cdot t)+\frac{2}{k^2\cdot\pi^2}\cdot t\cdot\cos(k\cdot\pi\cdot t)-\frac{2}{k^3\cdot\pi^3}\cdot\sin(k\cdot\pi\cdot t)$$

## *The Trigonometric System*

Let $T > 0$. System of functions

$$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{k\in\mathbb{N}}$$

is orthogonal (but not orthonormal) in $L_2([0,T])$. It is called the *trigonometric system*.

The Fourier coefficients of a function $x \in L_2([0,T])$ with respect to the trigonometric system are

$$\hat{a}_0 = \frac{1}{T} \cdot \int_0^T x(t)\,dt$$

$$\hat{a}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1,2,\dots$$

$$\hat{b}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1,2,\dots$$

and the *Fourier series* of $x$ with respect to the trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{a}_0 + \sum_{k=1}^{\infty} \hat{a}_k \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + \sum_{k=1}^{\infty} \hat{b}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)$$

If function $x$ is odd, then $\hat{a}_k = 0, k = 0,1,2, \ldots$, if $x$ is even, then $\hat{b}_k = 0, k = 1,2, \ldots$

In the special case $T = 2\pi$ the trigonometric system is

$$\{1, \cos(k \cdot t), \sin(k \cdot t)\}_{k \in \mathbb{N}}$$

and the Fourier coefficients are

$$\hat{a}_0 = \frac{1}{2\pi} \cdot \int_0^{2\pi} x(t)\, dt$$

$$\hat{a}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \cos(k \cdot t)\, dt, \quad k = 1,2, \ldots$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \sin(k \cdot t)\, dt, \quad k = 1,2, \ldots$$

Using the trigonometric equality

$$A \cdot \sin x + B \cdot \cos x = \sqrt{A^2 + B^2} \cdot \sin(x + \varphi)\text{, where } \varphi = \begin{cases} \operatorname{arctg}\dfrac{b}{a}, & \text{if } a \geq 0 \\ \operatorname{arctg}\dfrac{b}{a} + \pi, & \text{if } a < 0 \end{cases}$$

an alternative form of the Fourier series is

$$\mathcal{FS}(x)(t) = \hat{c}_0 + \sum_{k=1}^{\infty} \hat{c}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t + \varphi_k\right),$$

is obtained, where $\hat{c}_0 = \hat{a}_0$, $\hat{c}_k = \sqrt{\hat{a}_k^2 + \hat{b}_k^2}, k = 1,2, \ldots$ and

$$\varphi_k = \begin{cases} \operatorname{arctg}\dfrac{\hat{b}_k}{\hat{a}_k}, & \text{if } \hat{a}_k \geq 0 \\ \operatorname{arctg}\dfrac{\hat{b}_k}{\hat{a}_k} + \pi, & \text{if } \hat{a}_k < 0 \end{cases}$$

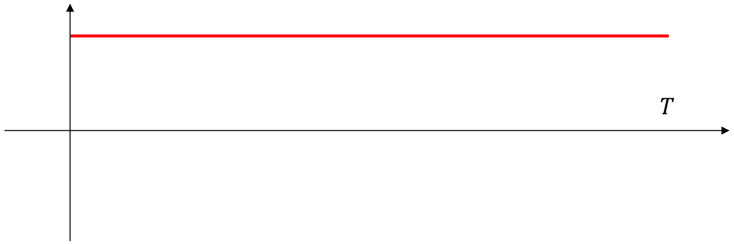
is the *phase* of the harmonic belonging to index $k$.

### *Remark*

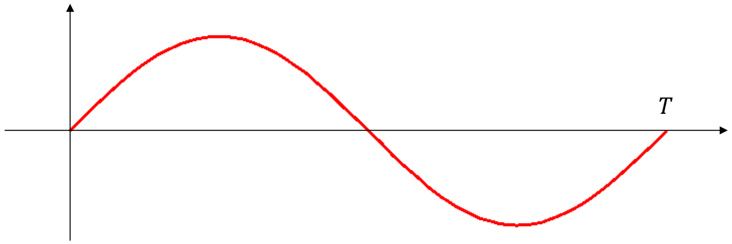
The use of this formula is that it contains at most one term (a sine function) of a certain frequency, while in the original formula two functions (a sine and a cosine) of the same frequency can appear. This fact is important in certain applications when the frequencies of the harmonic components (and the related amplitude) are in the focus. We have to note that the decompositions with respect to the exponential system also contains at most one term corresponding to a certain frequency.

The graph of the constant function and the first four cosine and the first four sine functions of the trigonometric system belonging to the period $T$ on the interval $[0, T]$ are

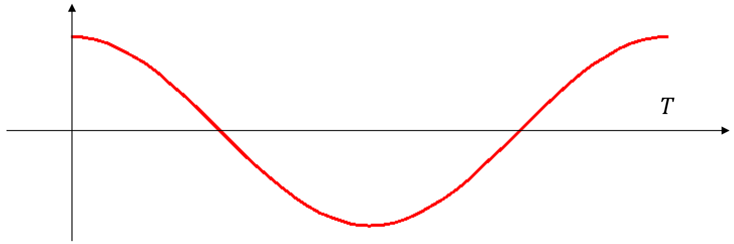
$$t \to 1$$



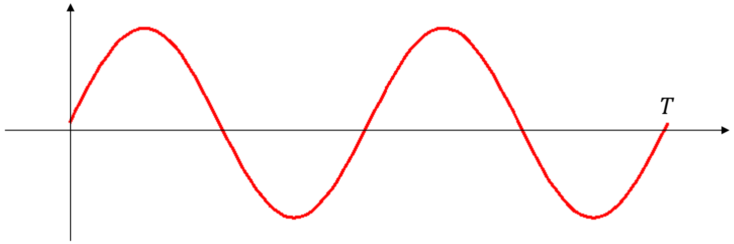

$$t \to \sin\left(\frac{2\pi}{T} \cdot t\right)$$



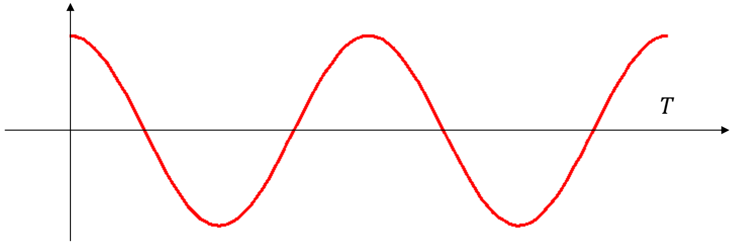

$$t \to \cos\left(\frac{2\pi}{T} \cdot t\right)$$



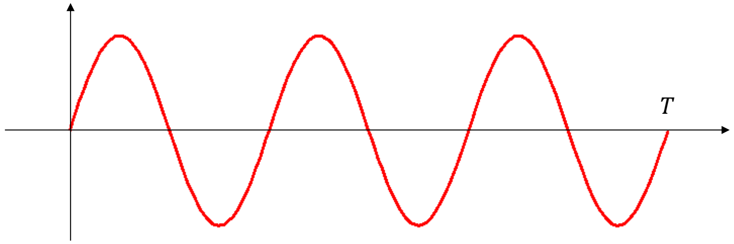
$$t \rightarrow \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



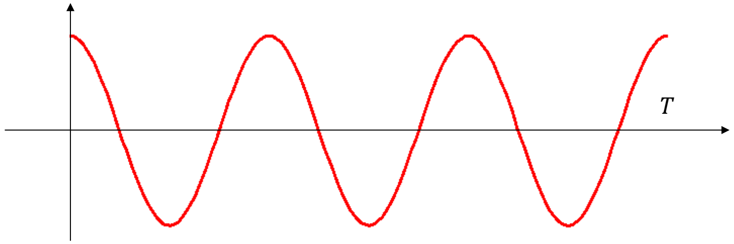

$$t \rightarrow \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



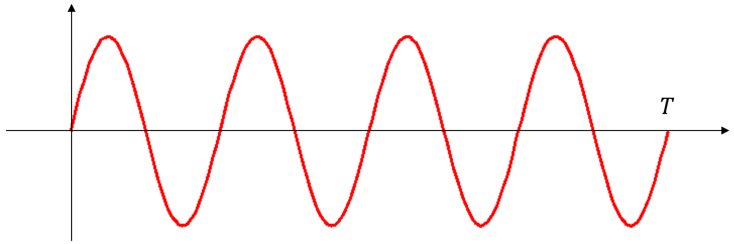

$$t \rightarrow \sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$



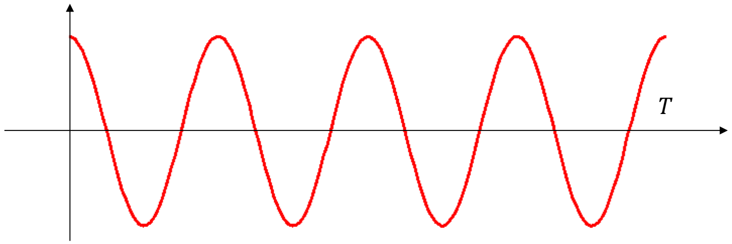

$$t \rightarrow \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$




$$t \rightarrow \sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



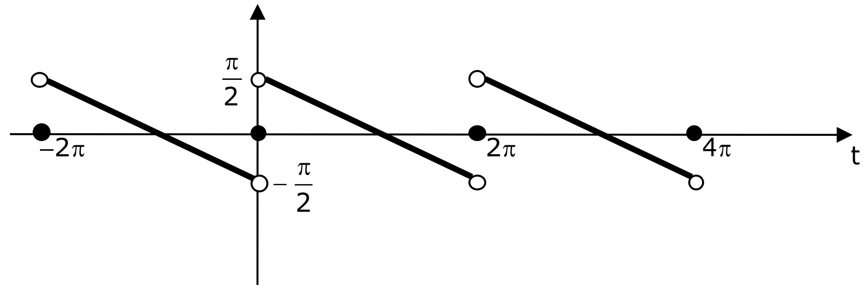

$$t \rightarrow \cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



### *Example*

Determine the Fourier coefficients of the $2\pi$-periodic function $x$ defined as

$$x(t) = \begin{cases} 0, & \text{if} \quad t = 0 \\ -\frac{1}{2}t + \frac{\pi}{2}, & \text{if} \quad 0 < t < 2\pi \end{cases}$$




with respect to the trigonometric system.

### *Solution*

Function $x$ is odd, so $\hat{a}_k = 0, k = 0,1,2, \ldots$

We can get the coefficients $\hat{b}_k$ by integration by parts

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \sin(k \cdot t)\, dt =$$

$$= \frac{1}{\pi} \cdot \left[\left(\frac{1}{2k}t - \frac{\pi}{2k}\right) \cdot \cos(k \cdot t) - \frac{1}{2k^2} \cdot \sin(k \cdot t)\right]_0^{2\pi} =$$

$$= \frac{1}{\pi} \cdot \left(\left(\left(\frac{\pi}{k} - \frac{\pi}{2k}\right) \cdot \cos(k \cdot 2\pi) - \frac{1}{2k^2} \cdot \sin(k \cdot 2\pi)\right) - \left(-\frac{\pi}{2k} \cdot \cos 0 - \frac{1}{2k^2} \cdot \sin 0\right)\right) = \frac{1}{k}$$

*Details of the calculation (integration by parts):*

$$\int\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\sin(k\cdot t)\,dt=-\frac{1}{k}\cdot\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\cos(k\cdot t)-\frac{1}{2k}\cdot\int\cos(k\cdot t)\,dt=$$

$$\begin{bmatrix} g(t)=-\frac{1}{2}t+\frac{\pi}{2} & \Longrightarrow & g'(t)=-\frac{1}{2} \\ f'(t)=\sin(k\cdot t) & \Longrightarrow & f(t)=-\frac{1}{k}\cdot\cos(k\cdot t)\end{bmatrix}$$

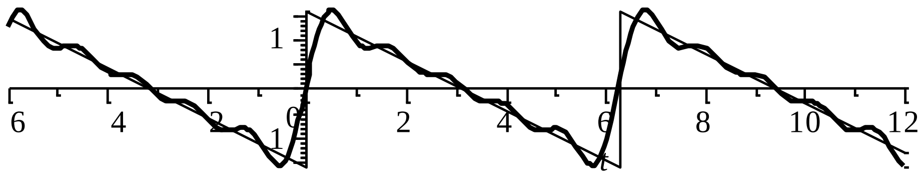
$$=-\frac{1}{k}\cdot\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\cos(k\cdot t)-\frac{1}{2k^2}\cdot\sin(k\cdot t)=\left(\frac{1}{2k}t-\frac{\pi}{2k}\right)\cdot\cos(k\cdot t)-\frac{1}{2k^2}\cdot\sin(k\cdot t)$$

Since $\hat{a}_k=0, k=0,1,2,\ldots$ and $\hat{b}_k=\frac{1}{k}, k=1,2,\ldots$ the Fourier series of $x$ is

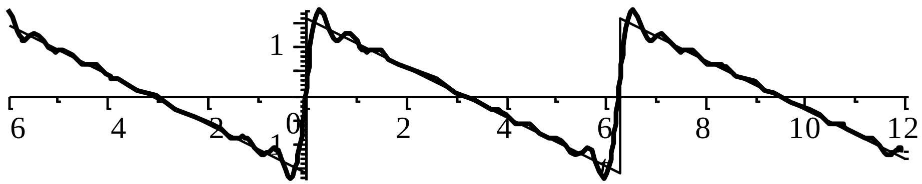
$$\mathcal{FS}x(t)=\sum_{k=1}^{\infty}\frac{\sin(k\cdot t)}{k}.$$

The sum of the first 5 terms and the sum of the first 10 terms in the Fourier series.

$$t\to\sum_{k=1}^{5}\frac{\sin(k\cdot t)}{k}$$




$$t\to\sum_{k=1}^{10}\frac{\sin(k\cdot t)}{k}$$



# 4th week – Questions

***Question 1***

Give the orthonormal trigonometric system which is used for the decomposition of $T$-periodic functions

***Answer***

$$\left\{\frac{1}{\sqrt{T}}, \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

## *Question 2*

Give the *Fourier series* and the *Fourier* coefficients of function $x \in L_2([0,T])$ with respect to the orthonormal trigonometric system.

## *Answer*

$$\mathcal{FS}(x)(t) = \hat{A}_0 \cdot \mathrm{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \mathrm{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \mathrm{SIN}_k(t),$$

$$\hat{A}_0 = \langle x, \mathrm{CONST} \rangle = \int_0^T x(t) \cdot \mathrm{CONST}(\mathrm{t})\, dt = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}}\, dt$$

$$\hat{A}_k = \langle x, \mathrm{COS}_k \rangle = \int_0^T x(t) \cdot \mathrm{COS}_k(t)\, dt = \int_0^T x(t) \cdot \left( \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \quad k = 1,2, \ldots$$

$$\hat{B}_k = \langle x, \mathrm{SIN}_k \rangle = \int_0^T x(t) \cdot \mathrm{SIN}_k(t)\, dt = \int_0^T x(t) \cdot \left( \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \quad k = 1,2, \ldots$$

## *Question 3*

Give the Parseval's equality in the case of the orthonormal trigonometric system.

## *Answer*

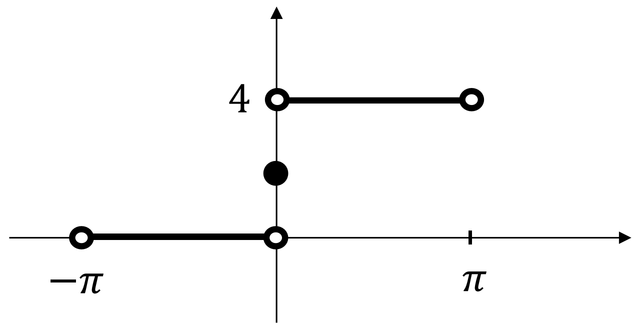
$$\|x\|^2 = \int_0^T x^2 = \hat{A}_0^2 + \sum_{k=1}^{\infty} \hat{A}_k^2 + \sum_{k=1}^{\infty} \hat{B}_k^2$$

# 4th week – Exercises

***Exercise***

Determine the Fourier coefficients of the $2\pi$-periodic function $x$ defined as

$$x(t) = \begin{cases} 0, & \text{if} \quad -\pi < t < 0 \\ 2, & \text{if} \quad t = 0 \\ 4, & \text{if} \quad 0 < t < \pi \\ 2, & \text{if} \quad t = \pi \end{cases}$$




with respect to the trigonometric system.

Give the sum of the first 4 terms and the sum of the first 8 terms in the Fourier series.

***Solution***

$$\hat{a}_0 = \frac{1}{2\pi} \cdot \int_0^{\pi} 4 \, dt = 2$$

$$\hat{a}_k = \frac{1}{\pi} \cdot \int_0^{\pi} 4 \cdot \cos(k \cdot t) \, dt = \frac{4}{k \cdot \pi} \cdot [\sin(k \cdot t)]_0^{\pi} = 0$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{\pi} 4 \cdot \sin(k \cdot t)\, dt = \frac{-4}{k \cdot \pi} \cdot [\cos(k \cdot t)]_0^{\pi} = \frac{4}{k \cdot \pi} \cdot (1 - \cos(k \cdot \pi))$$

We have that

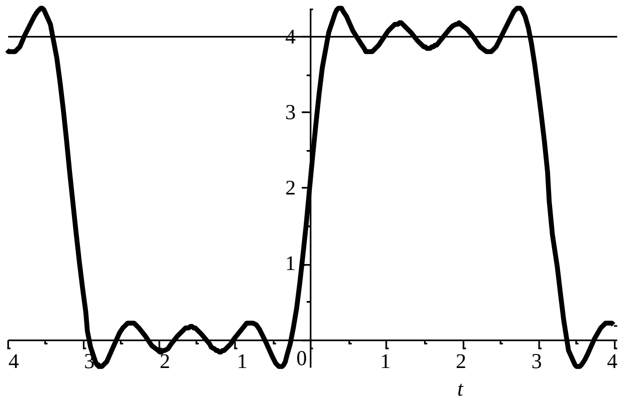
$$\hat{b}_k = \begin{cases} \dfrac{8}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

writing the odd numbers $k$ in the form $k = 2n - 1$ the Fourier series of $x$ is

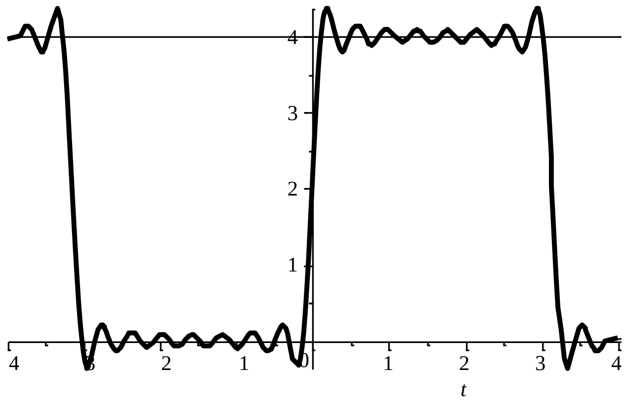
$$x(t) = 2 + \frac{8}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin((2n-1) \cdot t)}{2n - 1}$$

The two partial sums are

$$t \to 2 + \sum_{k=1}^{4} \frac{8}{(2k-1)\cdot\pi} \cdot \sin((2k-1)\cdot t)$$



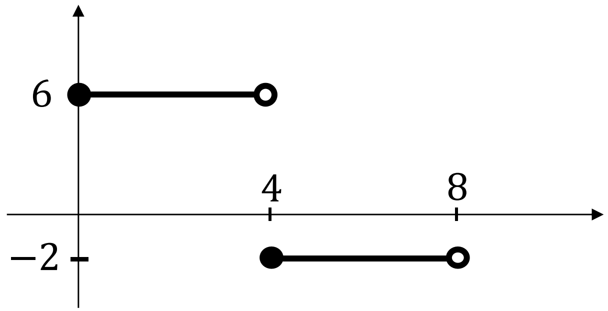

$$t \to 2 + \sum_{k=1}^{8} \frac{8}{(2k-1)\cdot\pi} \cdot \sin((2k-1)\cdot t)$$



## ***Exercise***

Determine the Fourier coefficients of the 8-periodic function $x$ defined as

$$x(t) = \begin{cases} 6 & \text{if } 0 \le t < 4 \\ -2 & \text{if } 4 \le t < 8 \end{cases}$$




## ***Solution***

$$\hat{a}_0 = \frac{1}{8} \cdot \int_0^4 6\,dt + \frac{1}{8} \cdot \int_4^8 -2\,dt = 2$$

$$\hat{a}_k = \frac{1}{4} \cdot \int_0^4 6 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt =$$

$$= \frac{6}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 - \frac{2}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 = 0$$

$$\hat{b}_k = \frac{1}{4} \cdot \int_0^4 6 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt =$$

$$= \frac{-6}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 + \frac{2}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 =$$

$$= \frac{-6}{k \cdot \pi} \cdot (\cos(k \cdot \pi) - 1) + \frac{2}{k \cdot \pi} \cdot (\cos(2k \cdot \pi) - \cos(k \cdot \pi))$$

$$\hat{b}_k = \begin{cases} \dfrac{16}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

writing the odd numbers $k$ in the form $k = 2n - 1$ the Fourier series of $x$ is

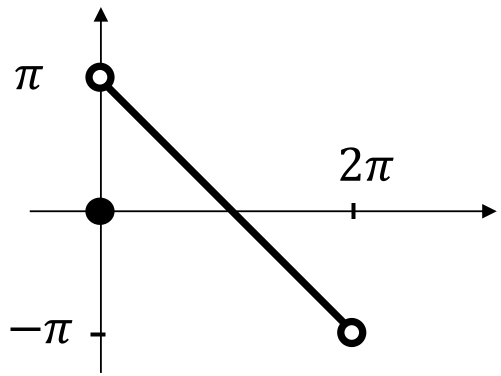
$$x(t) = 2 + \frac{16}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin\left((2n-1) \cdot \frac{\pi}{4} \cdot t\right)}{2n - 1}$$

## Exercise

Calculate the Fourier coefficient $\hat{b}_4$ of the $2\pi$-periodic function $x$ defined as

$$x(t) = \begin{cases} 0, & \text{if} \quad t = 0 \\ \pi - t, & \text{if} \quad 0 < t < 2\pi \end{cases}$$

with respect to the trigonometric system.




## Solution

$$\hat{b}_4 = \frac{1}{\pi} \cdot \int_0^{2\pi} (\pi - t) \cdot \sin(4 \cdot t)\, dt =$$

$$= \frac{1}{\pi} \cdot \left[ -\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{16} \cdot \sin(4 \cdot t) \right]_0^{2\pi} = \frac{1}{\pi} \cdot \left( \frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{1}{2}$$

*Details of the calculation (integration by parts):*

$$\int (\pi - t) \cdot \sin(4 \cdot t)\, dt = -\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{4} \cdot \int \cos(4 \cdot t)\, dt =$$

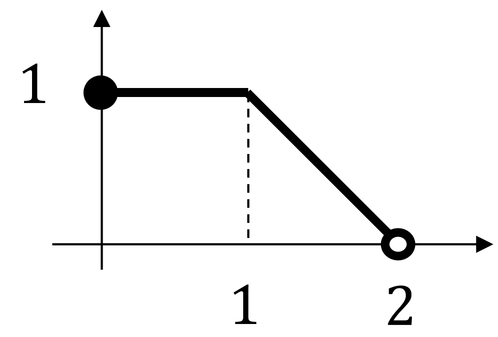
$$= -\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{16} \cdot \sin(4 \cdot t)$$

$$\begin{bmatrix} g(t) = \pi - t & \Longrightarrow & g'(t) = -1 \\ f'(t) = \sin(4 \cdot t) & \Longrightarrow & f(t) = -\frac{1}{4} \cdot \cos(4 \cdot t) \end{bmatrix}$$

## *Exercise*

Calculate the Fourier coefficient $\hat{b}_{10}$ of the 2-periodic function $x$ defined as

$$x(t) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ 2 - t, & \text{if } 1 < t < 2 \end{cases}$$




with respect to the trigonometric system.

## *Solution*

$$\hat{b}_{10} = \int_0^2 x(t) \cdot \sin(10 \cdot \pi \cdot t)\, dt = \int_0^1 \sin(10\pi \cdot t)\, dt + \int_1^2 (2 - t) \cdot \sin(10\pi \cdot t)\, dt =$$

$$= -\frac{1}{10\pi} \cdot [\cos(10\pi \cdot t)]_0^1 + \left[\frac{t-2}{10\pi} \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t)\right]_1^2 = \frac{1}{10\pi}$$

*Details of the calculation (integration by parts):*

$$\int (2 - t) \cdot \sin(10\pi \cdot t)\, dt = -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{10\pi} \cdot \int \cos(10\pi \cdot t)\, dt =$$

$$= -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t)$$

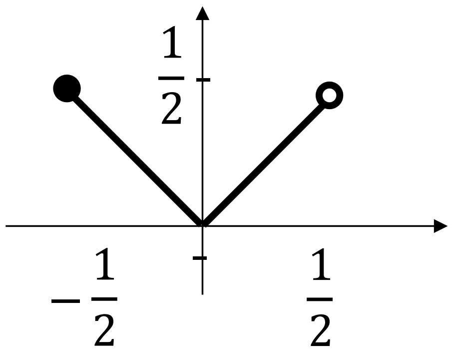
$$\begin{bmatrix} g(t) = 2 - t & \Longrightarrow & g'(t) = -1 \\ f'(t) = \sin(10\pi \cdot t) & \Longrightarrow & f(t) = -\dfrac{1}{10\pi} \cdot \cos(10\pi \cdot t) \end{bmatrix}$$

## *Exercise*

Calculate the Fourier coefficient $\hat{a}_2$ of the 1-periodic function $x$ defined as

$$x(t) = |t|, \quad -\frac{1}{2} \leq t < \frac{1}{2}$$

with respect to the trigonometric system.




## *Solution*

$$\hat{a}_2 = 2 \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| \cdot \cos(2 \cdot 2\pi \cdot t)\, dt = 4 \cdot \int_{0}^{\frac{1}{2}} t \cdot \cos(4\pi \cdot t)\, dt =$$

$$= 4 \cdot \left[\frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t)\right]_0^{1/2} =$$

$$= 4 \cdot \left(\frac{1}{4\pi} \cdot \frac{1}{2} \cdot \sin(2\pi) + \frac{1}{16\pi^2} \cdot \cos(2\pi) - \frac{1}{16\pi^2}\right) = 0$$

*Details of the calculation (integration by parts):*

$$\int t \cdot \cos(4\pi \cdot t)\, dt = \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) - \frac{1}{4\pi} \cdot \int \sin(4\pi \cdot t)\, dt =$$

$$= \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t)$$

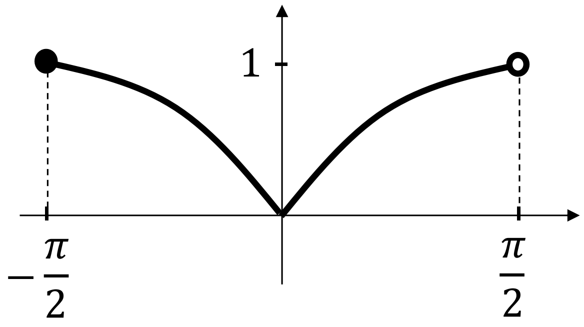
$$\begin{bmatrix} g(t) = t & \Longrightarrow & g'(t) = 1 \\ f'(t) = \cos(4\pi \cdot t) & \Longrightarrow & f(t) = \frac{1}{4\pi} \cdot \sin(4\pi \cdot t) \end{bmatrix}$$

## Exercise

Calculate the Fourier coefficient $\hat{a}_9$ of the $\pi$-periodic function $x$ defined as

$$x(t) = |\sin t|, \quad -\frac{\pi}{2} \le t < \frac{\pi}{2}$$

with respect to the trigonometric system.




## Solution

$$\hat{a}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt,$$

$$\hat{a}_9 = \frac{2}{\pi} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \cdot \cos(9 \cdot 2 \cdot t)\, dt = \frac{4}{\pi} \cdot \int_0^{\frac{\pi}{2}} \sin t \cdot \cos(18 \cdot t)\, dt =$$

$$= \frac{4}{\pi} \cdot \left[\frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t)\right]_0^{\pi/2} =$$

$$= \frac{4}{\pi} \cdot \left(\frac{18}{323} \cdot \sin\frac{\pi}{2} \cdot \sin(9\pi) + \frac{1}{323} \cdot \cos\frac{\pi}{2} \cdot \cos(9\pi) - \frac{1}{323}\right) = \frac{-4}{323 \cdot \pi}$$

*Details of the calculation (integration by parts):*

$$\int \sin t \cdot \cos(18 \cdot t)\,dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \int \cos t \cdot \sin(18 \cdot t)\,dt =$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \left(-\frac{1}{18} \cdot \cos t \cdot \cos(18 \cdot t) - \frac{1}{18} \cdot \int \sin t \cdot \cos(18 \cdot t)\,dt\right) =$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t)\,dt$$

$$\begin{bmatrix} g(t) = \sin t & \Longrightarrow & g'(t) = \cos t \\ f'(t) = \cos(18 \cdot t) & \Longrightarrow & f(t) = \frac{1}{18} \cdot \sin(18 \cdot t) \end{bmatrix}$$

$$\begin{bmatrix} g(t) = \cos t & \Longrightarrow & g'(t) = -\sin t \\ f'(t) = \sin(18 \cdot t) & \Longrightarrow & f(t) = -\frac{1}{18} \cdot \cos(18 \cdot t) \end{bmatrix}$$

$$\int \sin t \cdot \cos(18 \cdot t)\,dt$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t)\,dt$$

$$\left(1 - \frac{1}{324}\right) \cdot \int \sin t \cdot \cos(18 \cdot t)\,dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t)$$

$$\int \sin t \cdot \cos(18 \cdot t)\,dt = \frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t)$$

## Exercise

Determine the period of the signal

$$x(t) = 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$$

and give the Fourier coefficients $\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3$.

## Solution

Period of function $t \to 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)$ is 20, period of function $t \to 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$ is 30.

It is easy to see, that period of their sum is equal to the smallest common multiple of 20 and 30, that is $T = 60$.

Now it is evident that signal $x$ contains two harmonic components, namely

$$t \to 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) = 6 \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right)$$

and

$$t \to 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) = 12 \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right)$$

thus $\hat{b}_3 = 6$ and $\hat{a}_2 = 12$. All other Fourier coefficients are equal to zero.

We can calculate the Fourier coefficients according to the formulas. $T = 60$ thus

$$\hat{b}_3 = \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)\right) \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \int_0^{60} \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \left[\frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right)\right)\right]_0^{60} +$$

$$+ \frac{24}{60} \cdot \left[-\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)\right]_0^{60} = 6$$

*Details of the calculation*

$$\int \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt = \frac{1}{2} \cdot \int 1 - \cos\left(\frac{2\pi}{10} \cdot t\right) dt = \frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right)\right)$$

$$\int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$= -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \int \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$\left[\begin{array}{lll} g(t) = \cos\left(\frac{2\pi}{30}\cdot t\right) & \Longrightarrow & g'(t) = -\frac{2\pi}{30}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right) \\ f'(t) = \sin\left(\frac{2\pi}{20}\cdot t\right) & \Longrightarrow & f(t) = -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) \end{array}\right]$$

$$\left[\begin{array}{lll} g(t) = \sin\left(\frac{2\pi}{30}\cdot t\right) & \Longrightarrow & g'(t) = \frac{2\pi}{30}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right) \\ f'(t) = \cos\left(\frac{2\pi}{20}\cdot t\right) & \Longrightarrow & f(t) = \frac{20}{2\pi}\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) \end{array}\right]$$

$$= -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) -$$

$$-\frac{2}{3}\cdot\left(\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt\right) =$$

$$= -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) + \frac{4}{9}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt$$

$$\frac{5}{9}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt = -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)$$

$$\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt = -\frac{18}{\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{6}{5}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)$$

$$\hat{a}_2 = \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)\right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \int_0^{60} \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \left[\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)\right]_0^{60} +$$

$$+ \frac{24}{60} \cdot \left[\frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right)\right)\right]_0^{60} = 12$$

*Details of the calculation*

$$\int \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{1}{2} \cdot \int 1 + \cos\left(\frac{2\pi}{15} \cdot t\right) dt = \frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right)\right)$$

$$\int \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)$$

(see the same calculation above)

## Exercise

Give the spectrum of the following signals

**a** $x(t) = 0.2 \cdot \sin(250 \cdot t - 5.6) - 4.52 \cdot \sin(1250 \cdot t - 3.2) + 2.87 \cdot \sin(800 \cdot t)$

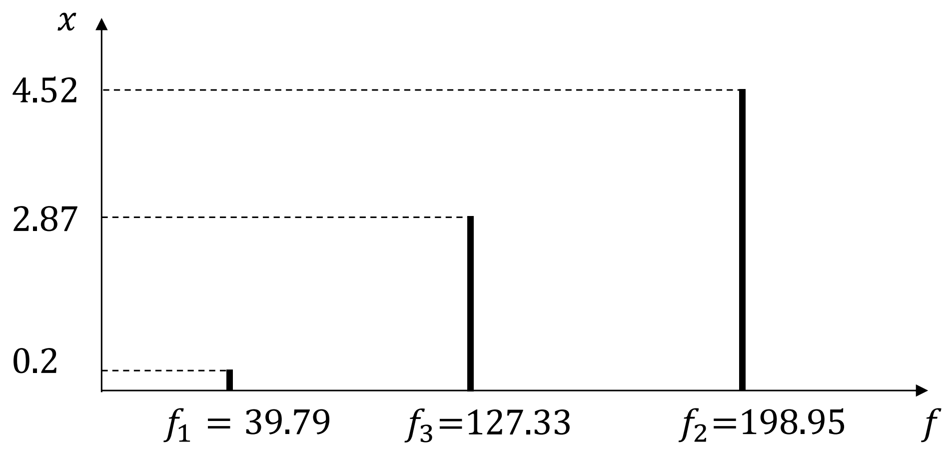
**b** $x(t) = 100 \cdot \sin(5.48 \cdot t - 0.6) + 55 \cdot \sin(6.28 \cdot t - 3) +$
$+21 \cdot \sin(7.27 \cdot t + 1) + 66 \cdot \sin(t - 1.9)$

## Solution a

$$\omega_1 = 250 \left[\frac{rad}{s}\right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 39.79[Hz]$$

$$\omega_2 = 1250 \left[\frac{rad}{s}\right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 198.95[Hz]$$

$$\omega_3 = 800 \left[\frac{rad}{s}\right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 127.33[Hz]$$



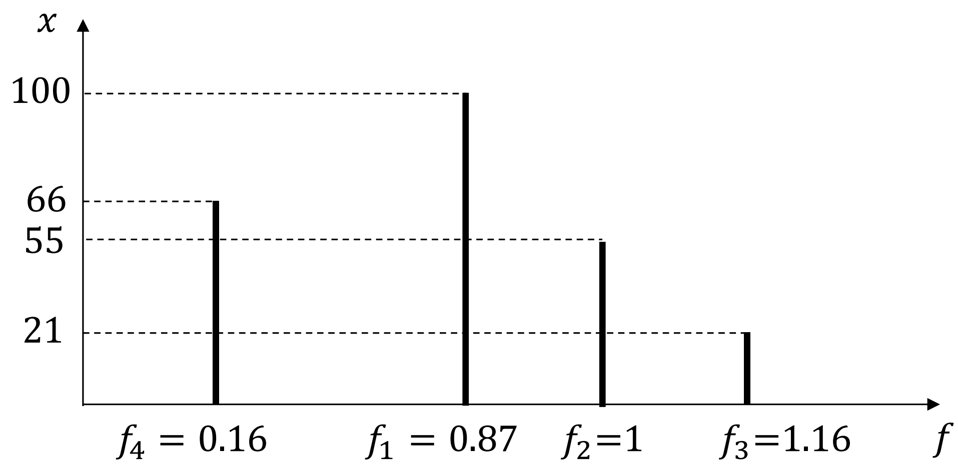
## Solution b

$$\omega_1 = 5.48\left[\frac{rad}{s}\right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 0.87[Hz]$$

$$\omega_2 = 6.28\left[\frac{rad}{s}\right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 1[Hz]$$

$$\omega_3 = 7.27\left[\frac{rad}{s}\right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 1.16[Hz]$$

$$\omega_4 = 1\left[\frac{rad}{s}\right] \Rightarrow f_4 = \frac{\omega_4}{2\pi} = 0.16[Hz]$$



# 5th week

# 5 Exponential System, Vibration Spectrum

# The Exponential System

## *The Orthonormal Exponential System*

Let $T > 0$. System of functions

$$\left\{\mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k \in \mathbb{Z}}$$

is orthonormal in $L_2([0, T])$. This system is called *orthonormal exponential system*.

***Remark***

In the exponential system index $k$ is from $\mathbb{Z}$, that is, there are negative indices as well.

***Remark***

Because of the periodic nature of the complex exponential function in the imaginary part of the variable the coefficients $k \cdot \frac{2\pi}{T}$ in the exponents have "frequency" meaning similarly to the elements of the trigonometric system (see in Appendix 1).

The Fourier coefficients of a function $x \in L_2([0, T])$ with respect to the orthonormal exponential system are

$$\hat{X}_k = \langle x, \mathrm{EXP}_k \rangle = \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) dt, \quad k \in \mathbb{Z},$$

and the Fourier series (decomposition) of $x$ is

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \mathrm{EXP}_k\right) = \sum_{k=-\infty}^{\infty} \left(\langle x, \mathrm{EXP}_k \rangle \cdot \mathrm{EXP}_k\right).$$

***Remark***

When calculating the Fourier coefficients of the $T$-periodic functions with respect to the orthonormal exponential system we can take the integrals on any interval of length $T$. E.g. we often do the calculations on interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

***Example***

Show that system of functions

$$\left\{\mathrm{EXP}_k(t)=\frac{1}{\sqrt{T}}\cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

is orthonormal in $L_2([0,T])$.

***Solution***

For arbitrary $k\in\mathbb{Z}$ we have

$$\|\mathrm{EXP}_k\|^2=\int_0^T\left(\frac{1}{\sqrt{T}}\cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot\frac{1}{\sqrt{T}}\cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right)dt=\int_0^T\frac{1}{T}dt=1$$

(we used that $\left(e^{i\cdot\alpha}\right)^*=e^{-i\cdot\alpha},\ \alpha\in\mathbb{R}$)

For arbitrary $k,l\in\mathbb{Z},k\neq l$ we have

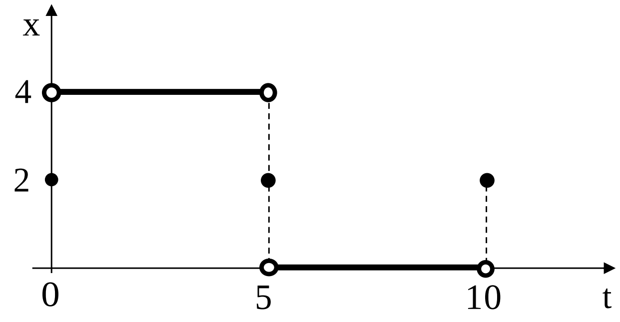
$$\langle\mathrm{EXP}_k,\mathrm{EXP}_l\rangle=\int_0^T\left(\frac{1}{\sqrt{T}}\cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\cdot\frac{1}{\sqrt{T}}\cdot e^{-i\cdot l\cdot\frac{2\pi}{T}\cdot t}\right)dt=\frac{1}{T}\cdot\int_0^T e^{i\cdot(k-l)\cdot\frac{2\pi}{T}\cdot t}\,dt=$$

$$=\frac{1}{T}\cdot\frac{1}{i\cdot(k-l)\cdot\frac{2\pi}{T}}\cdot\left[e^{i\cdot(k-l)\cdot\frac{2\pi}{T}\cdot t}\right]_0^T=\frac{1}{2\pi\cdot i\cdot(k-l)}\cdot\left(e^{2\pi\cdot i\cdot(k-l)}-1\right)=0$$

***Exercise***

Give the Fourier series of 10-periodic function $x$ defined as

$$x(t)=\begin{cases}4, & \text{if} \quad 0<t<5\\ 0, & \text{if} \quad 5<t<10\\ 2, & \text{if} \quad x\in\{0,5,10\}\end{cases}$$




with respect to the orthonormal exponential system.

***Solution***

$$\hat{X}_0=\langle x,\mathrm{EXP}_0\rangle=\int_0^5 4\cdot\frac{1}{\sqrt{10}}dt=\frac{20}{\sqrt{10}}$$

If $k\neq 0$

$$\hat{X}_k=\langle x,\mathrm{EXP}_k\rangle=\int_0^5 4\cdot\left(\frac{1}{\sqrt{10}}\cdot e^{-i\cdot k\cdot\frac{2\pi}{10}\cdot t}\right)dt=\frac{4}{\sqrt{10}}\cdot\frac{-10}{2\pi\cdot i\cdot k}\cdot\left[e^{-i\cdot k\cdot\frac{2\pi}{10}\cdot t}\right]_0^5=$$

$$=\frac{20\cdot i}{\sqrt{10}\cdot\pi\cdot k}\cdot\left(e^{-i\cdot k\cdot\pi}-1\right)=\begin{cases}\dfrac{-40\cdot i}{\sqrt{10}\cdot\pi\cdot k}, & \text{if} \quad k \text{ is odd}\\ 0, & \text{if} \quad k\text{ is even}, k\neq 0\end{cases}$$

Using the notation $k = 2l - 1, l \in \mathbb{Z}$ the Fourier series of $x$ is

$$\mathcal{FS}f(x)(t) = 2 + \sum_{l=-\infty}^{\infty} \left( \frac{-40 \cdot i}{\sqrt{10} \cdot \pi \cdot (2l-1)} \cdot \frac{1}{\sqrt{10}} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} \cdot t} \right) =$$

$$= 2 + \sum_{l=-\infty}^{\infty} \left( \frac{-4 \cdot i}{\pi \cdot (2l-1)} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} \cdot t} \right)$$

## *The Exponential System*

System of functions

$$\left\{e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

is orthogonal (but not orthonormal) in $L_2([0,T])$. It is called *exponential system*.

The (complex) Fourier coefficients of function $x \in L_2([0,T])$ with respect to the exponential system are

$$\hat{x}_k = \frac{1}{T}\cdot\int_0^T x(t)\cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\,dt,\quad k\in\mathbb{Z},$$

the Fourier series of $x$ is

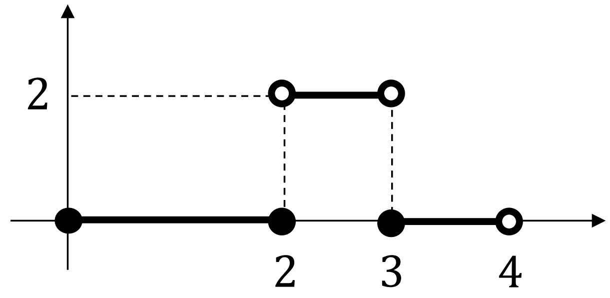
$$\mathcal{FS}(x)(t) = \sum_{k=-\infty}^{\infty}\left(\hat{x}_k\cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right).$$

Functions $k\to|\hat{x}_k|$, $k\to|\hat{x}_k|^2$, and $k\to\arg\hat{x}_k$ are called *amplitude spectrum*, *energy spectrum* and *phase spectrum*, respectively.

## *Example*

Calculate the Fourier coefficient $\hat{x}_5$ of the 4-periodic function $x$ defined as

$$x(t) = \begin{cases} 2, & \text{if } 2 < t < 3 \\ 0, & \text{otherwise on } [0,4[ \end{cases}$$




with respect to the exponential system.

## *Solution*

$$\hat{x}_5 = \frac{1}{4} \cdot \int_2^3 2 \cdot e^{-i \cdot 5 \cdot \frac{2\pi}{4} \cdot t}\, dt = \frac{1}{4} \cdot \frac{-2}{5\pi \cdot i} \cdot \left[ e^{-i \cdot \frac{5\pi}{2} \cdot t} \right]_0^4 =$$

$$= \frac{-1}{10\pi \cdot i} \cdot \left( e^{-i \cdot 10\pi} - 1 \right) = \frac{-1}{10\pi \cdot i} \cdot (\cos(-10\pi) + i \cdot \sin(-10\pi) - 1) = 0$$

## *Real and Complex Fourier Coefficients*

If $x \in L_2([0,T])$ is a real-valued function, we have

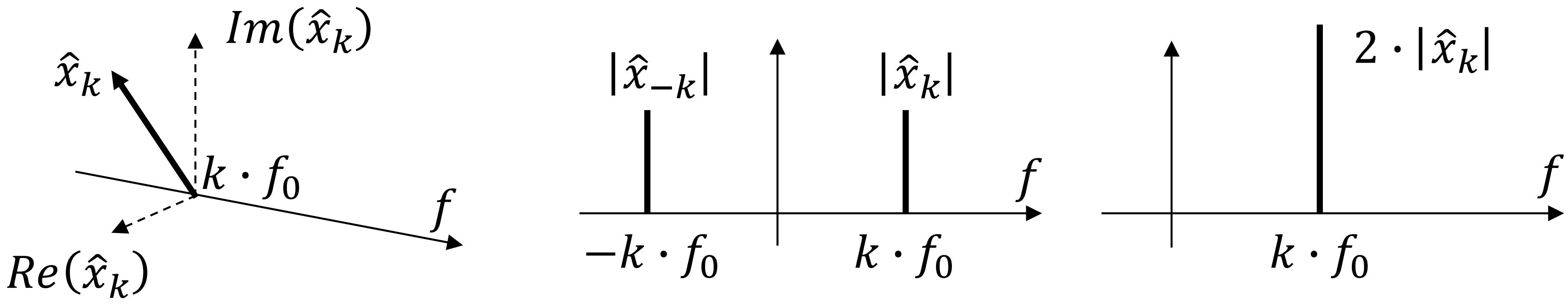
$$\hat{x}_{-k} = \hat{x}_k^* \quad k \in \mathbb{Z}$$

and, consequently

$$|\hat{x}_{-k}| = |\hat{x}_k|\,, \quad k \in \mathbb{Z}$$

showing that the complex spectrum has symmetric nature and the fact that the Fourier coefficients of a real-valued function belonging to 'negative frequencies' have not independent meaning.

The complex spectrum is displayed in different ways. We can draw a "3D" diagram showing the complex values (the real and the imaginary part of the coefficients), or we can plot only the values $|\hat{x}_k|$, and finally we can plot values $2 \cdot |\hat{x}_k|$ on the non-negative frequency axis.




Consider the orthonormal trigonometric system

$$\left\{\mathrm{CONST}(\mathrm{t}) = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

and the orthonormal exponential system

$$\left\{\mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k \in \mathbb{Z}}$$

in $L_2([0, T])$. Since both the real and the complex Fourier coefficients $(\hat{A}_k, \hat{B}_k, \hat{X}_k)$ belong to frequency $k \cdot \frac{2\pi}{T}$, they are expected to be connected. In fact

$$\hat{X}_0 = \hat{A}_0,$$

furthermore the properties of sine, cosine and exponential functions imply that for $k \in \mathbb{Z}, k > 0$ we have

$$\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k - \hat{B}_k \cdot i\right), \quad \hat{X}_{-k} = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k + \hat{B}_k \cdot i\right), \text{ and } \left|\hat{X}_k\right| = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}.$$

Considering the trigonometric system

$$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

and the exponential system

$$\left\{e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k \in \mathbb{Z}}$$

the connection between the real and complex Fourier coefficients $(\hat{a}_k, \hat{b}_k, \hat{x}_k)$ is as follows:

$$\hat{x}_0 = \hat{a}_0,$$

furthermore, for $k \in \mathbb{Z}, k > 0$, we have

$$\hat{x}_k = \frac{1}{2} \cdot \left(\hat{a}_k - \hat{b}_k \cdot i\right), \quad \hat{x}_{-k} = \frac{1}{2} \cdot \left(\hat{a}_k + \hat{b}_k \cdot i\right), \quad |\hat{x}_k| = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

***Example***

Using the Euler formula $e^{i\cdot t} = \cos t + i\cdot \sin t\,, t \in \mathbb{R}$ show that

$$\hat{X}_k = \frac{1}{\sqrt{2}}\cdot\left(\hat{A}_k - \hat{B}_k\cdot i\right)$$

$$\hat{X}_k = \frac{1}{\sqrt{2}}\cdot\left(\hat{A}_k + \hat{B}_k\cdot i\right)$$

and

$$\left|\hat{X}_k\right| = \frac{1}{2}\cdot\sqrt{\hat{A}_k^2 + \hat{B}_k^2},\ \ k \in \mathbb{Z}, k > 0$$

Express $\hat{X}_k + \hat{X}_{-k}$ and $i\cdot\left(\hat{X}_k - \hat{X}_{-k}\right), k \in \mathbb{Z}, k > 0$.

### *Solution*

Let $k \in \mathbb{Z}, k > 0$.

$$\hat{X}_k = \int_0^T x(t) \cdot \left( \frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) dt =$$

$$= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt$$

$$= \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$$

The sine function is odd while the cosine function is even thus

$$\hat{X}_{-k} = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt$$

$$= \frac{1}{\sqrt{2}} \cdot \hat{A}_k + \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$$

$$\hat{X}_k + \hat{X}_{-k} = \sqrt{2} \cdot \hat{A}_k, \quad k \in \mathbb{Z}, k > 0$$

$$i \cdot \left(\hat{X}_k - \hat{X}_{-k}\right) = \sqrt{2} \cdot \hat{B}_k, \quad k \in \mathbb{Z}, k > 0$$

In the formula $\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$ we can see that

$$\mathrm{Re}\left(\hat{X}_k\right) = \frac{1}{\sqrt{2}} \cdot \hat{A}_k \quad \text{and} \quad \mathrm{Im}\left(\hat{X}_k\right) = -\frac{1}{\sqrt{2}} \cdot \hat{B}_k,$$

thus

$$\left|\hat{X}_k\right| = \sqrt{\frac{1}{2} \cdot \hat{A}_k^2 + \frac{1}{2} \cdot \hat{B}_k^2} = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}$$

### *Example*

Using the formulas obtained in the previous exercise, manipulate the Fourier series of a function $x \in L_2([0,T])$ with respect to the orthonormal exponential system to get the Fourier series of $x$ with respect to the orthonormal trigonometric system.

### *Solution*

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \mathrm{EXP}_k\right) = \sum_{k=-\infty}^{\infty} \hat{X}_k \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right) =$$

$$= \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + i \cdot \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{X}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty}\left(\left(\hat{X}_k + \hat{X}_{-k}\right) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + \sum_{k=1}^{\infty}\left(i \cdot \left(\hat{X}_k - \hat{X}_{-k}\right) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{A}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty}\left(\hat{A}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + \sum_{k=1}^{\infty}\left(\hat{B}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{A}_0 \cdot \mathrm{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \mathrm{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \mathrm{SIN}_k(t)$$

***Example***

Using the Euler formula $e^{i\cdot t} = \cos t + i\cdot\sin t\,, t\in\mathbb{R}$ show that

$$\hat{x}_k = \frac{1}{2}\cdot\left(\hat{a}_k - \hat{b}_k\cdot i\right),\quad \hat{X}_{-k} = \frac{1}{2}\cdot\left(\hat{a}_k + \hat{b}_k\cdot i\right),\quad \left|\hat{X}_k\right| = \frac{1}{2}\cdot\sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

***Solution***

Let $k\in\mathbb{Z}, k>0$.

$$\hat{x}_k = \frac{1}{T}\cdot\int_0^T x(t)\cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\,dt$$

$$= \frac{1}{T}\cdot\int_0^T x(t)\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt - i\cdot\frac{1}{T}\cdot\int_0^T x(t)\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$= \frac{1}{2}\cdot\frac{2}{T}\cdot\int_0^T x(t)\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt - \frac{1}{2}\cdot i\cdot\frac{2}{T}\cdot\int_0^T x(t)\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt = \frac{1}{2}\cdot\hat{a}_k - \frac{1}{2}\cdot i\cdot\hat{b}_k.$$

The sine function is odd while the cosine function is even thus

$$\hat{x}_{-k} = \frac{1}{T}\cdot\int_0^T x(t)\cdot\cos\left(-k\cdot\frac{2\pi}{T}\cdot t\right)dt - i\cdot\frac{1}{T}\cdot\int_0^T x(t)\cdot\sin\left(-k\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$= \frac{1}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \frac{1}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

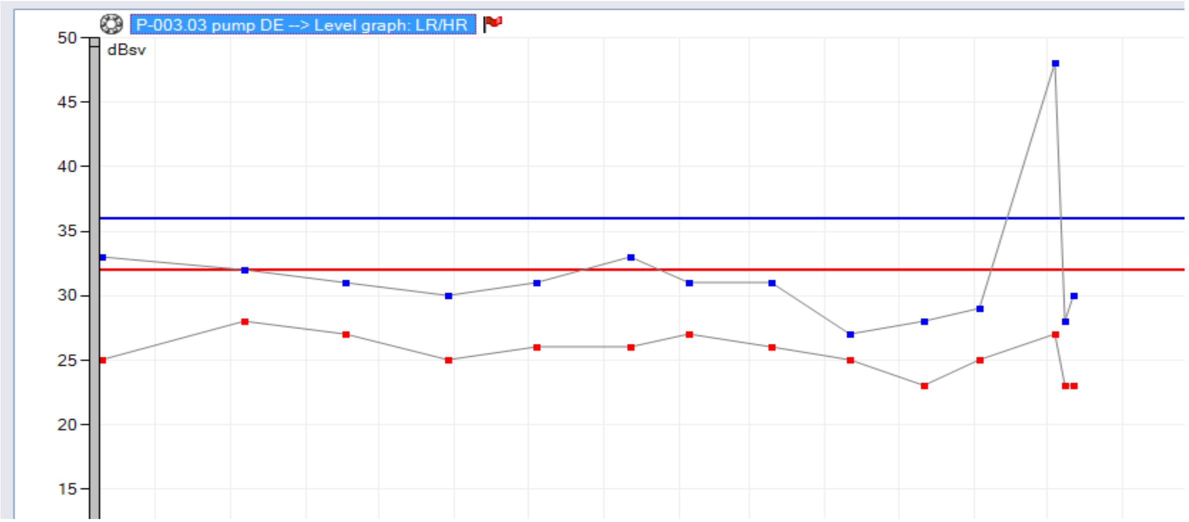
$$= \frac{1}{2} \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{2} \cdot i \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \hat{a}_k + \frac{1}{2} \cdot i \cdot \hat{b}_k.$$

In the formula $\hat{x}_k = \frac{1}{2} \cdot \hat{a}_k - \frac{1}{2} \cdot i \cdot \hat{b}_k$ we can see that $\mathrm{Re}(\hat{x}_k) = \frac{1}{2} \cdot \hat{a}_k$ and $\mathrm{Im}(\hat{x}_k) = -\frac{1}{2} \cdot \hat{b}_k$ thus

$$|\hat{x}_k| = \sqrt{\frac{1}{4} \cdot \hat{a}_k^2 + \frac{1}{4} \cdot \hat{b}_k^2} = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

In predictive maintenance of machinery the control of the propagation of failures in time is even more important than the determination of the actual status. The SPM system contains many graphical functions providing information about changes in time.

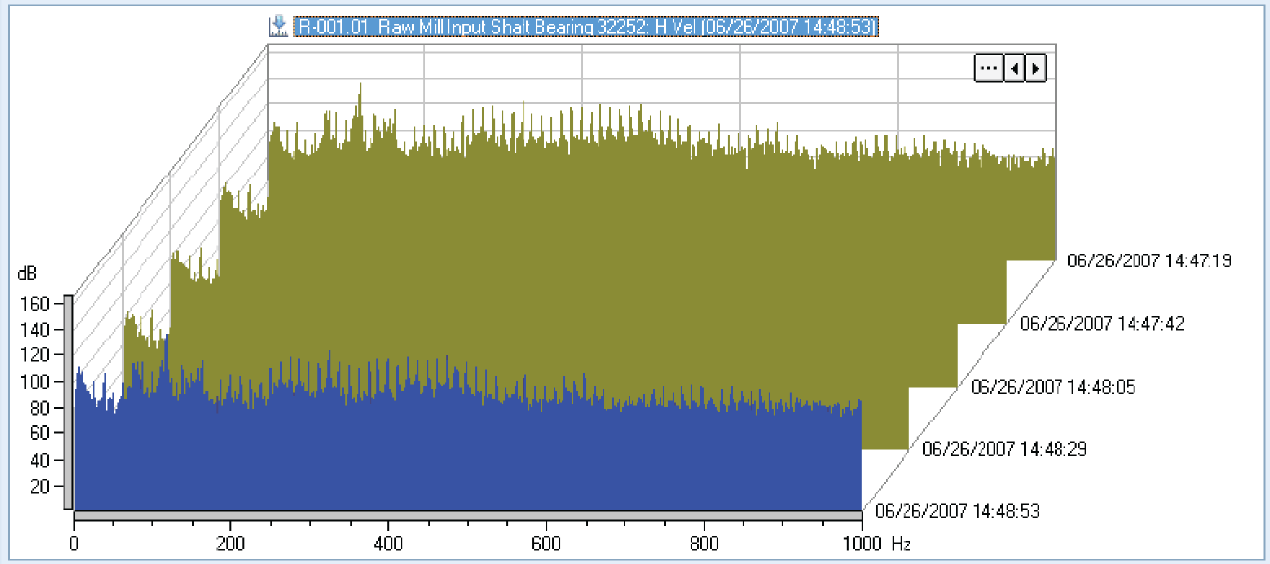
One type of these diagrams shows some numerical values as a function of time and also the related control limits. These diagrams enable the visual control, when a high value of a parameter appears relevant maintenance actions can be scheduled.



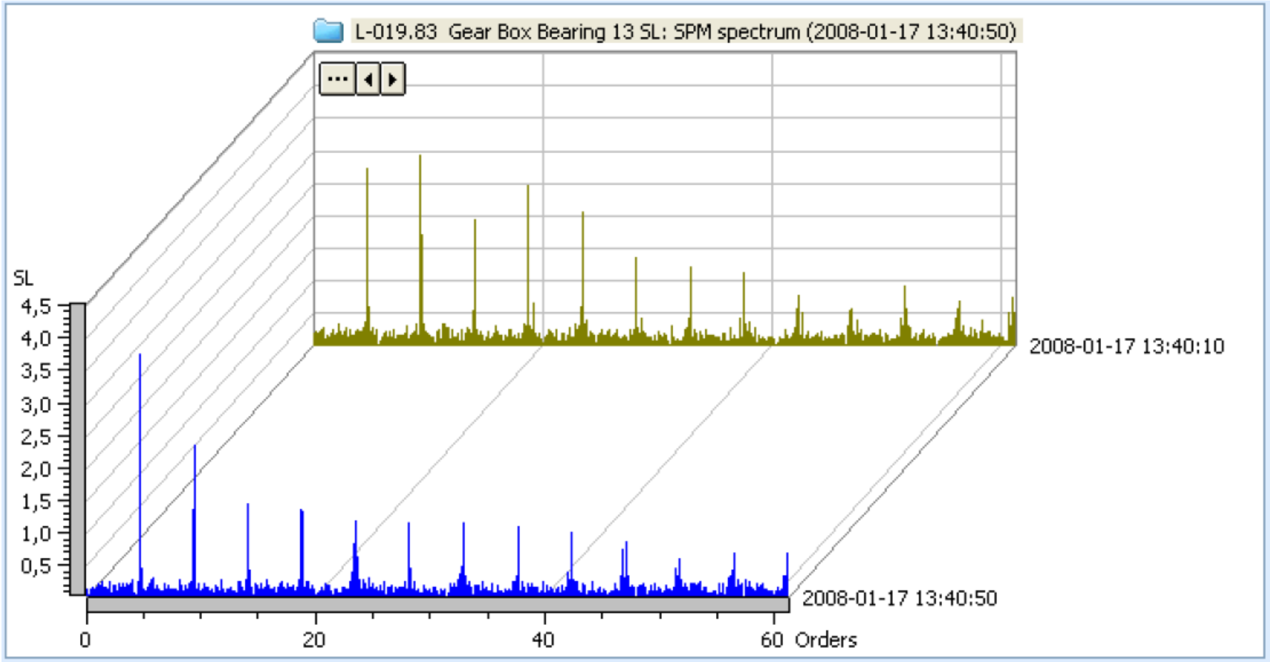

[5]

Another type of diagrams shows the change of graphs, for instance the change of the spectrum. When the amplitudes belonging to critical frequencies increase or new frequencies appear in the spectrum the root cause of the change must be identified to avoid the further propagation of the failure.

An important example is the so-called **Waterfall diagram** which is a three-dimensional display of up to 50 vibration spectra. The different readings are displayed along an axis, with the latest being the nearest the viewer.

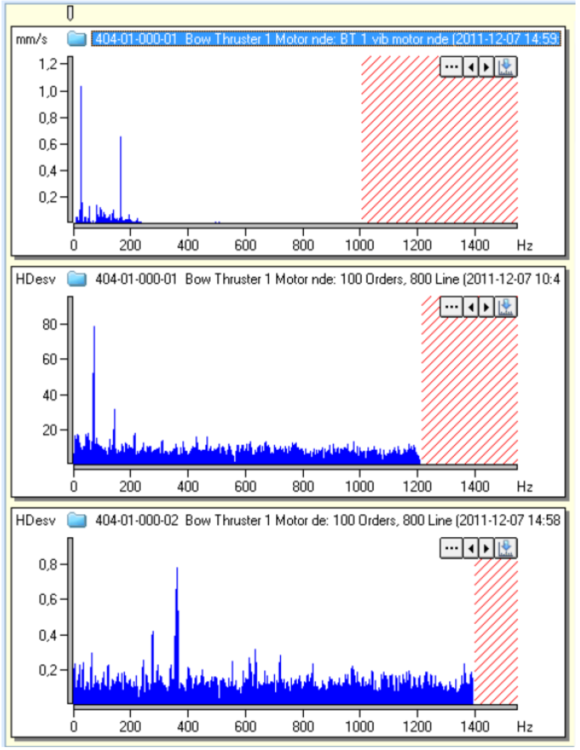



[5]


L-019.83 Gear Box Bearing 13 SL: SPM spectrum (2008-01-17 13:40:50)
SL
4,5
4,0
3,5
3,0
2,5
2,0
1,5
1,0
0,5
2008-01-17 13:40:10
2008-01-17 13:40:50
0
20
40
60
Orders

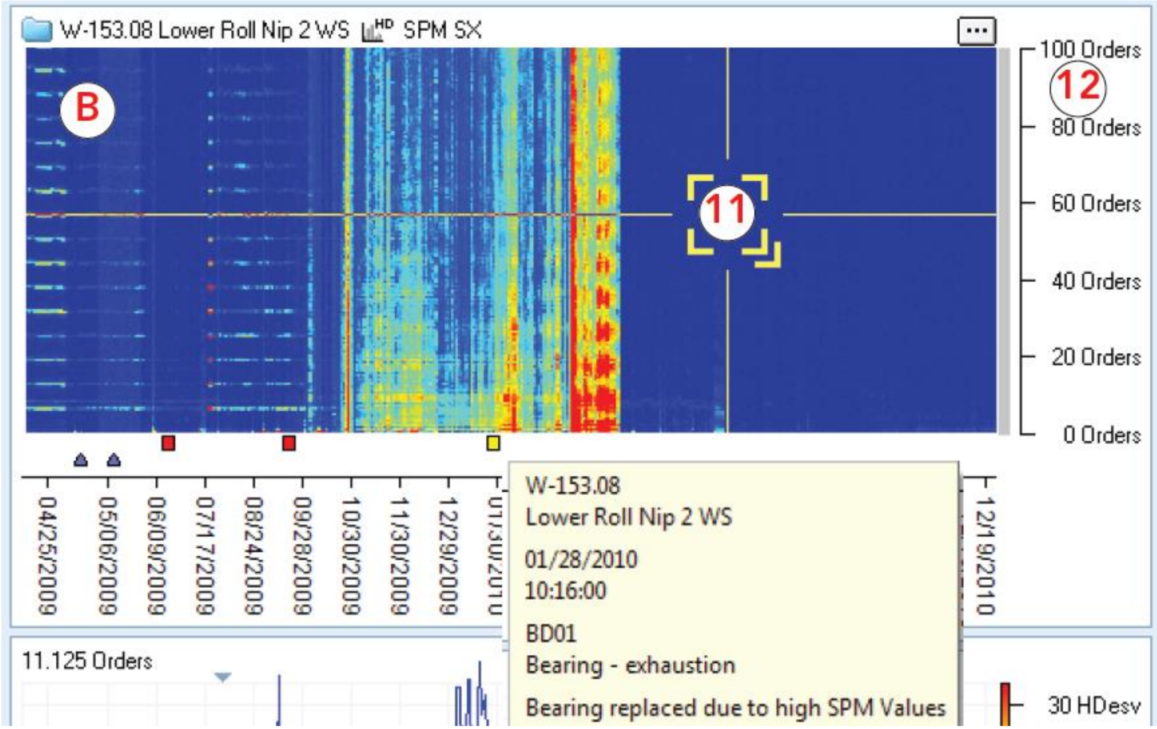
[5]

With the **Compare spectrum** function we can view more than one frequency range and/or resolution at a time. This means that we can implement a variable frequency range from one measuring assignment to another and also between measuring points.

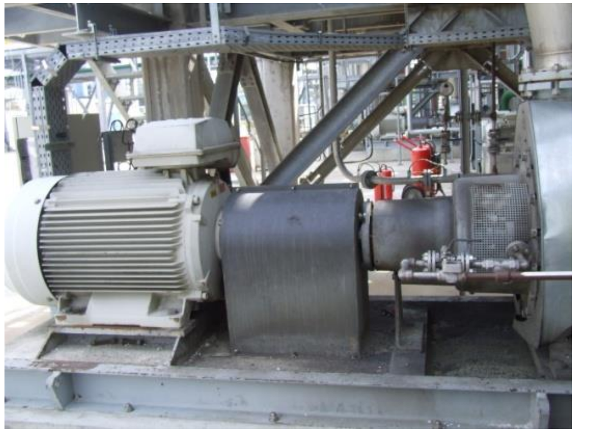


[5]

The **Coloured Spectrum Overview** is a three-dimensional view of all spectra under a particular measuring assignment. Its purpose is to simplify the process of identifying in spectra the patterns and trends which indicate damages. In the Coloured Spectrum Overview, signals which are always present in the machine are clearly distinguished from signals caused by developing damages. The Coloured Spectrum Overview provides a very good overall picture of machine condition development.

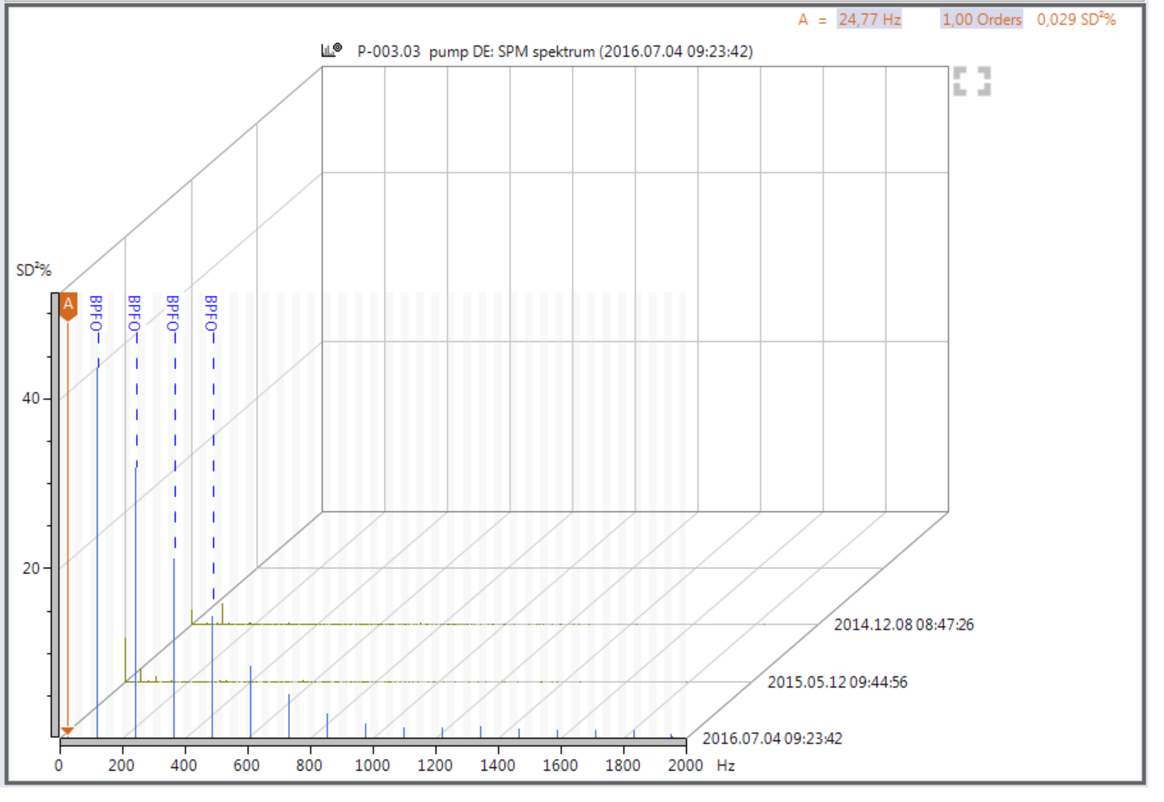

W-153.08 Lower Roll Nip 2 WS
SPM SX
B
11
12
100 Orders
80 Orders
60 Orders
40 Orders
20 Orders
0 Orders
04/25/2009
05/06/2009
06/09/2009
07/17/2009
08/24/2009
09/28/2009
10/30/2009
11/30/2009
12/29/2009
12/19/2010
W-153.08
Lower Roll Nip 2 WS
01/28/2010
10:16:00
BD01
Bearing - exhaustion
Bearing replaced due to high SPM Values
11.125 Orders
30 HDesv
[5]

# A Case Study: Condition Monitoring of a Pump Bearing



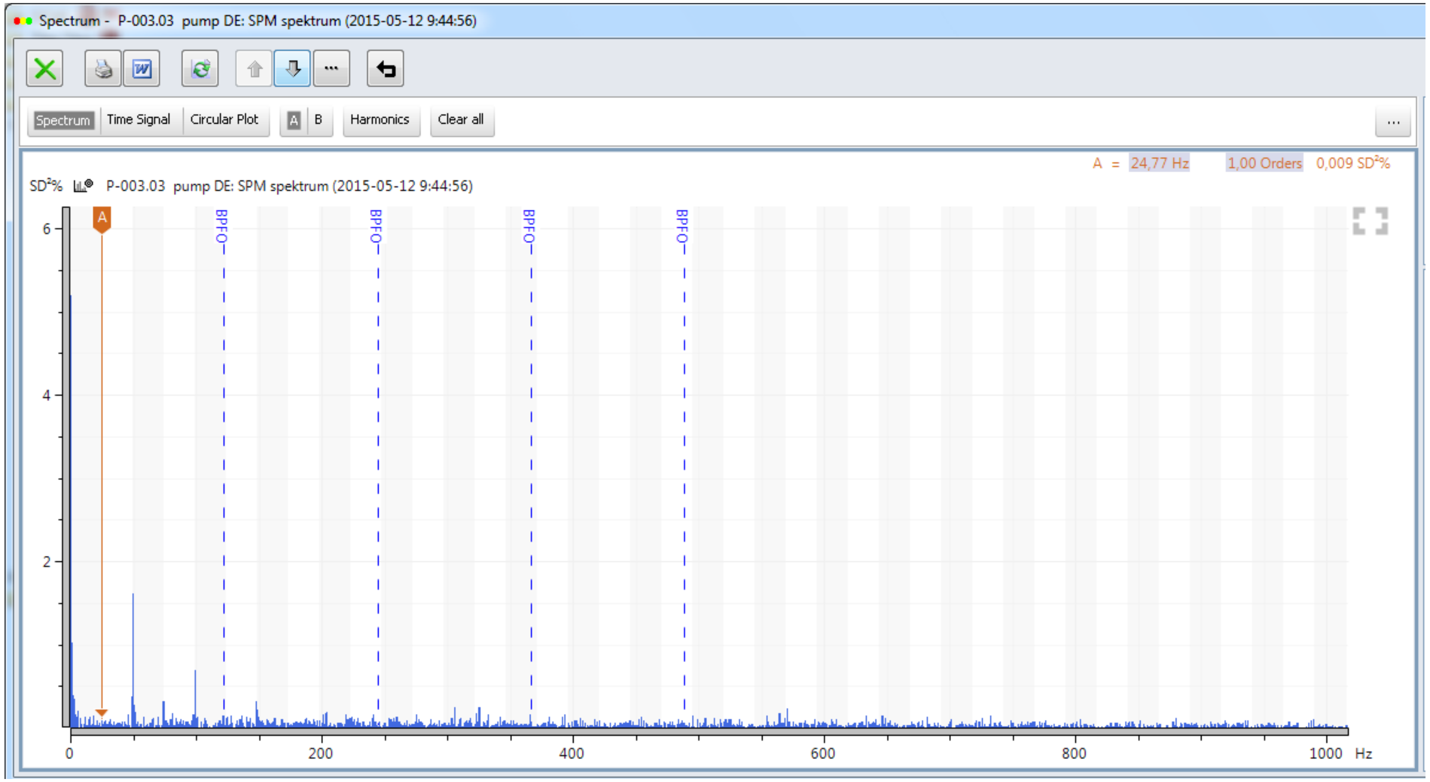
The drive-end bearing of a pump used in a chemical production process was investigated. Several measuring techniques were assigned to the measuring point, among others, the shock pulse measurement.

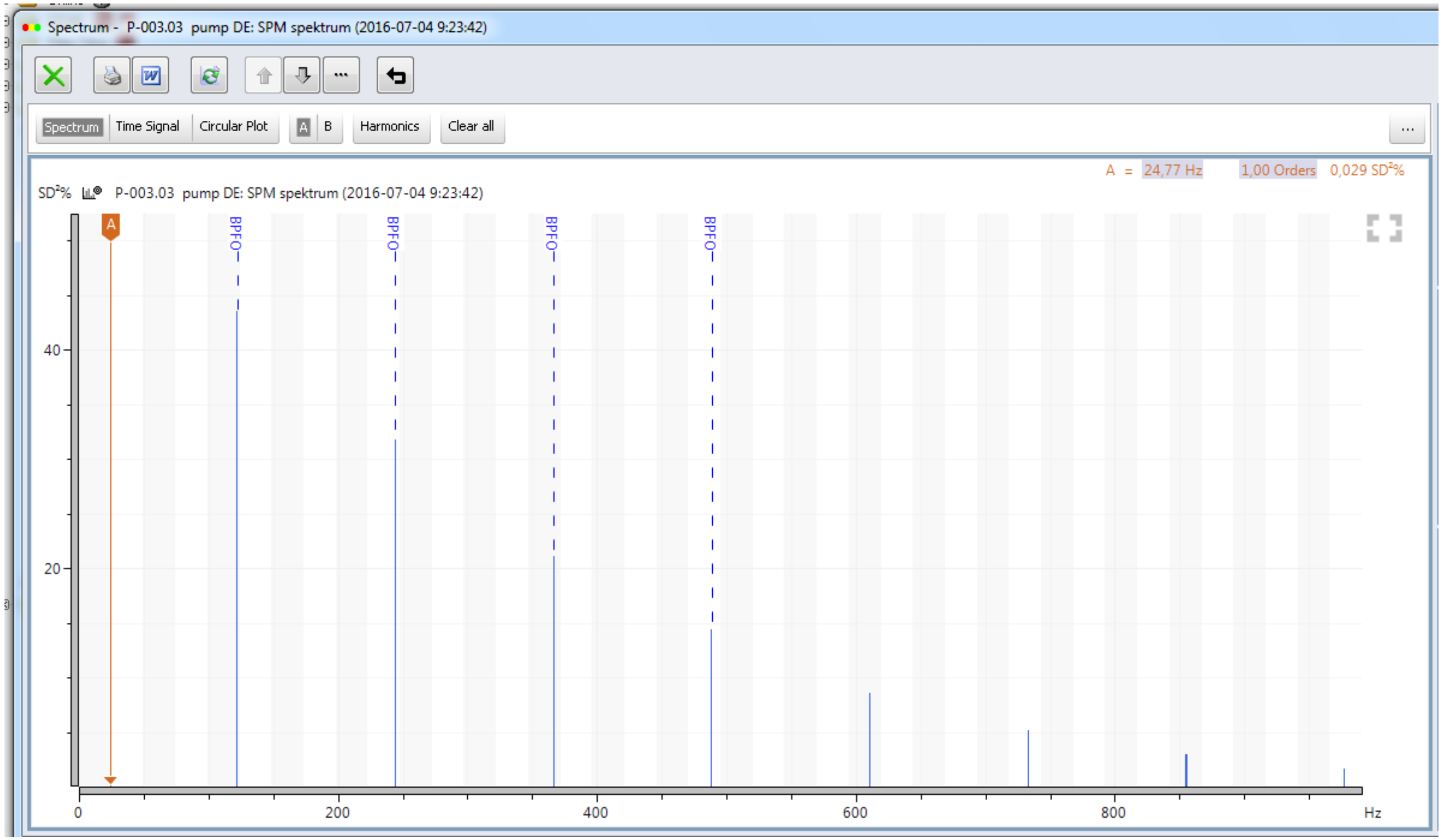
The waterfall diagram shows clearly, that the measure of amplitude enhancement was significant at certain frequencies.

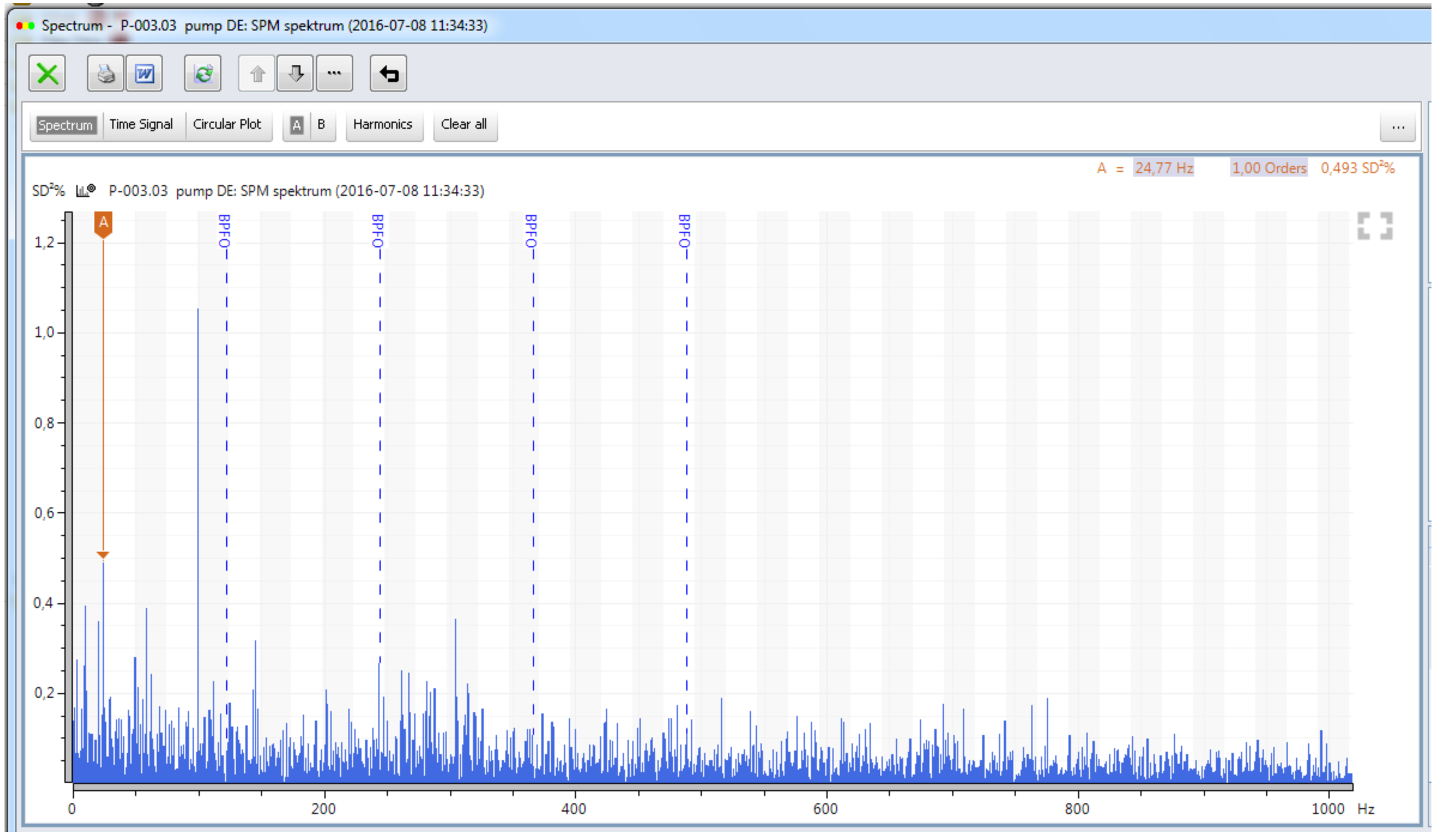


Further investigations showed that the high lines matched the symptom lines belonging to the outer ring fault (BPFO), that is, a failure of the outer ring was detected.

The following three diagrams show the spectrum measured before outer ring fault appeared (good condition), when the problem has developed (presence of failure), and after installing a new bearing (good condition again).




Spectrum - P-003.03 pump DE: SPM spektrum (2016-07-04 9:23:42)
Spectrum
Time Signal
Circular Plot
A
B
Harmonics
Clear all
A = 24,77 Hz
1,00 Orders
0,029 SD²%
SD²%
P-003.03 pump DE: SPM spektrum (2016-07-04 9:23:42)
BPFO
BPFO
BPFO
BPFO
40
20
0
200
400
600
800
Hz



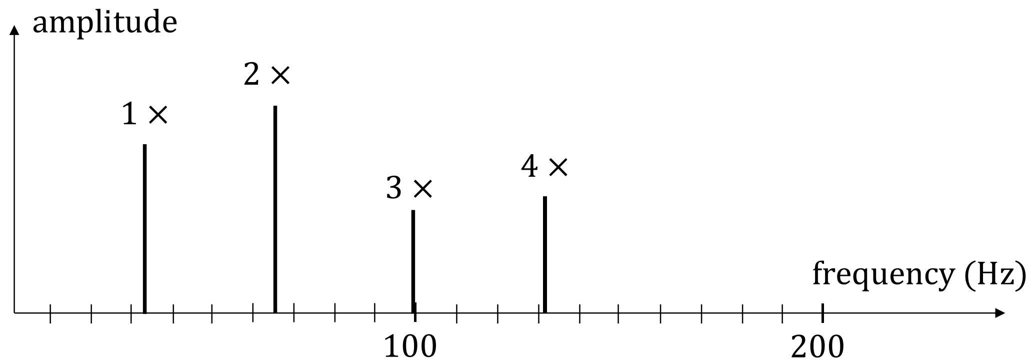

The figure shows the severe fault on the surface of the outer ring.

## *Example*

The bearing fault coefficients for the bearing of type ISO6302 are the following

| | | | | |
|---|---|---|---|---|
| Bearing number | 6302 | | | |
| Manufacturer | NSK | | | |
| TYPE no. | 1 | | | |
| Inner diameter | 15 | mm | BPO | 2,558 |
| Outer diameter | 42 | mm | BPI | 4,442 |
| Mean diameter | 28,5 | mm | BS | 1,724 |
| Width | 13 | mm | FT | 0,365 |

The rotational speed of the shaft during the measurement is 1140 $rpm$. The sketch of the spectrum provided by SPM Ruby is




Which element of the bearing is damaged: outer ring, inner ring, ball, cage, or none of them?

### *Solution*

The bearing fault frequencies belonging to the rotational speed of 1140 $rpm = 19\ rps$ are

| fault type | coefficient | rotational speed ($rps$) | fault frequency ($Hz$) |
|---|---|---|---|
| outer ring | 2.558 | 19 | 48.60 |
| inner ring | 4.442 | 19 | 84.40 |
| ball | 1.724 | 19 | 32.76 |
| cage | 0.365 | 19 | 6.94 |

The spectrum contains a frequency near to 32.765 $Hz$ (ball spin frequency) and its harmonics. It suggests that there is a fault on a ball.

# 5th week – Questions

***Question 1***

Give the orthonormal exponential system which is used for the decomposition of $T$-periodic functions

***Answer***

$$\left\{\mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k \in \mathbb{Z}}$$

## *Question 2*

Give the *Fourier series* and the *Fourier* coefficients of function $x \in L_2([0,T])$ with respect to the orthonormal exponential system.

## *Answer*

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \mathrm{EXP}_k\right) = \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right)\right)$$

$$\hat{X}_k = \langle x, \mathrm{EXP}_k \rangle = \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) dt, \quad k \in \mathbb{Z}$$

## *Question 3*

Give the connection between trigonometric and exponential *Fourier* coefficients.

## *Answer*

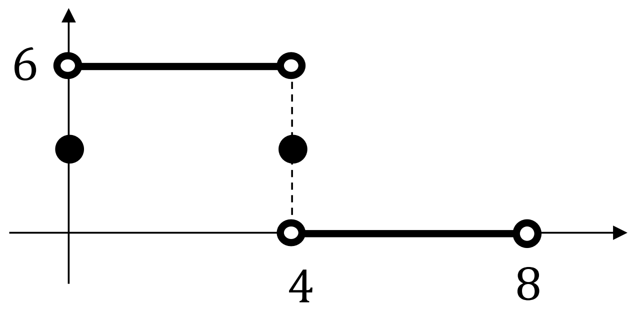
$$\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k - \hat{B}_k \cdot i\right), \quad \hat{X}_{-k} = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k + \hat{B}_k \cdot i\right), \; and \; \left|\hat{X}_k\right| = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}.$$

# 5th week – Exercises

### *Exercise*

Calculate the Fourier coefficient $\hat{x}_3$ of the 8-periodic function $x$ defined as

$$x(t) = \begin{cases} 6, & \text{if} \quad 0 < t < 4 \\ 0, & \text{if} \quad 4 < t < 8 \\ 3, & \text{if} \quad x \in \{0,4\} \end{cases}$$




with respect to the orthonormal exponential system.

### *Solution*

$$\hat{X}_3 = \langle x(t), \frac{1}{\sqrt{8}} \cdot e^{-i \cdot 3 \cdot \frac{2\pi}{8} \cdot t} \rangle = \int_0^4 6 \cdot \frac{1}{\sqrt{8}} \cdot e^{-i \cdot \frac{3\pi}{4} \cdot t} \, dt = \frac{6}{\sqrt{8}} \cdot \frac{-4}{3\pi \cdot i} \cdot \left[ e^{-i \cdot \frac{3\pi}{4} \cdot t} \right]_0^4 =$$

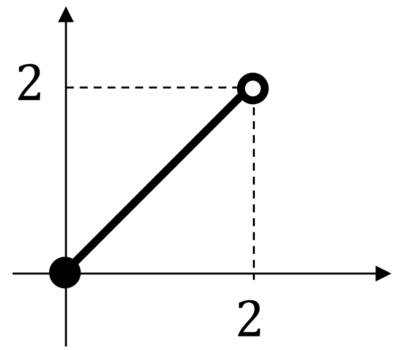
$$= \frac{-\sqrt{8}}{\pi \cdot i} \cdot \left( e^{-i \cdot 3\pi} - 1 \right) = \frac{-\sqrt{8}}{\pi \cdot i} \cdot (\cos(-3\pi) + i \cdot \sin(-3\pi) - 1) = \frac{2 \cdot \sqrt{8}}{\pi \cdot i}$$

## Exercise

Calculate the Fourier coefficient $\hat{x}_4$ of the 2-periodic function $x$ defined as

$$x(t) = t, \quad 0 \le t < 2$$

with respect to the orthonormal exponential system.




## Solution

$$\hat{X}_4 = \langle x(t), \frac{1}{\sqrt{2}} \cdot e^{-i\cdot 4\cdot\frac{2\pi}{2}\cdot t} \rangle = \int_0^2 t \cdot \frac{1}{\sqrt{2}} \cdot e^{-i\cdot 4\cdot\pi\cdot t}\, dt = \frac{1}{\sqrt{2}} \cdot \frac{-1}{4\pi\cdot i} \cdot \left[\left(t + \frac{1}{4\pi i}\right) \cdot e^{-i\cdot 4\pi\cdot t}\right]_0^4 =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{-1}{4\pi\cdot i} \cdot \left(\left(4 + \frac{1}{4\pi i}\right) \cdot e^{-i\cdot 16\pi} - \frac{1}{4\pi i}\right)$$

*Details of the calculation*

$$\int t \cdot e^{-i\cdot 4\pi\cdot t}\, dt = \frac{-1}{4\pi\cdot i} \cdot t \cdot e^{-i\cdot 4\pi\cdot t} + \frac{1}{4\pi\cdot i} \cdot \int e^{-i\cdot 4\pi\cdot t}\, dt = \frac{-1}{4\pi\cdot i} \cdot t \cdot e^{-i\cdot 4\pi\cdot t} + \frac{1}{16\pi^2} \cdot e^{-i\cdot 4\pi\cdot t} =$$

$$\left[\begin{matrix} g(t) = & t \\ f'(t) = & e^{-i\cdot 4\pi\cdot t} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} g'(t) = & 1 \\ f(t) = & \frac{1}{-4\pi\cdot i} \cdot e^{-i\cdot 4\pi\cdot t} \end{matrix}\right] \qquad = \frac{-1}{4\pi\cdot i} \cdot \left(t + \frac{1}{4\pi i}\right) \cdot e^{-i\cdot 4\pi\cdot t}$$

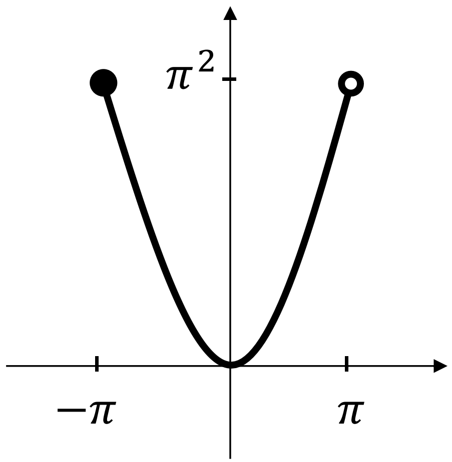
## Exercise

Calculate the Fourier coefficients of the $2\pi$-periodic function $x$ defined as

$$x(t) = t^2, \quad -\pi \le t < \pi$$

with respect to the orthonormal exponential system.

Use the Parseval equality to give the sum $\sum_{k=1}^{\infty} \frac{1}{k^4}$.




## Solution

$$\hat{X}_0 = \int_{-\pi}^{\pi} t^2 \cdot \frac{1}{\sqrt{2\pi}} dt = \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{3} \cdot \pi^3$$

If $k \neq 0$

$$\hat{X}_k = \int_{-\pi}^{\pi} t^2 \cdot \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-i \cdot k \cdot t}\right) dt = \frac{1}{\sqrt{2\pi}} \cdot \left[\left(\frac{1}{k} \cdot i \cdot t^2 + \frac{2}{k^2} \cdot t - \frac{2}{k^3} \cdot i\right) \cdot e^{-i \cdot k \cdot t}\right]_{-\pi}^{\pi} =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left(\left(\frac{1}{k} \cdot i \cdot \pi^2 + \frac{2}{k^2} \cdot \pi - \frac{2}{k^3} \cdot i\right) \cdot e^{-i \cdot k \cdot \pi} - \left(\frac{1}{k} \cdot i \cdot \pi^2 - \frac{2}{k^2} \cdot \pi - \frac{2}{k^3} \cdot i\right) \cdot e^{-i \cdot k \cdot \pi}\right) =$$

$$= \frac{4\pi}{\sqrt{2\pi}} \cdot (-1)^k \cdot \frac{1}{k^2}$$

*Details of the calculation (integration by parts):*

$$\int t^2 \cdot e^{-i\cdot k\cdot t}\,dt = \frac{1}{-k\cdot i}\cdot t^2 \cdot e^{-i\cdot k\cdot t} + \frac{2}{k\cdot i}\cdot \int t \cdot e^{-i\cdot k\cdot t}\,dt =$$

$$\begin{bmatrix} g(t) = & t^2 & \Longrightarrow & g'(t) = & 2t \\ f'(t) = & e^{-i\cdot k\cdot t} & & f(t) = & \frac{1}{-k\cdot i}\cdot e^{-i\cdot k\cdot t} \end{bmatrix} \begin{bmatrix} g(t) = & t & \Longrightarrow & g'(t) = & 1 \\ f'(t) = & e^{-i\cdot k\cdot t} & & f(t) = & \frac{1}{-k\cdot i}\cdot e^{-i\cdot k\cdot t} \end{bmatrix}$$

$$= \frac{1}{-k\cdot i}\cdot t^2 \cdot e^{-i\cdot k\cdot t} + \frac{2}{k\cdot i}\cdot \left(\frac{1}{-k\cdot i}\cdot t\cdot e^{-i\cdot k\cdot t} + \frac{1}{k\cdot i}\cdot \int e^{-i\cdot k\cdot t}\,dt\right) =$$

$$= \frac{1}{-k\cdot i}\cdot t^2 \cdot e^{-i\cdot k\cdot t} + \frac{2}{k^2}\cdot t\cdot e^{-i\cdot k\cdot t} + \frac{2}{k^3\cdot i}\cdot e^{-i\cdot k\cdot t} = \left(\frac{1}{k}\cdot i\cdot t^2 + \frac{2}{k^2}\cdot t - \frac{2}{k^3}\cdot i\right)\cdot e^{-i\cdot k\cdot t}$$

We used that

$$e^{-i\cdot k\cdot \pi} = \cos(-k\cdot\pi) + i\cdot\sin(-k\cdot\pi) = \cos(k\cdot\pi) = (-1)^k$$

$$e^{i\cdot k\cdot \pi} = \cos(k\cdot\pi) + i\cdot\sin(k\cdot\pi) = \cos(k\cdot\pi) = (-1)^k$$

According to the Parseval's equality

$$\|x\|^2 = \int_{-\pi}^{\pi} t^4\,dt = \sum_{k=-\infty}^{\infty} \hat{X}_k^2 = \hat{X}_0^2 + 2\cdot\sum_{k=1}^{\infty}\hat{X}_k^2 = \frac{2}{9}\cdot\pi^5 + 2\cdot 8\pi\cdot\sum_{k=1}^{\infty}\frac{1}{k^4}$$

Since $\int_{-\pi}^{\pi} t^4\, dt = \frac{1}{5} \cdot [t^5]_{-\pi}^{\pi} = \frac{2}{5} \cdot \pi^5$ we have

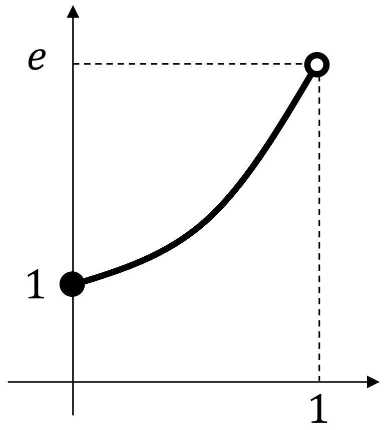
$$\frac{2}{5} \cdot \pi^5 = \frac{2}{9} \cdot \pi^5 + 16\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^4} \qquad \Rightarrow \qquad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

## Exercise

Calculate the Fourier coefficient $\hat{x}_3$ of the 1-periodic function $x$ defined as

$$x(t) = e^t, \quad 0 \le t < 1$$

with respect to the exponential system.




## Solution

$$\hat{x}_3 = \int_0^1 e^t \cdot e^{-i\cdot 3\cdot 2\pi\cdot t}\, dt = \int_0^1 e^{(1-6\pi i)\cdot t}\, dt = \frac{1}{1-6\pi i} \cdot \left[e^{(1-6\pi i)\cdot t}\right]_0^1 = \frac{1}{1-6\pi i} \cdot \left(e^{1-6\pi i} - 1\right) =$$

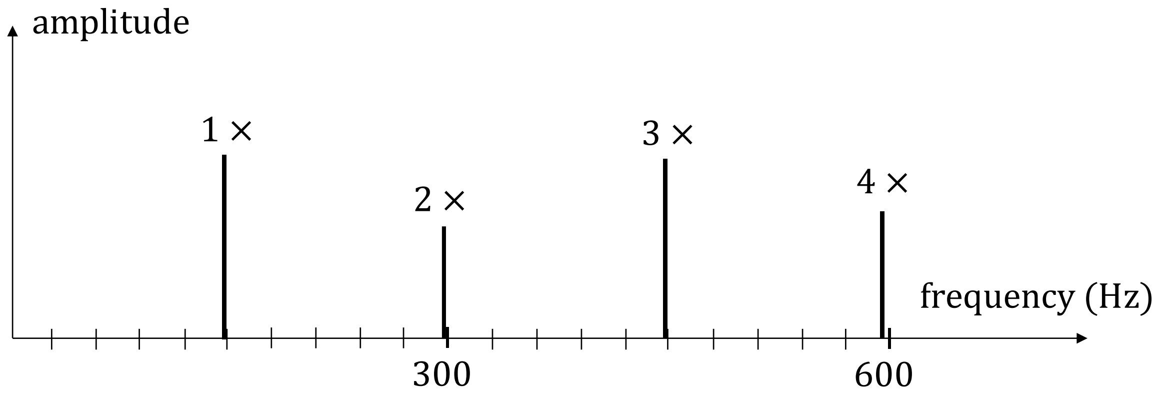
$$= \frac{1}{1-6\pi i} \cdot \left(e \cdot e^{-6\pi i} - 1\right) = \frac{e}{1-6\pi i} \cdot (\cos(-6\pi) + i \cdot \sin(-6\pi)) - \frac{1}{1-6\pi i} = \frac{e-1}{1-6\pi i}$$

## *Exercise*

The bearing fault coefficients for the bearing of type ISO30210 are the following

| | | | | |
|---|---|---|---|---|
| Bearing number | 30210 | | | |
| Manufacturer | SKF | | | |
| TYPE no. | 6 | | | |
| Inner diameter | 50 | mm | BPO | 8,571 |
| Outer diameter | 90 | mm | BPI | 11,429 |
| Mean diameter | 70 | mm | BS | 3,301 |
| Width | 21,75 | mm | FT | 0,429 |

The rotational speed of the shaft during the measurement is 780 *rpm*. The sketch of the spectrum provided by SPM Ruby is




Which element the symptom lines in the figure belong to (outer ring, inner ring, ball, cage, or none of them)?

## ☑ *Solution*

The bearing fault frequencies belonging to the rotational speed of 780 $rpm = 13\ rps$ are

| fault type | coefficient | rotational speed ($rps$) | fault frequency ($Hz$) |
|---|---|---|---|
| outer ring | 8.571 | 13 | 111.42 |
| inner ring | 11.429 | 13 | 148.58 |
| ball | 3.301 | 13 | 42.92 |
| cage | 0.429 | 13 | 5.577 |

The spectrum contains a frequency near to 148.58 $Hz$ (inner ring frequency) and its harmonics. It suggests that there is a fault on the inner ring.

## ***Exercise***

It is known that the specific symptom of a coupling problem is a line in the frequency spectrum at 2nd order. Determine the specific frequency belonging to the coupling problem if the rotational speed of the shaft is 1800 $rpm$.

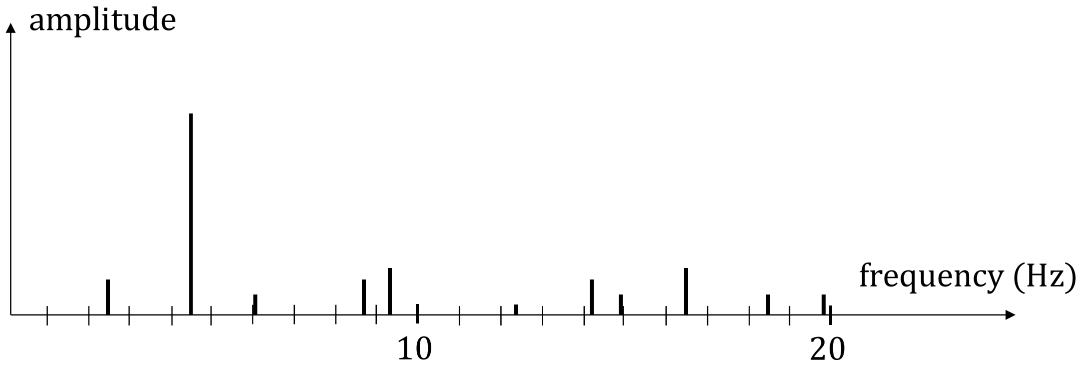
## ***Solution***

The rotational speed is $1800\ rpm = 50\ rps$.

The line belonging to the coupling problem is at $2 \times 50 = 100\ Hz$ in the frequency spectrum.

## *Exercise*

Sometimes a possible way to determine the rotational speed of the shaft is to find the frequency of 1st order (frequency belonging to unbalance). Since this frequency generally belongs to the highest energy harmonic component of the signal, the shaft speed can be identified finding the highest line in the frequency spectrum. Give the probable value of the rotational speed of the shaft using the following spectrum.




## *Solution*

The highest line in the frequency spectrum is near to 4.5 $Hz$. It suggests that the rotational speed of the shaft is 4.5 $rps = 4.5 \times 60 = 225\ rpm$.

# 6th week

# 6 Continuous Fourier Transform, Discrete Fourier Transform, Fast Fourier Transform

# Integral Transforms, Convolution

Let $K: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a given integrable function. Function

$$F(s) = \int_a^b f(t) \cdot K(s,t)\, dt, \quad s \in \mathbb{C}$$

is called the *integral transform* of function $f: [a,b] \to \mathbb{C}$ if the integral is convergent.

The *Fourier transform* and the *Laplace transform* are two well-known integral transforms, which are frequently used in different fields of engineering and sciences. Some special transformations appear in special applications, e.g. the wavelet transform is important tool, for example in technical diagnostics.

Some transformations (e.g. Fourier and wavelet) have continuous and discrete forms. Discrete transformations are used in discrete signal processing where only a sampled signal is available rather than the formula of the function (signal).

A basic concept in signal analysis is the convolution. It has different forms depending on the field of the application. *Convolution* of integrable functions $x: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ can be defined as

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau, \quad t \in \mathbb{R}.$$

The convolution formula can be considered as a weighted average of the function $x(\tau)$ at the moment $t$ where the weighting is given by $h(t-\tau)$. As $t$ changes, the weighting function emphasizes different parts of the input function.

For functions $x$ and $h$ supported only on the non-negative real line a modified formula is used. *Convolution* of integrable functions $x: [0,\infty[\to \mathbb{R}$ and $h: [0,\infty[\to \mathbb{R}$ is

$$(x * h)(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau, \quad t \in [0,\infty[.$$

If $h(t), t \in \mathbb{R}$ is the impulse response (response to the delta function input) of a linear time-invariant system, then the response of the system for an input $x(t), t \in \mathbb{R}$ is

$$y(t) = (x * h)(t), t \in \mathbb{R}.$$

***Example***

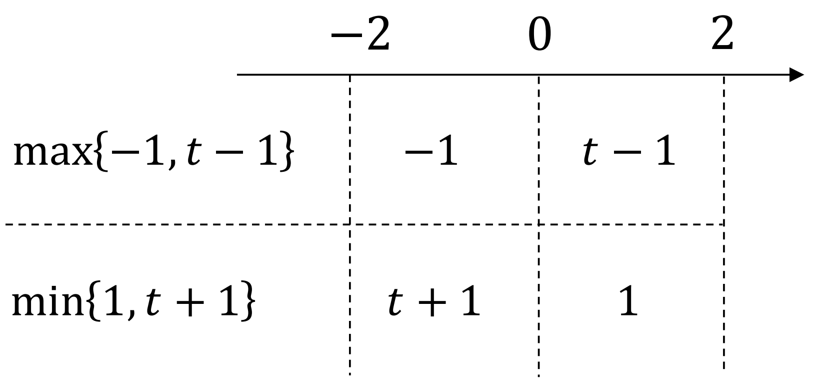
Determine the convolution $\Pi * \Pi$ where $\Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$ is the rectangular pulse function on $[-1,1]$.

***Solution***

$$\Pi * \Pi(\mathrm{t}) = \int_{-\infty}^{\infty} \Pi(\tau) \cdot \Pi(t-\tau)\, d\tau$$

The integral is different from zero when both $\Pi(\tau)$ and $\Pi(t-\tau)$ are different from zero, that is, when $-1 \leq \tau \leq 1$ and $-1 \leq t-\tau \leq 1$. The latter system of inequalities can be written as $t-1 \leq \tau \leq t+1$.

The two requirements imply that the limits of the integration are $\max\{-1, t-1\}$ and $\min\{1, t+1\}$ assuming that $\max\{-1, t-1\} \leq \min\{1, t+1\}$, that is, when $-2 \leq t \leq 2$.




$$\Pi * \Pi(\mathrm{t}) = \int_{\max\{-1,t-1\}}^{\min\{1,t+1\}} \Pi(\tau) \cdot \Pi(t-\tau)\, d\tau$$

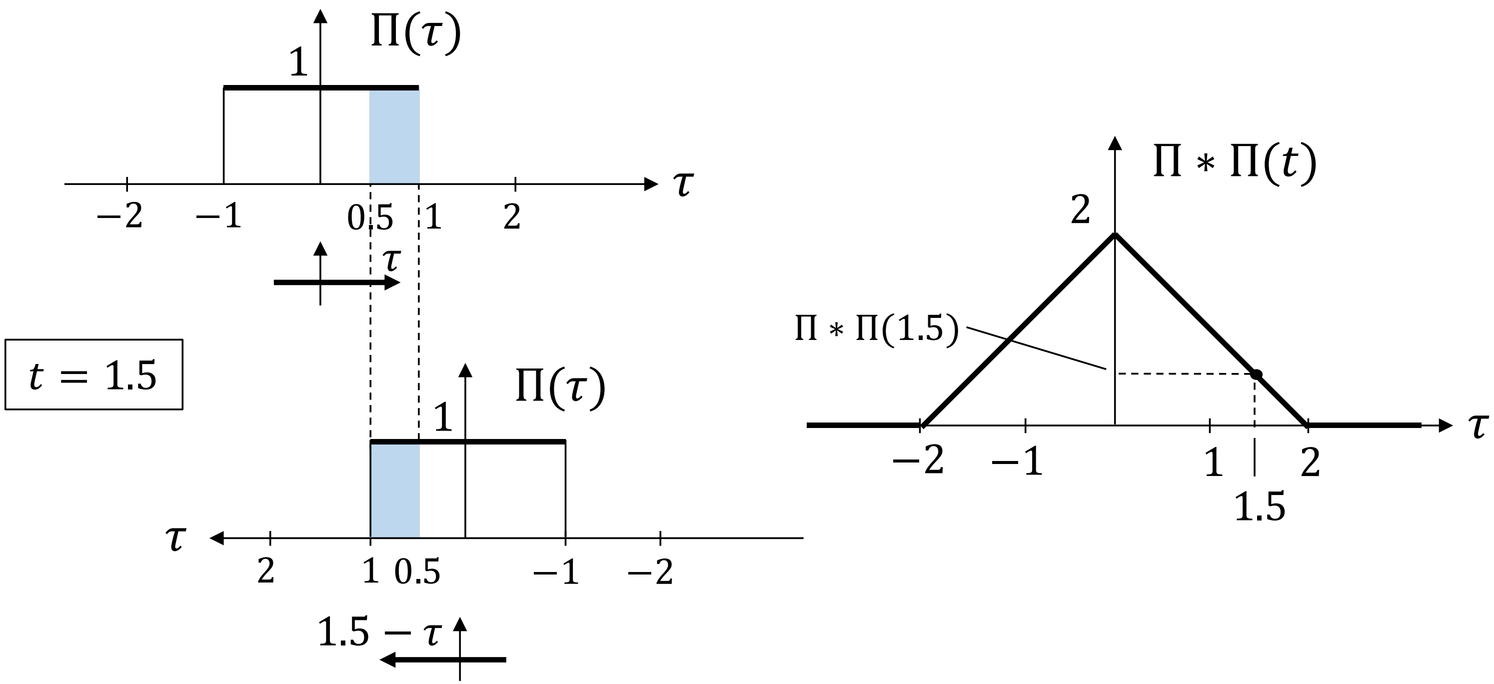
If $-2 \le t \le 0$, then

$$\Pi * \Pi(\mathrm{t}) = \int_{\tau=-1}^{t+1} 1\, d\tau = [\tau]_{\tau=-1}^{t+1} = t + 2$$

If $0 \le t \le 2$, then

$$\Pi * \Pi(\mathrm{t}) = \int_{\tau=t-1}^{1} 1\, d\tau = [\tau]_{\tau=t-1}^{1} = -t + 2$$

The following figure, as an example, shows the calculation of $\Pi(1.5)$:



The convolution is the triangle function $\mathrm{tri}\, t = \begin{cases} t+2 & \text{if} \quad -2 \leq t \leq 0 \\ -t+2 & \text{if} \quad 0 \leq t \leq 2 \\ 0 & \text{if} \quad |t| > 2 \end{cases}$.

# Continuous Fourier Transform

Function

$$\mathcal{FT}(x)(\omega) = \hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\cdot\omega\cdot t}\, dt, \quad \omega \in \mathbb{R}$$

is the *Fourier transform* of function $x: \mathbb{R} \to \mathbb{R}$ if the integral is convergent.

The *Fourier integral* of $x: \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{FI}(x)(t) = \mathcal{FT}^{-1}(\hat{x})(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\cdot\omega\cdot t}\, d\omega, \quad t \in \mathbb{R}.$$

Functions

$$\omega \to |\hat{x}(\omega)|, \quad \omega \to |\hat{x}(\omega)|^2, \quad \text{and} \quad \omega \to \angle\, \hat{x}(\omega)$$

are called *amplitude spectrum*, *energy spectrum* and *phase spectrum*, respectively, in engineering literature.

***Remark***

Different formulas can be found in different textbooks for the Fourier transform and for the Fourier integral. Here we use the angular frequency $\omega$ as a variable and we write coefficient $\frac{1}{2\pi}$ in the Fourier integral rather than in the Fourier transform to harmonize it with the formula of the Laplace transform. Instead of $\omega$ the frequency $f$ can also be used as a variable and the coefficients in the two formulas can be written in different way.

***Remark***

The Fourier coefficients of periodic functions have discrete nature, values of $\hat{x}_k$ are linked to the 'discrete angular frequencies' $k \cdot \omega_0 = k \cdot \frac{2\pi}{T}$, this is why this spectrum is called '*discrete'.* Furthermore, the Fourier series gives the function in the form of a (countable) infinite summation.

The spectrum of non-periodic functions provided by the Fourier transform is called continuous since the domain of the Fourier transform is the real line, that is, any real angular frequencies can appear in the spectrum. Instead of summation there is an integration in the Fourier integral.

Using the Euler's formula $e^{i \cdot t} = \cos t + i \cdot \sin t\,, t \in \mathbb{R}$ we can write the Fourier transform of $x\colon \mathbb{R} \to \mathbb{R}$ as

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t}\, dt =$$

$$= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(-\omega \cdot t)\, dt + i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(-\omega \cdot t)\, dt =$$

$$= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt - i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt = \hat{a}(\omega) - i \cdot \hat{b}(\omega), \quad \omega \in \mathbb{R}.$$

When $x$ is even, then $\hat{b}_x = 0$ and we have

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}.$$

When $x$ is odd, then $\hat{a}_x = 0$ and we have

$$\hat{x}(\omega) = -i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt = -i \cdot 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}.$$

Integrals

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

and

$$\mathcal{FT}_{\sin}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

are called the *cosine Fourier transform* and the *sine Fourier transform* of function $x: [0, \infty[ \to \mathbb{R}$.

The *Fourier cosine integral* of $x$ is

$$\mathcal{FI}_{\cos}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\cos}(x)(\omega) \cdot \cos(\omega \cdot t)\, d\omega, \quad t \in \mathbb{R}, t \geq 0,$$

while the *Fourier sine integral* of $x$ is

$$\mathcal{FI}_{sin}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\sin}(x)(\omega) \cdot \sin(\omega \cdot t)\, d\omega, \quad t \in \mathbb{R}, t \geq 0.$$

Each real function $x: \mathbb{R} \to \mathbb{R}$ (having Fourier transform) can be analysed with its cosine and sine Fourier transform since $x$ can be written as

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = g(t) + h(t), \quad t \in \mathbb{R}.$$

where $g$ is even and $h$ is odd. Thus

$$\mathcal{FT}(x) = \mathcal{FT}(g) + \mathcal{FT}(h) = \mathcal{FT}_{\cos}(g) - i \cdot \mathcal{FT}_{\sin}(h)$$

If function $x$ is piecewise continuous then $\mathcal{FI}(x)$ is equal to $x$ wherever $x$ is continuous, and $\mathcal{FI}(x)$ is the average the left- and right-hand limits wherever $x$ is discontinuous.

### *Remark*

Since a piecewise continuous function (signal) can be reconstructed from its Fourier transform (through its Fourier integral) we can say that the Fourier transform contains all information about the function, and can be considered as an alternative representation.

For instance, a vibration process can be described in the 'time domain' (e.g. vibration velocity vs. time function) and also in 'frequency domain' (e.g. vibration frequency spectrum).

Parseval's equality (energy of a signal):

$$\int_{t=-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} |\hat{x}(\omega)|^2\,d\omega.$$

The following table shows some functions with their Fourier transforms.

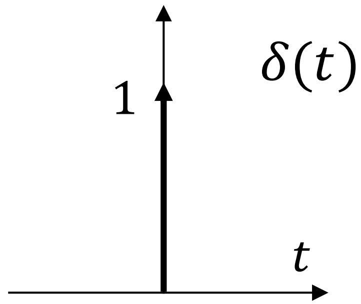
We can find some 'dual' properties of the Fourier transform which show how the Fourier transform changes (in the frequency domain) when the function is changed in the time domain, and vice versa.

For $\alpha, \beta, T, \omega_0 \in \mathbb{R}$

| | **time domain** | **frequency domain** |
|---|---|---|
| | $t \to \boldsymbol{x(t)} = \mathcal{FT}^{-1}(\hat{x})(t)$ | $\omega \to \boldsymbol{\hat{x}(\omega)} = \mathcal{FT}(x)(\omega)$ |
| linearity | $t \to \alpha \cdot x(t) + \beta \cdot y(t)$ | $\omega \to \alpha \cdot \hat{x}(\omega) + \beta \cdot \hat{y}(\omega)$ |
| shift in the time domain | $t \to x(t - T)$ | $\omega \to \hat{x}(\omega) \cdot e^{-i \cdot T \cdot \omega}$ |
| shift in the frequency domain (modulation) | $t \to x(t) \cdot e^{i \cdot \omega_0 \cdot t}$ | $\omega \to \hat{x}(\omega - \omega_0)$ |
| scaling | $t \to x(\alpha \cdot t)$ | $\omega \to \frac{1}{\lvert\alpha\rvert} \cdot \hat{x}\left(\frac{\omega}{\alpha}\right)$ |
| convolution | $t \to (x * y)(t)$ | $\omega \to \hat{x}(\omega) \cdot \hat{y}(\omega)$ |

In the theory of integral transforms *unit impulse function* (or *Dirac delta function*) has important role. Dirac delta is not a common function, its value is different from zero only at 0, and its integral on $\mathbb{R}$ is 1:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



## *Example*

Determine the complex Fourier transform and the Fourier integral of the rectangular pulse function

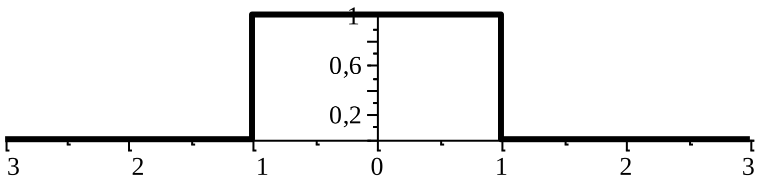
$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

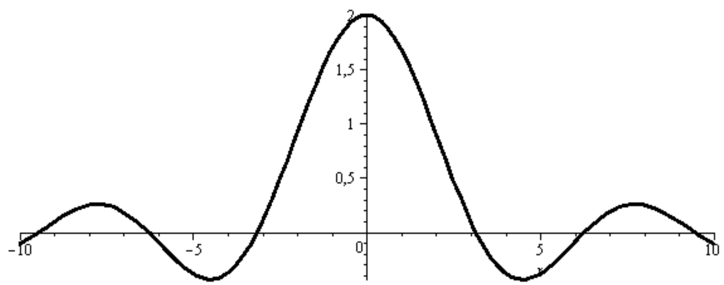
## *Solution*

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t}\, dt = \int_{t=-1}^{1} e^{-i \cdot \omega \cdot t}\, dt = \left[ \frac{e^{-i \cdot \omega \cdot t}}{-i \cdot \omega} \right]_{t=-1}^{1} =$$

$$= \frac{1}{-i \cdot \omega} \cdot \left( e^{-i \cdot \omega} - e^{i \cdot \omega} \right) = \frac{2}{\omega} \cdot \frac{e^{i \cdot \omega} - e^{-i \cdot \omega}}{2i} = 2 \cdot \frac{\sin \omega}{\omega} = 2 \cdot \operatorname{sinc} \omega$$

$$t \to x(t) = \Pi(t) \qquad\qquad \omega \to \hat{x}(\omega) = 2 \cdot \frac{\sin \omega}{\omega}$$



The Fourier integral of $x$ is

$$\mathcal{FI}(x)(t) = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i \cdot \omega \cdot t} \, d\omega = \frac{1}{\pi} \cdot \int_{\omega=-\infty}^{\infty} \frac{\sin(\omega)}{\omega} \cdot e^{i \cdot \omega \cdot t} \, d\omega$$

***Example***

Determine the sine and cosine Fourier transform of the of the rectangular pulse function

$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

***Solution***

Since $x$ is even $\mathcal{FT}_{\sin}(x) = 0$.

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt = 2 \cdot \int_{t=0}^{1} \cos(\omega \cdot t)\, dt = 2 \cdot \frac{\sin \omega}{\omega}$$

The Fourier cosine integral of $x$ is

$$\mathcal{FI}(x)(t) = \frac{2}{\pi} \cdot \int_{\omega=0}^{\infty} \frac{\sin \omega}{\omega} \cdot \cos(\omega \cdot t)\, d\omega$$

***Example***

Determine the Fourier transform of the unit impulse (Dirac delta) function

***Solution***

$$\mathcal{FT}(\delta)(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-i\cdot\omega\cdot t}\, dt = e^{-i\cdot 0\cdot t} \cdot \int_{-\infty}^{\infty} \delta(t) \cdot dt = 1$$

## Example

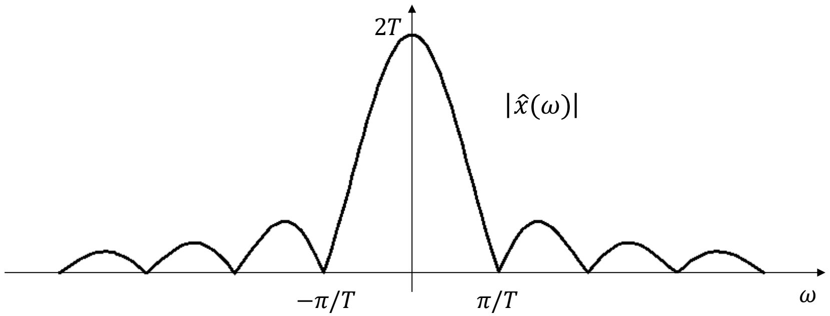
Determine the Fourier transform of the shifted rectangular pulse function

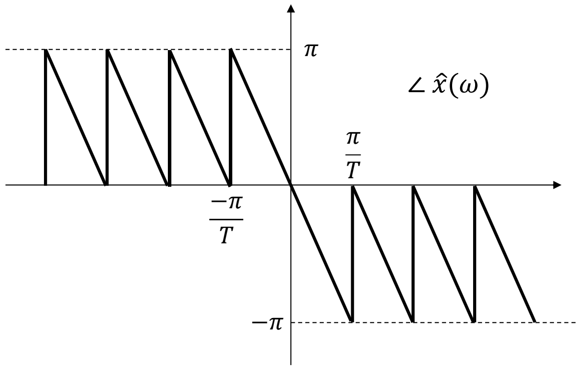
$$x(t) = \begin{cases} 1 & \text{if} \quad 1 - T \le t \le 1 + T \\ 0 & \text{if} \quad t < 1 - T \text{ or } t > 1 + T \end{cases}$$

## Solution

$$\hat{x}(\omega) = \int_{t=1-T}^{1+T} e^{-i\cdot\omega\cdot t}\, dt = \frac{-1}{i\cdot\omega}\cdot\left[e^{-i\cdot\omega\cdot t}\right]_{t=1-T}^{1+T} = \frac{-1}{i\cdot\omega}\cdot\left(e^{-i\cdot\omega\cdot(1+T)} - e^{i\cdot\omega\cdot(1-T)}\right) =$$

$$= \frac{-1}{i\cdot\omega}\cdot e^{-i\cdot\omega}\cdot\left(e^{-i\cdot\omega\cdot T} - e^{i\cdot\omega\cdot T}\right) = 2\cdot e^{-i\cdot\omega}\cdot\frac{1}{\omega}\cdot\frac{e^{i\cdot\omega\cdot T} - e^{-i\cdot\omega\cdot T}}{2i} = 2\cdot\frac{\sin(\omega\cdot T)}{\omega}\cdot e^{-i\cdot\omega}$$



## *Example*

Calculate $\lim_{a \to 0+0} \hat{x}(\omega)$ when $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0$

## *Solution*

$$\lim_{a \to 0+0} \hat{x}(\omega) = \lim_{a \to 0+0} \frac{a}{a^2 + \omega^2} + \lim_{a \to 0+0} \frac{-i \cdot \omega}{a^2 + \omega^2} = \pi \cdot \delta(\omega) + \frac{1}{i \cdot \omega}$$

The unit step function $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ has not Fourier transform since the integral

$$\int_{t=0}^{\infty} e^{-i\cdot\omega\cdot t}\,dt = \int_{t=0}^{\infty} \cos(\omega \cdot t)\,dt - i \cdot \int_{t=0}^{\infty} \sin(\omega \cdot t)\,dt$$

is not convergent.

The unit step function can be considered as the limit of $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a\cdot t} & \text{if } t \geq 0 \end{cases}$, $a > 0$ as $a \to 0 + 0$, thus we can define, symbolically, the 'Fourier transform' of unit step function as $\pi \cdot \delta(\omega) + \frac{1}{i\cdot\omega}$.

This definition yields Fourier transform of further important functions.

***Example***

Determine symbolically the Fourier transform of the negative unit step function $x(t) = \begin{cases} 1 & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases}$ using the Fourier transform of the unit step function.

***Solution***

$$\int_{t=-\infty}^{0} e^{-i\cdot\omega\cdot t}\, dt = \int_{t=0}^{\infty} e^{-i\cdot(-\omega)\cdot t}\, dt = \pi \cdot \delta(\omega) - \frac{1}{i \cdot \omega}$$

### *Example*

Determine symbolically the Fourier transform of periodic function (with basic frequency $\omega_0$)

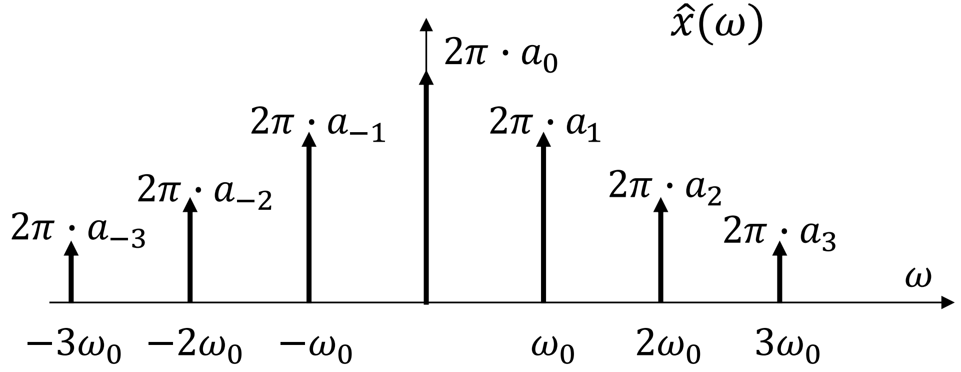
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{i \cdot k \cdot \omega_0 \cdot t}$$

using that $\int_{t=-\infty}^{\infty} e^{-i \cdot \omega \cdot t}\, dt = 2\pi \cdot \delta(\omega)$.

### *Solution*

$$\hat{x}(\omega) = \sum_{k=-\infty}^{\infty} a_k \cdot \int_{t=-\infty}^{\infty} e^{i \cdot k \cdot \omega_0 \cdot t} \cdot e^{-i \cdot \omega \cdot t}\, dt = \sum_{k=-\infty}^{\infty} a_k \cdot \int_{t=-\infty}^{\infty} e^{i \cdot (k \cdot \omega_0 - \omega) \cdot t}\, dt =$$

$$= 2\pi \cdot \sum_{k=-\infty}^{\infty} a_k \cdot \delta(\omega - k \cdot \omega_0)$$



***Example***

Use time shift property of Fourier transform $\mathcal{FT}\big(x(t-T)\big) = e^{-i\cdot\omega\cdot T}\cdot\mathcal{FT}(x(t))$ to determine the Fourier transform of function $x(t) = \cos(\omega_0 \cdot t + \varphi)$.

***Solution***

$$x(t) = \cos(\omega_0 \cdot t + \varphi) = \cos\left(\omega_0 \cdot \left(t + \frac{\varphi}{\omega_0}\right)\right).$$

Using the transform of $x(t) = \cos(\omega_0 \cdot t)$ and the time shift property we have

$$\hat{x}(\omega) = e^{i\cdot\omega\cdot\frac{\varphi}{\omega_0}}\cdot\pi\cdot\big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big)$$

***Example***

Determine the Fourier transform of triangle function

$$x(t) = \mathrm{tri}(t) = \begin{cases} t+2 & if \quad -2 \leq t \leq 0 \\ -t+2 & if \quad 0 \leq t \leq 2 \\ 0 & if \quad |t| > 2 \end{cases}$$

using the convolution theorem $\mathcal{FT}(x * h) = \mathcal{FT}(x) \cdot \mathcal{FT}(h)$.

***Solution***

We have that

$$x(t) = \mathrm{tri}(t) = \Pi * \Pi(t)$$

where $\Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$, is the rectangular impulse function.

Using the convolution theorem we get

$$\hat{x}(\omega) = \mathcal{FT}(\Pi)(\omega) \cdot \mathcal{FT}(\Pi)(\omega) = 4 \cdot \frac{\sin^2 \omega}{\omega^2}$$

# Discrete Convolution

*Convolution* of two discrete-time signals $\{x[n]\}_{n\in\mathbb{N}}$ and $\{h[n]\}_{n\in\mathbb{N}}$ is

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]\,, \quad n \in \mathbb{N}.$$

***Remark***

If $h[n]$ is the impulse response (response to the delta function input) of a linear time-invariant system, then the response of the system for a discrete input $\{x[n]\}_{n\in\mathbb{N}}$ is

$$y[n] = (x * h)[n], \quad n \in \mathbb{N}.$$

### Example

Determine the convolution of functions $x: \mathbb{N} \to \mathbb{R}$ and $h: \mathbb{N} \to \mathbb{R}$ defined as

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x[n]$ | 4 | 1 | 2 | 5 |

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $h[n]$ | 1 | 2 | $-1$ |

$x[n] = 0, h[n] = 0$ otherwise.

### Solution

$$(x * h)[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[-k] = x[0] \cdot h[0] = 4 \cdot 1 = 4$$

$$(x * h)[1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k] = x[0] \cdot h[1] + x[1] \cdot h[0] = 4 \cdot 2 + 1 \cdot 1 = 9$$

$$(x * h)[2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[2-k] = x[0] \cdot h[2] + x[1] \cdot h[1] + x[2] \cdot h[0] =$$

$$= 4 \cdot (-1) + 1 \cdot 2 + 2 \cdot 1 = 0$$

$$(x * h)[3] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[3-k] = x[1] \cdot h[2] + x[2] \cdot h[1] + x[3] \cdot h[0] =$$

$$= 1 \cdot (-1) + 2 \cdot 2 + 5 \cdot 1 = 8$$

$$(x * h)[4] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[4-k] = x[2] \cdot h[2] + x[3] \cdot h[1] = 2 \cdot (-1) + 5 \cdot 2 = 8$$

$$(x * h)[5] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[5-k] = x[3] \cdot h[2] = 5 \cdot (-1) = -5$$

$(x * h)[n] = 0$ otherwise.

| $n$ | ... | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $(x * h)[n]$ | ... | 0 | 4 | 9 | 0 | 8 | 8 | $-5$ | 0 | ... |

# Discrete Fourier Transform

Since digital measuring systems are used in practice and the data are stored and processed with digital computers, sampled signals are available for signal processing. Consequently, the continuous transforms requiring the formula of the signal are unusable, we need algorithms (discrete transforms) that provide (or approximate) the spectrum from sampled signals. (Since the sampled signals contain limited measure of information about the signal, the exact spectrum cannot be determined from the sample.)

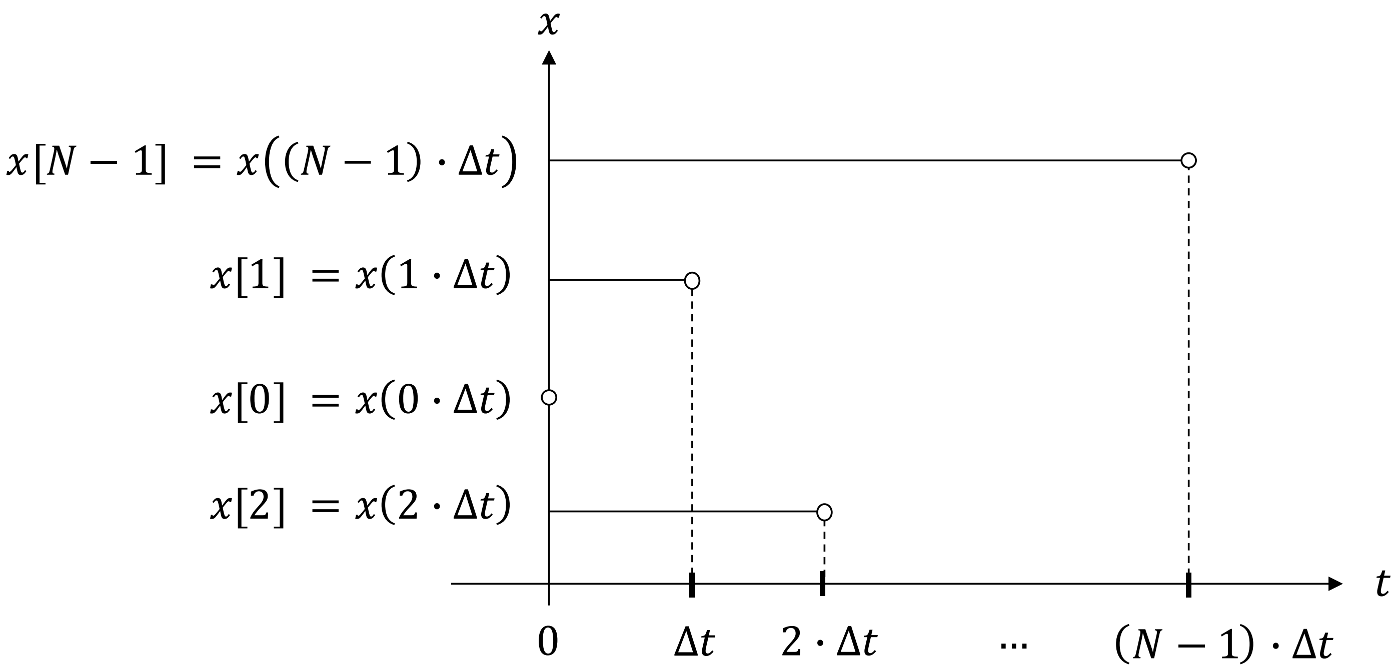
Let $T > 0$ be a fixed real number and $N$ be a fixed positive integer and suppose that values

$$x[n] = x[n \times \Delta T], \quad n = 0,1, \dots, N-1$$

of signal $x$ are provided by a sampling process.

The *discrete Fourier transform* of sampled signal $x[0], \dots, x[N-1]$ is

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}, \quad k = 0,1, \dots, N-1$$




Mathematically, both the input and the output of the discrete Fourier transform consist of $N$ pure numbers. In practice, when the circumstances of sampling are known, these values have physical meaning, and values $X[0], \ldots, X[N-1]$ provide a 'discrete spectrum'.

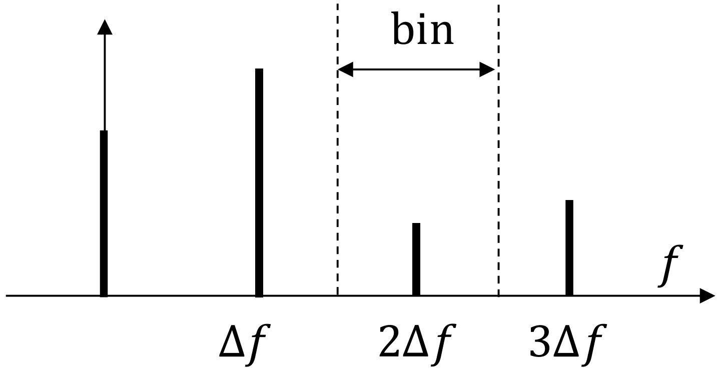
Considering a sampling process, $T$ is the *sampling time*, $N$ is the *sample size* (number of elements in the sample), $\Delta T$ is time between two measurements.

Further quantities are the *sampling frequency* $f_s = N/T = 1/\Delta T$, the *frequency resolution* $\Delta f = 1/T = f_s/N$. The potential frequency values in the discrete spectrum are $k \times \Delta f,\ k = 0,1, \ldots, N-1$.

### Example

If the sampling frequency is $f_s = 1000\,Hz$, and the sample size is $N = 1024$, then the frequency resolution is

$$\Delta f = \frac{1000}{1024} = 0.9766\ \frac{Hz}{bin}$$



The discrete Fourier transform can also be calculated as a matrix multiplication.

Introducing the notation $W_N = e^{-i\cdot\frac{2\pi}{N}}$ (then $e^{-i\cdot k\cdot n\cdot\frac{2\pi}{N}} = W_N^{k\cdot n}$), the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2\cdot(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix}$$

If $N = 2$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

If $N = 4$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

If $N = 8$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & r & -i & -i \cdot r & -1 & -r & i & i \cdot r \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i \cdot r & i & r & -1 & i \cdot r & -i & -r \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -r & -i & i \cdot r & -1 & r & i & -i \cdot r \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i \cdot r & i & -r & -1 & -i \cdot r & -i & r \end{pmatrix}, \quad r = \frac{1}{\sqrt{2}} \cdot (1 - i)$$

Calculation with the matrix:

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2\cdot(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

The inverse transformation is

$$x[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot e^{i\cdot k\cdot n\cdot\frac{2\pi}{N}} = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot W_N^{-k\cdot n} =$$

$$= \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot \left(W_N^{k\cdot n}\right)^*, \quad n = 0,1,\ldots,N-1$$

or in matrix form

$$\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^4 & \cdots & W_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2\cdot(N-1)} & \cdots & W_N^{-(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}$$

# Fast Fourier Transform (FFT)

Formula of DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] \cdot W_N^{k \cdot n}, \quad k = 0,1,\dots,N-1$$

where $W_N = e^{-i \cdot \frac{2\pi}{N}}$, or in matrix form

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2 \cdot (N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2 \cdot (N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}.$$

It is clear from the formulas that a DFT requires the evaluation of polynomial

$$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_{N-1} \cdot x^{N-1}$$

where

$$a_0 = x[0], a_1 = x[1], \dots, a_{N-1} = x[N-1]$$

on a special set

$$\{1, W_N, W_N^2, \dots, W_N^{N-1}\}, \qquad W_N = e^{-i \cdot \frac{2\pi}{N}}, \quad (W^n = 1)$$

which is a so-called collapse set.

($X$ is a collapse set if $|X^2| = \frac{1}{2} \cdot |X|$ or $X = \{1\}$, where $|X|$ denotes the number of elements in $X$.)

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}.$$

To reduce the computational time (number of steps) we use recursively that

$$A(x) = A_{\text{even}}(x^2) + x \cdot A_{\text{odd}}(x^2)$$

where

$$A_{\text{even}}(x) = a_0 + a_2 \cdot x + a_4 \cdot x^2 + \ldots + a_{N-2} \cdot x^{\frac{N}{2}-1} = \sum_{k=0}^{\frac{N}{2}-1} a_{2k} \cdot x^k$$

$$A_{\text{odd}}(x) = a_1 + a_3 \cdot x + a_5 \cdot x^2 + \ldots + a_{N-1} \cdot x^{\frac{N}{2}-1} = \sum_{k=0}^{\frac{N}{2}-1} a_{2k+1} \cdot x^k$$

# Decimation in time

Here we suppose that $N$ is even (in practice $N$ is a power of 2).

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{k \cdot n} = \sum_{n\ is\ even} x[n] \cdot W_N^{k \cdot n} + \sum_{n\ is\ odd} x[n] \cdot W_N^{k \cdot n} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_N^{2 \cdot r \cdot k} + \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_N^{(2 \cdot r + 1) \cdot k} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot (W_N^{2 \cdot})^{r \cdot k} + W_N^k \cdot \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot (W_N^{2 \cdot})^{r \cdot k} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_{\frac{N}{2}}^{r \cdot k} + W_N^k \cdot \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_{\frac{N}{2}}^{r \cdot k}$$

Since

$$G[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_{\frac{N}{2}}^{r \cdot k} \quad and \quad H[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_{\frac{N}{2}}^{r \cdot k}$$

are $\frac{N}{2}$ point DFTs, we have that the calculation of an $N$ point DFTs can be led back to the calculation of two $\frac{N}{2}$ point DFTs:

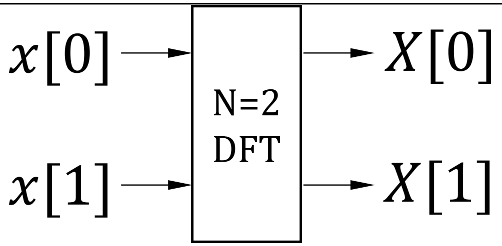
$$X[k] = G[k] + W_N^k \cdot H[k]$$

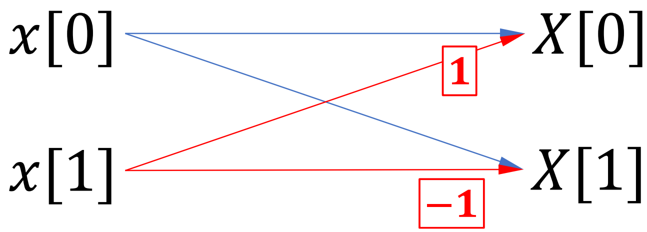
where $G[k]$ is calculated from values $X[0], X[2], X[4], \dots, X[N-2]$, while $H[k]$ is calculated from values $X[1], X[3], X[5], \dots, X[N-1]$.

# 2-point DFT

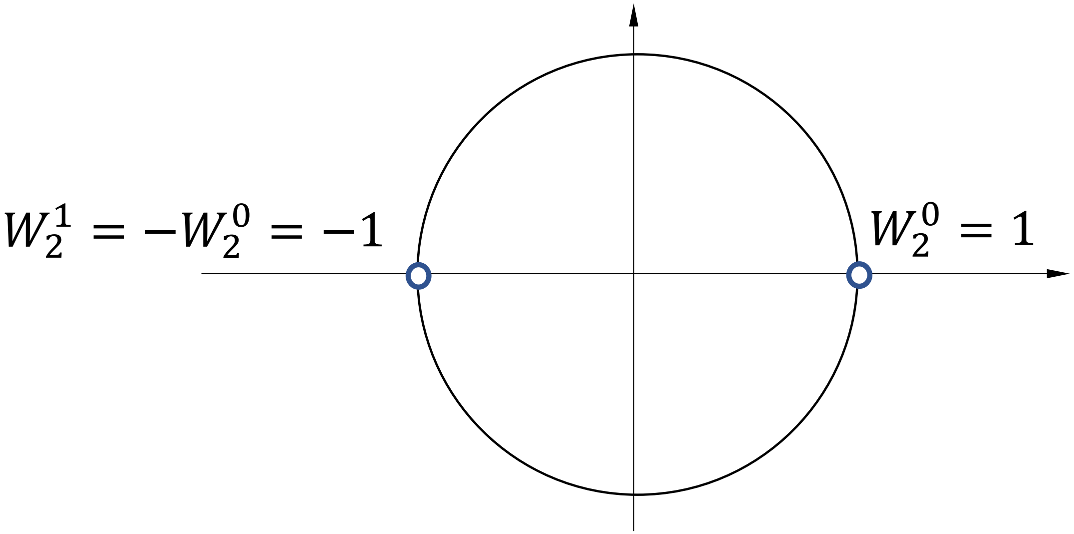
$$\begin{pmatrix} X[0] \\ X[1] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \end{pmatrix}$$

$$X[0] = x[0] + x[1]$$
$$X[1] = x[0] - x[1]$$




# 2nd roots of the unity



# 2-point inverse DFT

$$\begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \end{pmatrix}$$

$$x[0] = \frac{1}{2} \cdot (X[0] + X[1])$$

$$X[1] = \frac{1}{2} \cdot (X[0] - X[1])$$

# 4-point DFT

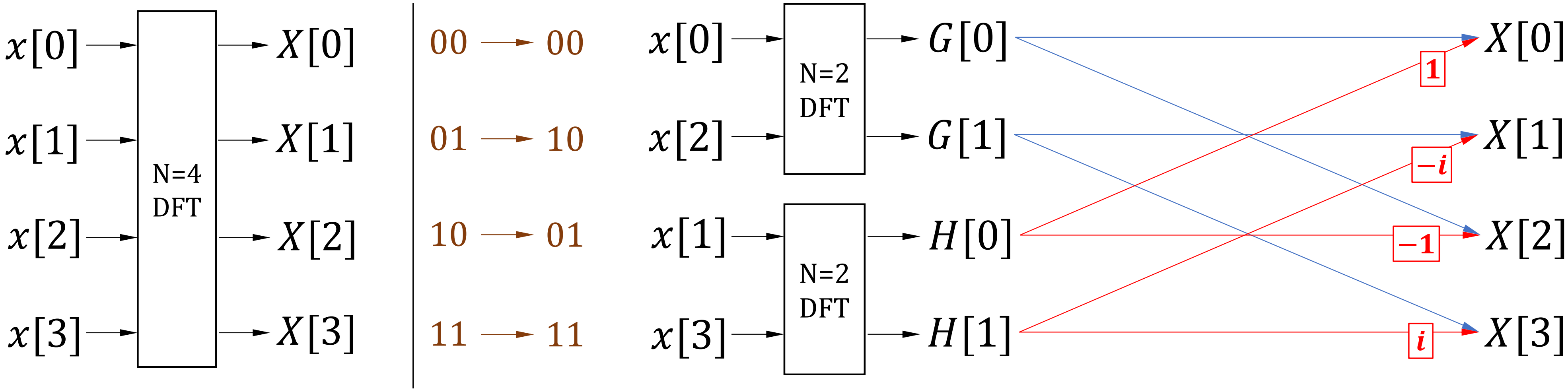
$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix}$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$



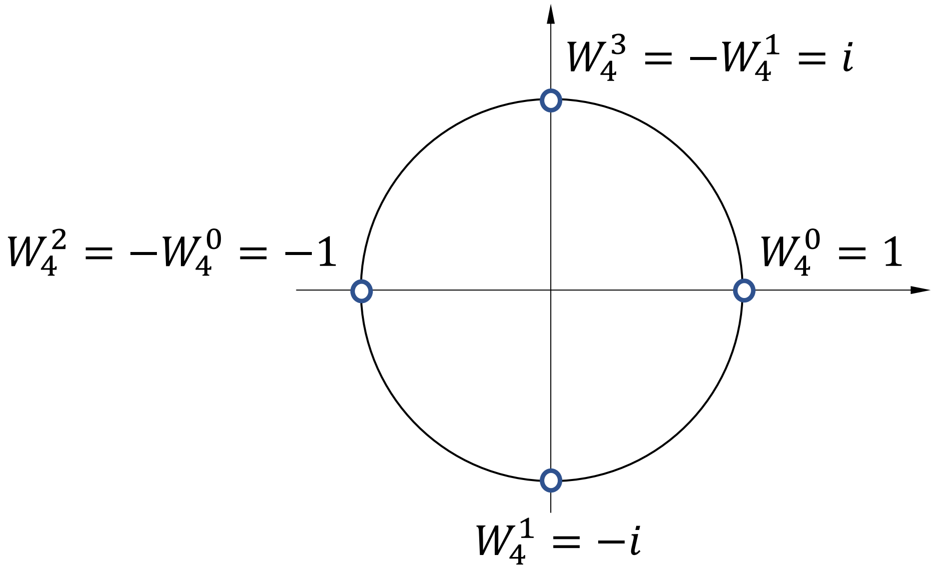

$$X[0] = G[0] + 1 \cdot H[0] = x[0] + x[2] + x[1] + x[3] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = G[1] - i \cdot H[1] = x[0] - x[2] - i \cdot (x[1] - x[3]) = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

$$X[2] = G[0] - 1 \cdot H[0] = x[0] + x[2] - (x[1] + x[3]) = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = G[1] - i \cdot H[1] = x[0] - x[2] + i \cdot (x[1] - x[3]) = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$

# 4th roots of the unity




# 4-point inverse DFT

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

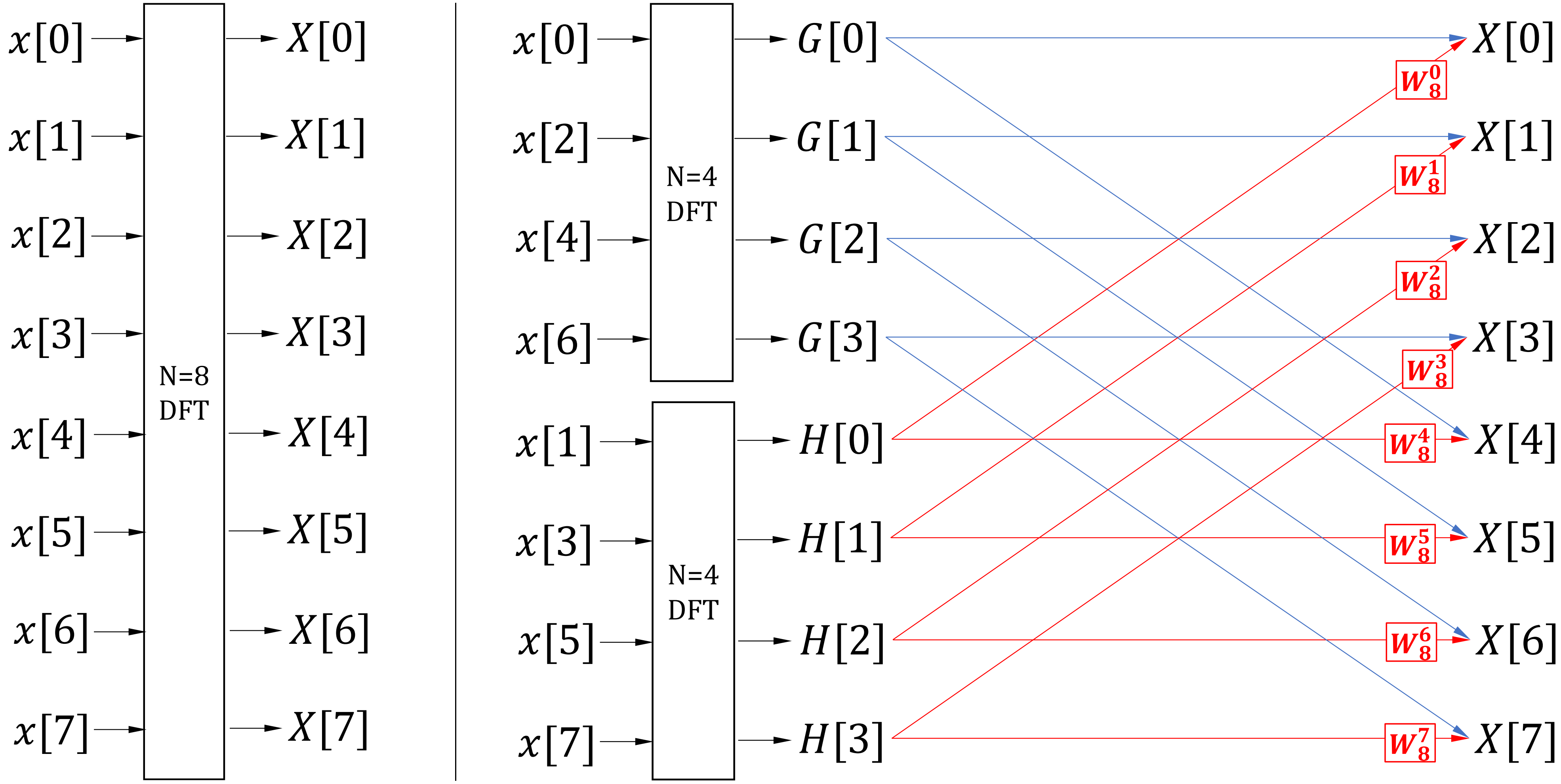
$$X[1] = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$

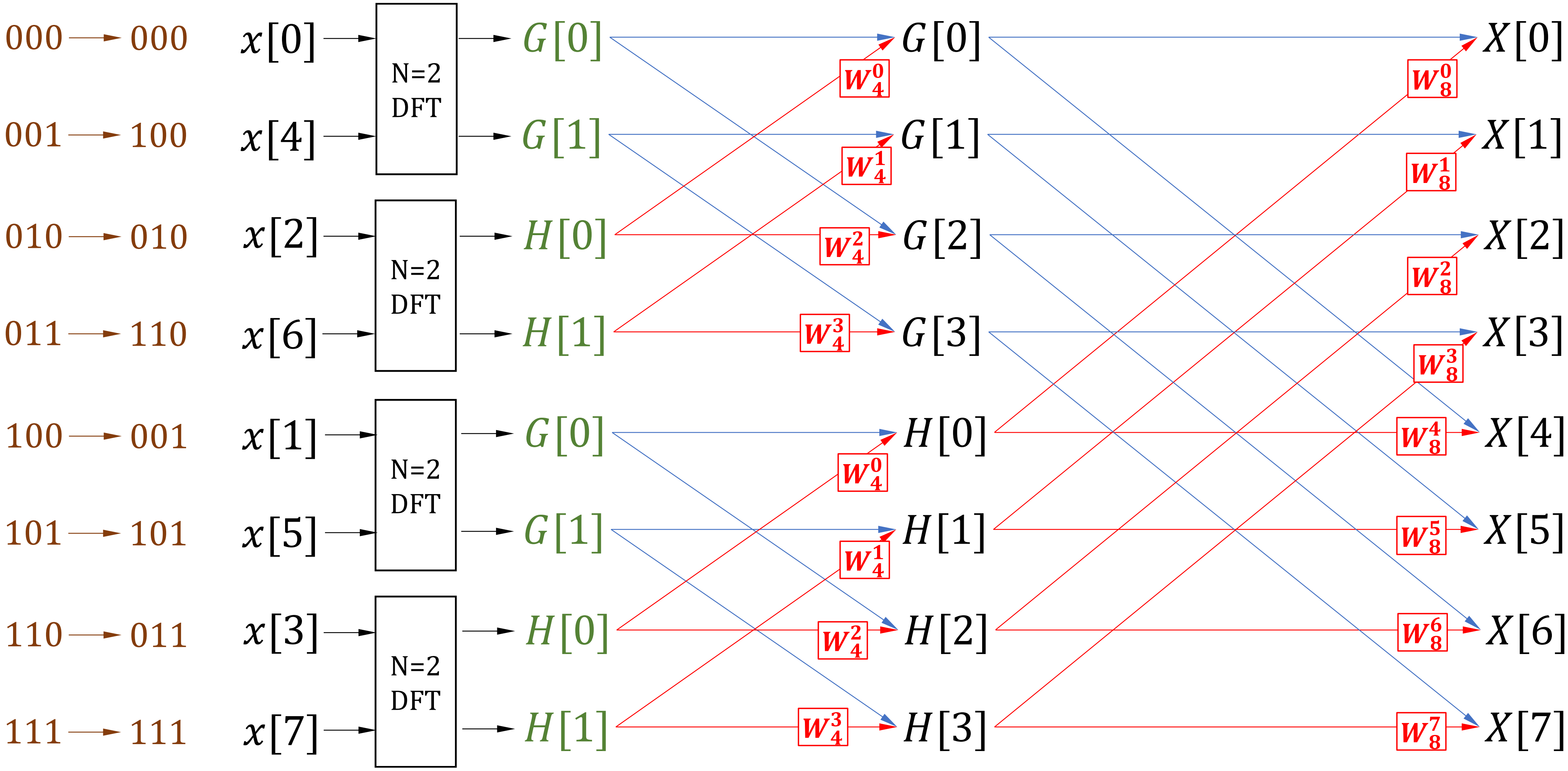
$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

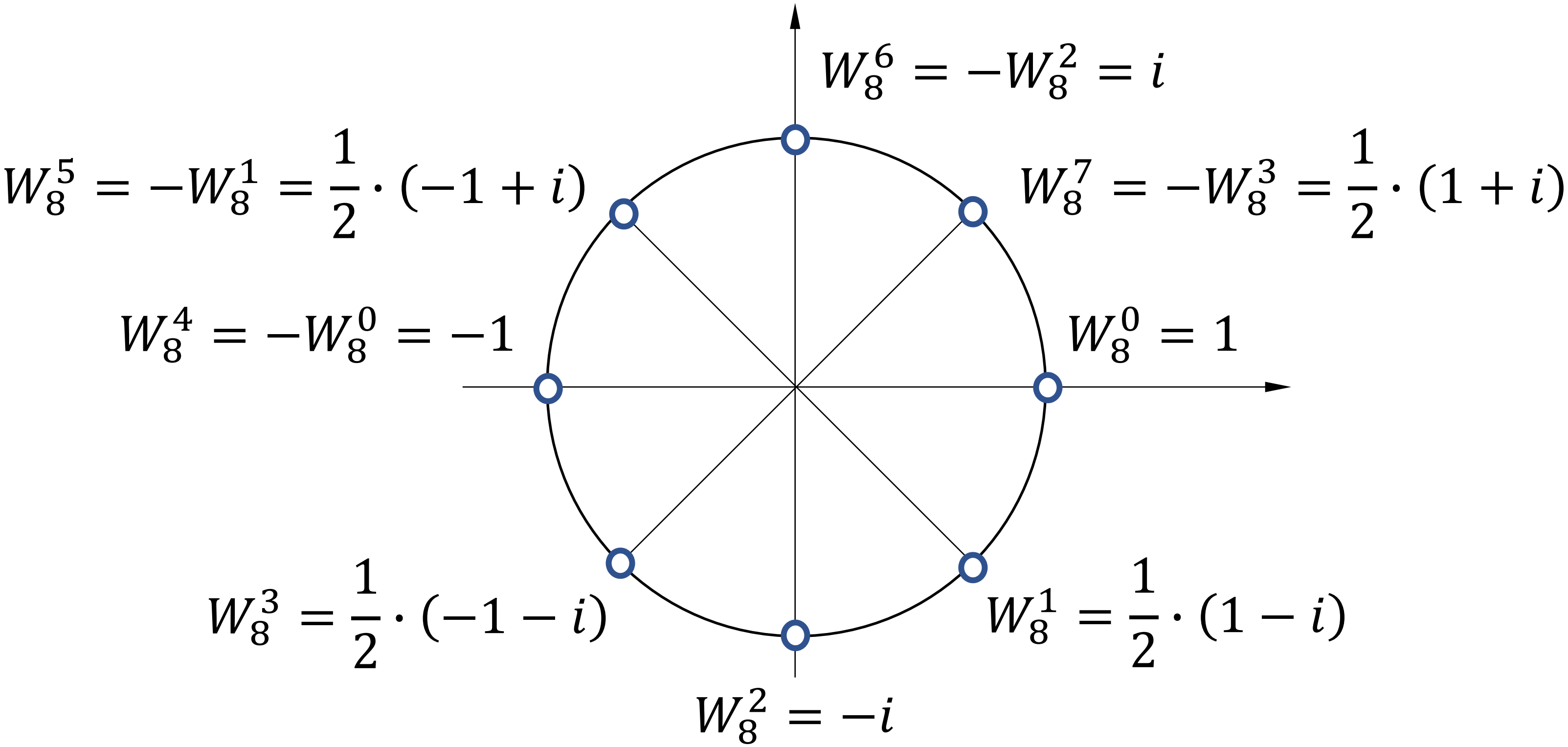
# 8-point DFT

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}}\cdot(1-i) & -i & \frac{1}{\sqrt{2}}\cdot(-1-i) & -1 & \frac{1}{\sqrt{2}}\cdot(-1+i) & i & \frac{1}{\sqrt{2}}\cdot(1+i) \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1}{\sqrt{2}}\cdot(-1-i) & i & \frac{1}{\sqrt{2}}\cdot(1-i) & -1 & \frac{1}{\sqrt{2}}\cdot(1+i) & -i & \frac{1}{\sqrt{2}}\cdot(-1+i) \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{1}{\sqrt{2}}\cdot(-1+i) & -i & \frac{1}{\sqrt{2}}\cdot(1+i) & -1 & \frac{1}{\sqrt{2}}\cdot(1-i) & i & \frac{1}{\sqrt{2}}\cdot(-1-i) \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1}{\sqrt{2}}\cdot(1+i) & i & \frac{1}{\sqrt{2}}\cdot(-1+i) & -1 & \frac{1}{\sqrt{2}}\cdot(-1-i) & -i & \frac{1}{\sqrt{2}}\cdot(1-i) \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$


x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
N=8
DFT
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
x[0]
x[2]
x[4]
x[6]
N=4
DFT
x[1]
x[3]
x[5]
x[7]
N=4
DFT
G[0]
G[1]
G[2]
G[3]
H[0]
H[1]
H[2]
H[3]
$W_8^0$
$W_8^1$
$W_8^2$
$W_8^3$
$W_8^4$
$W_8^5$
$W_8^6$
$W_8^7$
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]


000 → 000
001 → 100
010 → 010
011 → 110
100 → 001
101 → 101
110 → 011
111 → 111
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
N=2
DFT
G[0]
G[1]
H[0]
H[1]
G[2]
G[3]
H[2]
H[3]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
$W_4^0$
$W_4^1$
$W_4^2$
$W_4^3$
$W_8^0$
$W_8^1$
$W_8^2$
$W_8^3$
$W_8^4$
$W_8^5$
$W_8^6$
$W_8^7$

8th roots of the unity



# $6^{th}$ week – Questions

### *Question 1*

Give the formula of the continuous Fourier transform

***Answer***

$$\mathcal{FT}(x)(\omega) = \hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t} \, dt, \quad \omega \in \mathbb{R}$$

## *Question 2*

Give the formula of the discrete Fourier transform

## *Answer*

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}, \quad k = 0,1,\ldots,N-1$$

## Question 3

Give the transformation matrix of the 4-point DFT

## Answer

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

# 6th week – Exercises

### *Exercise*

Determine the convolution of functions $f: [0,\infty[\to \mathbb{R}$ and $g: [0,\infty[\to \mathbb{R}$ defined as

$$f(t) = e^{-2\cdot t}, \quad g(t) = e^{5\cdot t}.$$

### *Solution*

$$(f * g)(t) = e^{-2\cdot t} * e^{5\cdot t} = \int_0^t e^{-2\cdot\tau} \cdot e^{5\cdot(t-\tau)}\, d\tau = e^{5\cdot t} \cdot \int_0^t e^{-7\cdot\tau}\, d\tau =$$

$$= \frac{-1}{7} \cdot e^{5\cdot t} \cdot \left[e^{-7\cdot\tau}\right]_0^t = \frac{-1}{7} \cdot e^{5\cdot t} \cdot \left(e^{-7\cdot t} - 1\right) = \frac{-1}{7} \cdot e^{-2\cdot t} + \frac{1}{7} e^{5\cdot t}$$

## ***Exercise***

Determine the convolution of functions $x\colon [0,\infty[\to \mathbb{R}$ and $h\colon [0,\infty[\to \mathbb{R}$ defined as

$$x(t) = 5t - 3, \quad h(t) = e^{-\frac{1}{2}\cdot t}.$$

## ***Solution***

$$(x * h)(t) = (5t - 3) * e^{-\frac{1}{2}\cdot t} = \int_0^t (5\tau - 3) \cdot e^{-(t-\tau)}\, d\tau =$$

$$= \left[(5\tau - 8) \cdot e^{-(t-\tau)}\right]_{\tau=0}^{t} = 5t - 8 + 8 \cdot e^{-t}$$

*Details of the calculation (integration by parts):*

$$\int (5\tau - 3) \cdot e^{-(t-\tau)}\, d\tau = (5\tau - 3) \cdot e^{-(t-\tau)} - 5 \cdot \int e^{-(t-\tau)}\, dt =$$

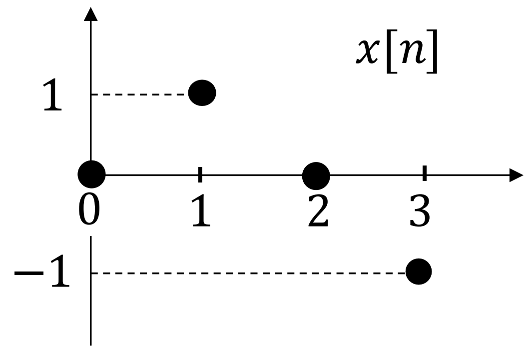
$$\begin{bmatrix} g(\tau) = & 5\tau - 3 & \Longrightarrow & g'(\tau) = & 5 \\ f'(\tau) = & e^{-(t-\tau)} & & f(\tau) = & e^{-(t-\tau)} \end{bmatrix}$$

$$= (5\tau - 3) \cdot e^{-(t-\tau)} - 5 \cdot e^{-(t-\tau)} = (5\tau - 8) \cdot e^{-(t-\tau)}$$

## *Exercise*

Determine the discrete Fourier transform of the sampled signal.

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x[n]$ | 0 | 1 | 0 | -1 |




Plot the complex numbers in the complex plane appearing in the sums.

## *Solution*

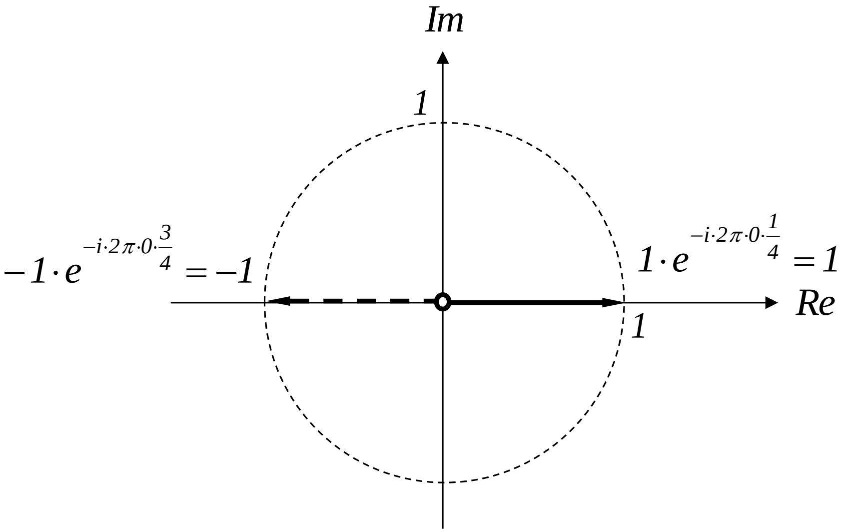
$$X[0] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 0 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] = 0 + 1 + 0 - 1 = 0$$

$$X[1] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 1 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \frac{\pi}{2}} =$$

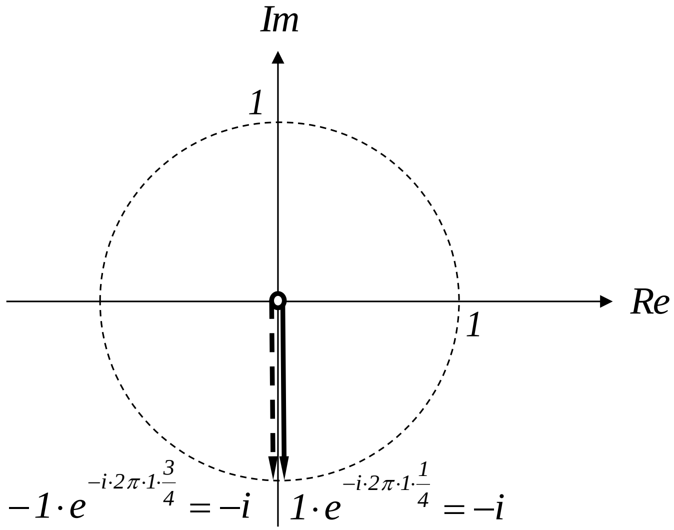
$$= 0 \cdot e^{-i \cdot 0 \cdot \frac{\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{\pi}{2}} = e^{-i \cdot \frac{\pi}{2}} - e^{-i \cdot 3 \cdot \frac{\pi}{2}} =$$

$$= \left(\cos\left(-\frac{\pi}{2}\right) + i \cdot \sin\left(-\frac{\pi}{2}\right)\right) - \left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right)\right) = 0 - i + 0 - i = -2 \cdot i$$

Values in the sum giving $X[0]$




Values in the sum giving $X[1]$




$$X[2] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 2 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \pi} =$$

$$= 0 \cdot e^{-i \cdot 0 \cdot \pi} + 1 \cdot e^{-i \cdot 1 \cdot \pi} + 0 \cdot e^{-i \cdot 2 \cdot \pi} - 1 \cdot e^{-i \cdot 3 \cdot \pi} = e^{-i \cdot \pi} - e^{-i \cdot 3 \cdot \pi} =$$

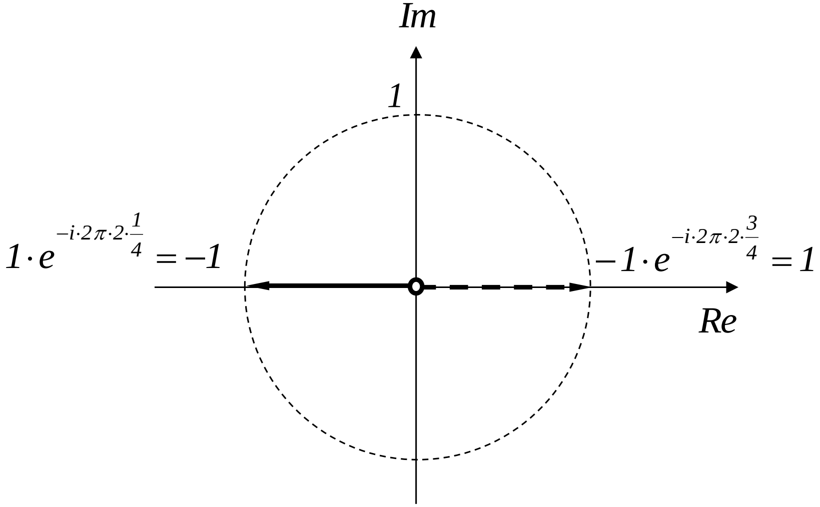
$$= \big(\cos(-\pi) + i \cdot \sin(-\pi)\big) - \big(\cos(-3\pi) + i \cdot \sin(-3\pi)\big) = 0 - 1 + 0 + 1 = 0$$

$$X[3] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 3 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \frac{3\pi}{2}} =$$

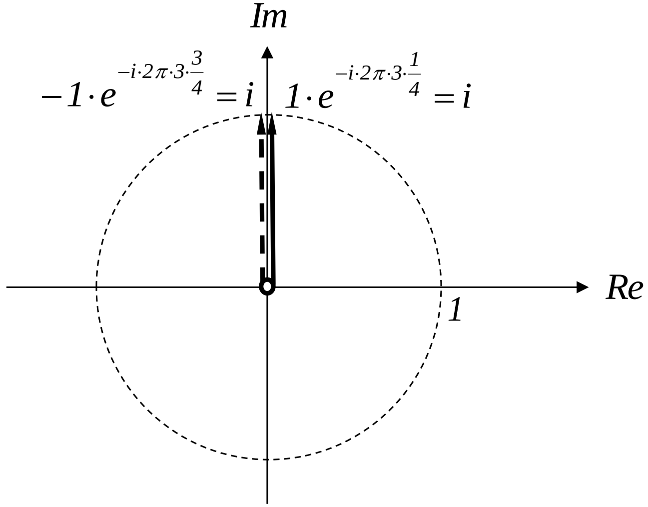
$$= 0 \cdot e^{-i \cdot 0 \cdot \frac{3\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{3\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{3\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{3\pi}{2}} = e^{-i \cdot \frac{3\pi}{2}} - e^{-i \cdot 3 \cdot \frac{3\pi}{2}} =$$

$$= \left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right)\right) - \left(\cos\left(-\frac{9\pi}{2}\right) + i \cdot \sin\left(-\frac{9\pi}{2}\right)\right) = 0 + i + 0 + i = 2 \cdot i$$

Values in the sum giving $X[2]$




Values in the sum giving $X[3]$




| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $X[k]$ | 0 | $-2 \cdot i$ | 0 | $2 \cdot i$ |
| $\lvert X[k] \rvert$ | 0 | 2 | 0 | 2 |

Calculation with the transformation matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \cdot i \\ 0 \\ 2 \cdot i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

## Exercise

Determine the discrete Fourier transform of the sampled signal

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x[n]$ | 8 | 4 | 8 | 0 |

using the transformation matrix.

## Solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 4 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ -4 \cdot i \\ 12 \\ 4 \cdot i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

## Exercise

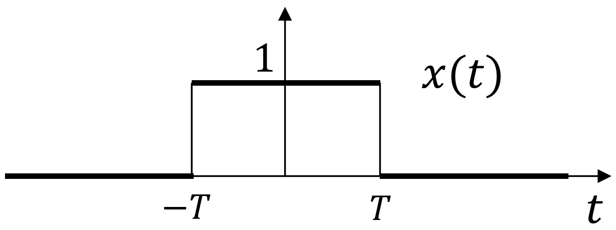
Determine the Fourier transform of the rectangular pulse function

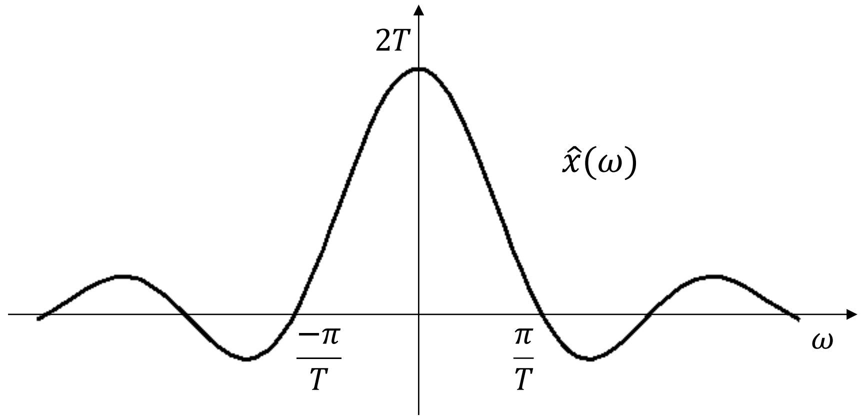
$$x(t) = \begin{cases} 1 & \text{if } |t| \le T \\ 0 & \text{if } |t| > T \end{cases}$$

## Solution

$$\hat{x}(\omega) = \int_{t=-T}^{T} e^{-i\cdot\omega\cdot t}\, dt = \frac{-1}{i\cdot\omega}\cdot\left[e^{-i\cdot\omega\cdot t}\right]_{t=-T}^{T} = \frac{-1}{i\cdot\omega}\cdot\left(e^{-i\cdot\omega\cdot T} - e^{i\cdot\omega\cdot T}\right) =$$

$$= 2\cdot\frac{1}{\omega}\cdot\frac{e^{i\cdot\omega\cdot T} - e^{-i\cdot\omega\cdot T}}{2i} = 2\cdot\frac{\sin(\omega\cdot T)}{\omega}$$



## Exercise

Determine the Fourier transform of the one-sided decaying exponential function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0$$

## Solution

$$\hat{x}(\omega) = \int_0^\infty e^{-a \cdot t} \cdot e^{-i \cdot \omega \cdot t}\, dt = \int_0^\infty e^{-(a+i \cdot \omega) \cdot t}\, dt = \lim_{b \to \infty} \cdot \int_0^b e^{-(a+i \cdot \omega) \cdot t}\, dt =$$

$$= \frac{-1}{a + i \cdot \omega} \cdot \lim_{b \to \infty} \left[ e^{-(a+i \cdot \omega) \cdot t} \right]_{t=0}^{b} = \frac{-1}{1 + i \cdot \omega} \cdot \lim_{b \to \infty} \left( e^{-(a+i \cdot \omega) \cdot b} - 1 \right) = \frac{1}{a + i \cdot \omega} = \frac{a - i \cdot \omega}{a^2 + \omega^2}$$

## Exercise

Determine the Fourier transform of the one-sided growing exponential function

$$x(t) = \begin{cases} 0 & if \quad t < 0 \\ e^{a \cdot t} & if \quad t \geq 0 \end{cases}, \quad a > 0$$

## Solution

The Fourier transform doesn’t exist.

## Exercise

Determine the Fourier transform of function

$$x(t) = e^{-a\cdot|t|}, \quad a > 0$$

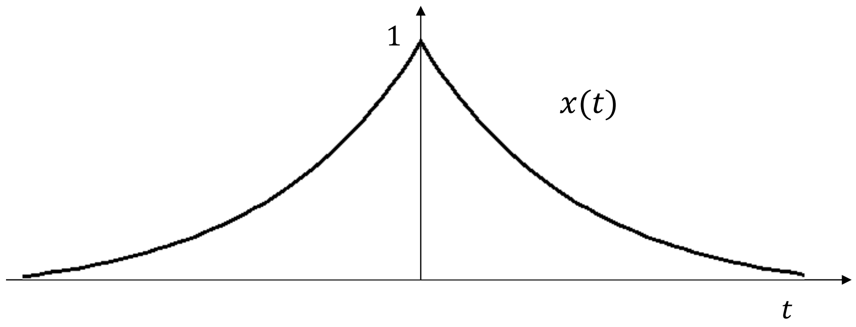
## Solution

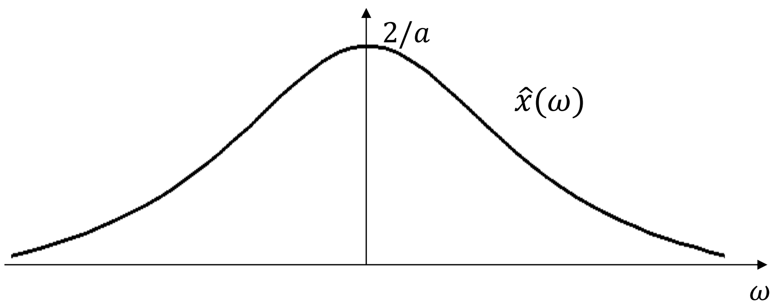
$$\hat{x}(\omega) = \int_{0}^{\infty} e^{-a\cdot|t|} \cdot e^{-i\cdot\omega\cdot t}\, dt = \int_{-\infty}^{0} e^{a\cdot t} \cdot e^{-i\cdot\omega\cdot t}\, dt + \int_{0}^{\infty} e^{-a\cdot t} \cdot e^{-i\cdot\omega\cdot t}\, dt =$$

$$= \lim_{b\to-\infty} \int_{b}^{0} e^{(a-i\cdot\omega)\cdot t}\, dt + \lim_{b\to\infty} \cdot \int_{0}^{b} e^{-(a+i\cdot\omega)\cdot t}\, dt =$$

$$= \frac{1}{a - i\cdot\omega} \cdot \lim_{b\to-\infty} \left[e^{(a-i\cdot\omega)\cdot t}\right]_{t=b}^{0} + \frac{-1}{a + i\cdot\omega} \cdot \lim_{b\to\infty} \left[e^{-(a+i\cdot\omega)\cdot t}\right]_{t=0}^{b} =$$

$$= \frac{1}{a - i\cdot\omega} \cdot \lim_{b\to\infty} \left(1 - e^{(a-i\cdot\omega)\cdot b}\right) - \frac{1}{a + i\cdot\omega} \cdot \lim_{b\to\infty} \left(1 - e^{-(a+i\cdot\omega)\cdot b}\right) = \frac{1}{a - i\cdot\omega} + \frac{1}{a + i\cdot\omega} = \frac{2a}{a^2 + \omega^2}$$



### *Exercise*

Determine symbolically the Fourier transform of constant function $x(t) = 1$ using the Fourier transform of the unit step function.

### *Solution*

The constant function $x(t) = 1$ is the sum of the unit step function and the negative unit step function, thus

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} e^{-i\cdot\omega\cdot t}\,dt = \int_{t=-\infty}^{0} e^{-i\cdot\omega\cdot t}\,dt + \int_{t=0}^{\infty} e^{-i\cdot\omega\cdot t}\,dt = 2\pi\cdot\delta(\omega)$$

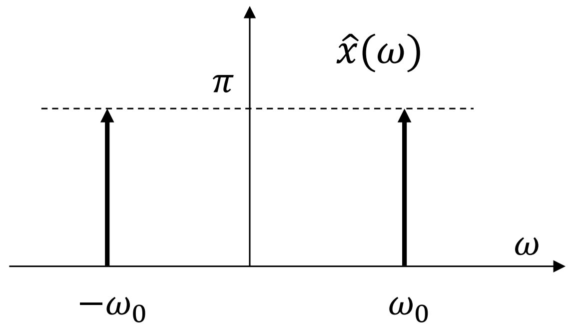
## *Exercise*

Determine symbolically the Fourier transform of periodic function $x(t) = \cos(\omega_0 \cdot t)$ using that $\int_{t=-\infty}^{\infty} e^{-i\cdot\omega\cdot t}\, dt = 2\pi \cdot \delta(\omega)$.

## *Solution*

$$\hat{x}(\omega) = \int\limits_{t=-\infty}^{\infty} \cos(\omega_0 \cdot t) \cdot e^{-i\cdot\omega\cdot t}\, dt = \frac{1}{2} \cdot \int\limits_{t=-\infty}^{\infty} \left(e^{i\cdot\omega_0\cdot t} + e^{-i\cdot\omega_0\cdot t}\right) \cdot e^{-i\cdot\omega\cdot t}\, dt =$$

$$= \frac{1}{2} \cdot \int\limits_{t=-\infty}^{\infty} \left(e^{-i\cdot(\omega-\omega_0)\cdot t} + e^{-i\cdot(\omega+\omega_0)\cdot t}\right) dt = \pi \cdot \delta(\omega - \omega_0) + \pi \cdot \delta(\omega + \omega_0)$$



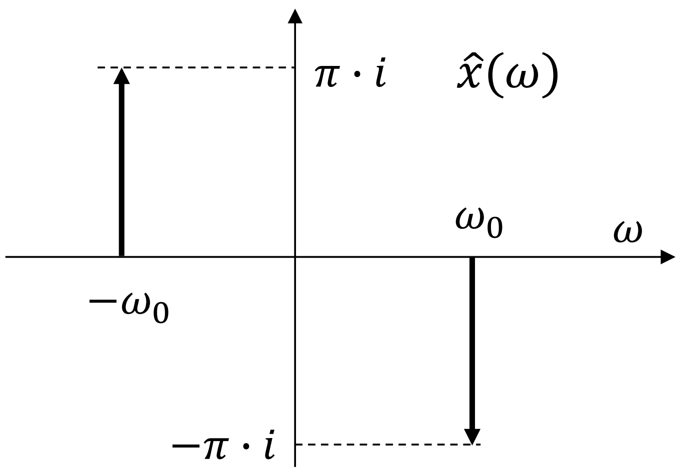
## Exercise

Determine symbolically the Fourier transform of periodic function $x(t) = \sin(\omega_0 \cdot t)$ using that $\int_{t=-\infty}^{\infty} e^{-i\cdot\omega\cdot t}\,dt = 2\pi \cdot \delta(\omega)$.

## Solution

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} \sin(\omega_0 \cdot t) \cdot e^{-i\cdot\omega\cdot t}\,dt = \frac{1}{2i} \cdot \int_{t=-\infty}^{\infty} \left(e^{i\cdot\omega_0\cdot t} - e^{-i\cdot\omega_0\cdot t}\right) \cdot e^{-i\cdot\omega\cdot t}\,dt =$$

$$= \frac{1}{2i} \cdot \int_{t=-\infty}^{\infty} \left(e^{-i\cdot(\omega-\omega_0)\cdot t} - e^{-i\cdot(\omega+\omega_0)\cdot t}\right) dt = -i \cdot \pi \cdot \delta(\omega - \omega_0) + i \cdot \pi \cdot \delta(\omega + \omega_0)$$

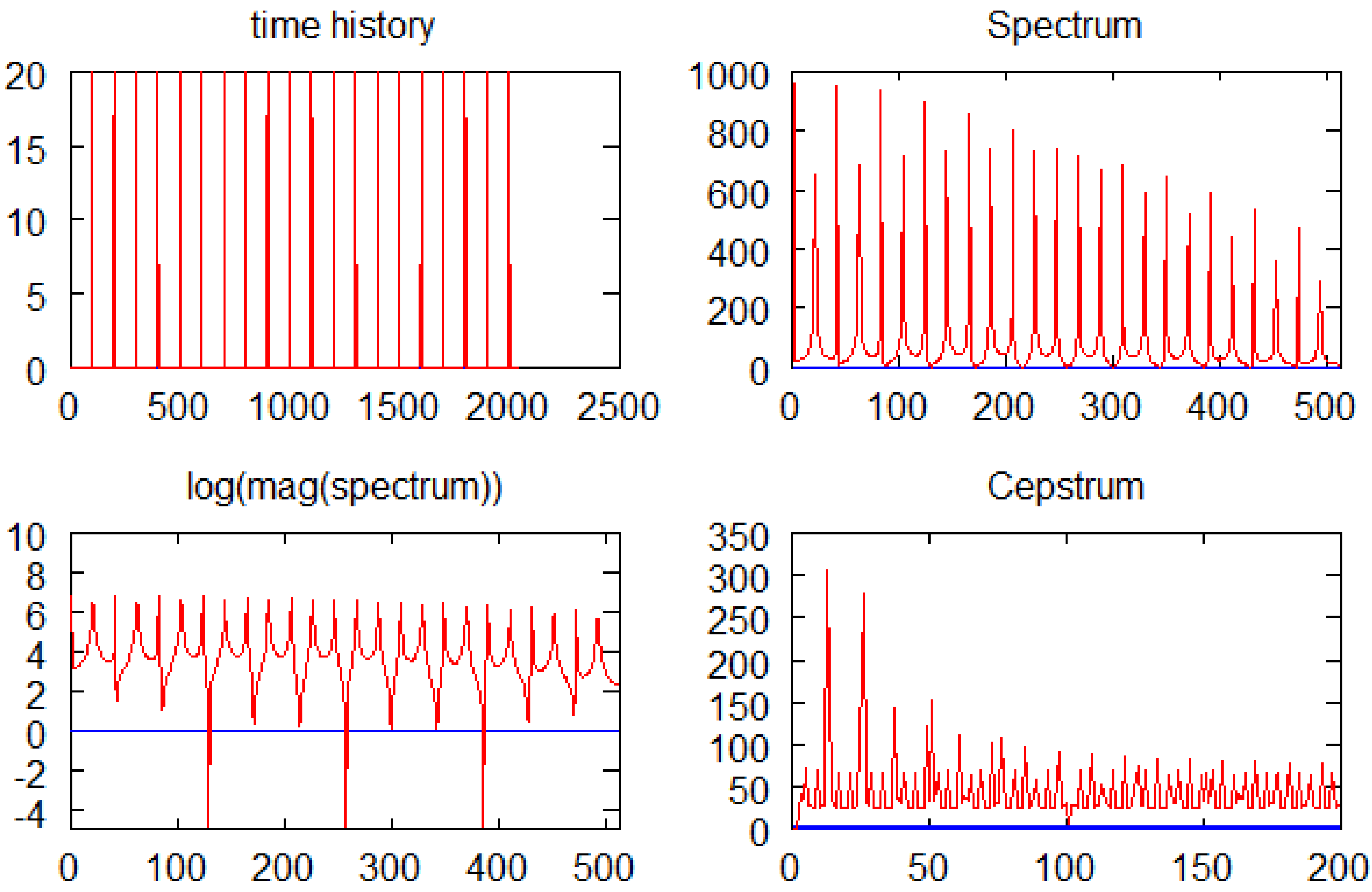


# 7th week

# 7 Cepstrum Analysis, Envelope Analysis

In Fourier analysis, the cepstrum is the result of computing the inverse Fourier transform (IFT) of the logarithm of the estimated signal spectrum. The method is a tool for investigating periodic structures in frequency spectra. The power cepstrum has applications in the analysis of human speech.

The term cepstrum was derived by reversing the first four letters of spectrum. Operations on cepstra are labelled quefrency analysis (or quefrency analysis), liftering, or cepstral analysis.


time history
20
15
10
5
0
0
500
1000
1500
2000
2500
Spectrum
1000
800
600
400
200
0
0
100
200
300
400
500
log(mag(spectrum))
10
8
6
4
2
0
-2
-4
0
100
200
300
400
500
Cepstrum
350
300
250
200
150
100
50
0
0
50
100
150
200

[6]

The cepstrum is the result of following sequence of mathematical operations:

transformation of a signal from the time domain to the frequency domain, computation of the logarithm of the spectral amplitude, transformation to quefrency domain, where the final independent variable, the quefrency, has a time scale.

The cepstrum is used in many variants. Most important are:

Power cepstrum: The logarithm is taken from the "power spectrum"

Complex cepstrum: The logarithm is taken from the spectrum, which is calculated via Fourier analysis .

The "cepstrum" was originally defined as power cepstrum by the following relationship

$$C_p = |F^{-1}\{log(|F\{f(t)\}|^2)\}|^2$$

The power cepstrum has main applications in analysis of sound and vibration signals. It is a complementary tool to spectral analysis

$$C_p = |F\ \{log(|F\{f(t)\}|^2)\}|^2$$

Due to this formula, the cepstrum is also sometimes called the spectrum of a spectrum. It can be shown that both formulas are consistent with each other as the frequency spectral distribution remains the same, the only difference being a scaling factor which can be applied afterwards. Some articles prefer the second formula.

Other notations are possible due to the fact, that the log of the power spectrum is equal to the log of the spectrum, if a scaling factor 2 is applied

$$log|F|^2 = 2\ log|F|$$

$$C_p = |F^{-1}\{2\ log|F|\}|^2 \quad , \quad or$$

$$C_p = 4 \cdot |F^{-1}\{log|F|\}|^2 \ ,$$

which provides a relationship to the real cepstrum.

The real cepstrum is directly related to the power cepstrum:

$$C_p = 4 \cdot C_r^2$$

It is derived from the complex cepstrum (defined below) by discarding the phase information (contained in the imaginary part of the complex logarithm).[5] It has a focus on periodic effects in the amplitudes of the spectrum:

$$C_r = F^{-1}\{log(|F\{f(t)\}|)\}$$

The independent variable of a cepstral graph is called the quefrency.[11] The quefrency is a measure of time, though not in the sense of a signal in the time domain. For example, if the sampling rate of an audio signal is 44100 Hz and there is a large peak in the cepstrum whose quefrency is 100 samples, the peak indicates the presence of a fundamental frequency that is 44100/100 = 441 Hz. This peak occurs in the cepstrum because the harmonics in the spectrum are periodic and the period corresponds to the fundamental frequency, since harmonics are integer multiples of the fundamental frequency.

The kepstrum, which stands for "Kolmogorov-equation power-series time response", is similar to the cepstrum and has the same relation to it as expected value has to statistical average, i.e. cepstrum is the empirically measured quantity, while kepstrum is the theoretical quantity. It was in use before the cepstrum.

The autocepstrum is defined as the cepstrum of the autocorrelation. The autocepstrum is more accurate than the cepstrum in the analysis of data with echoes.

Playing further on the anagram theme, a filter that operates on a cepstrum might be called a lifter. A low-pass lifter is similar to a low-pass filter in the frequency domain. It can be implemented by multiplying by a window in the quefrency domain and then converting back to the frequency domain, resulting in a modified signal, i.e. with signal echo being reduced.

The cepstrum can be seen as information about the rate of change in the different spectrum bands. It was originally invented for characterizing the seismic echoes resulting from earthquakes and bomb explosions. It has also been used to determine the fundamental frequency of human speech and to analyze radar signal returns. Cepstrum pitch determination is particularly effective because the effects of the vocal excitation (pitch) and vocal tract (formants) are additive in the logarithm of the power spectrum and thus clearly separate.

**Applications:**

The concept of the cepstrum has led to numerous applications:

dealing with reflection inference (radar, sonar applications, earth seismology)

estimation of speaker fundamental frequency (pitch)

speech analysis and recognition

medical applications in analysis of electroencephalogram (EEG) and brain waves

machine vibration analysis based on harmonic patterns (gearbox faults, turbine blade failures.

Recently cepstrum based deconvolution was used to remove the effect of the stochastic impulse trains, which originates an sEMG signal, from the power spectrum of sEMG signal itself. In this way, only information on motor unit action potential (MUAP) shape and amplitude were maintained, and then, used to estimate the parameters of a time-domain model of the MUAP itself.

A short-time cepstrum analysis was proposed by Schroeder and Noll for application to pitch determination of human speech.

## Envelope Analysis

Envelope Detection or Amplitude

Demodulation is the technique of extracting the modulating signal from an amplitude-modulated signal. The result is the time history of the modulating signal. This signal may be studied/interpreted as it is in the time domain or it may be subjected to a subsequent frequency analysis. Envelope Analysis is the FFT (Fast Fourier Transform) frequency spectrum of the modulating signal.

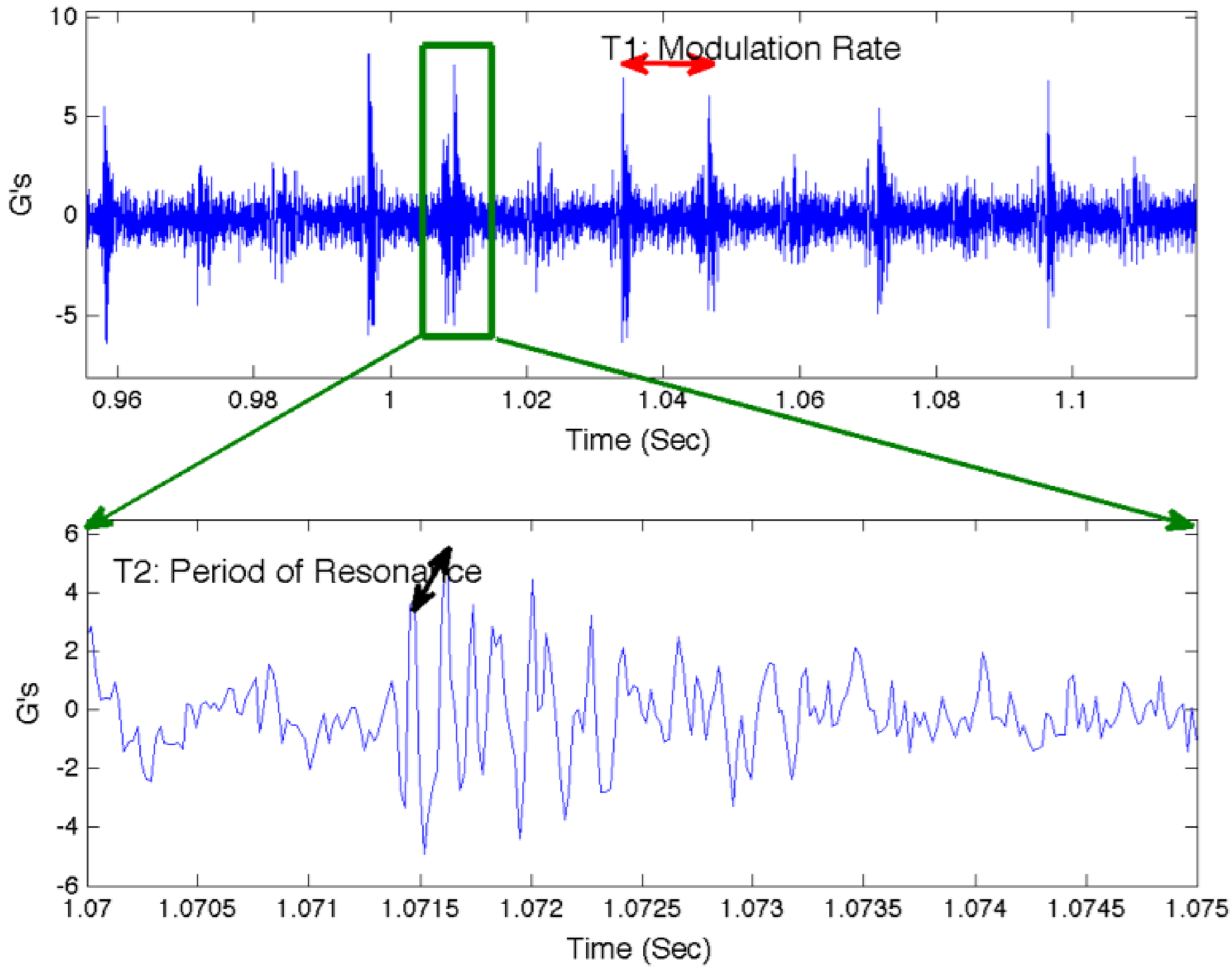
Envelope Analysis can be used for diagnostics/investigation of machinery where faults have an amplitude modulating effect on the characteristic frequencies of the machinery. Examples include faults in gearboxes, turbines and induction motors. Envelope Analysis is also an excellent tool for diagnostics of local faults like cracks and spallings in Rolling Element Bearings (REB).

Because the vibration signals of a faulty bearing are small compared to shaft order and gear mesh frequency, detection of faults at the bearing rate frequencies using Fourier analysis is difficult. Fault detection at the baseband frequencies of the bearing rate is “Stage 1” fault detection. Bearing faults detected using these types of analysis are Late Stage; the bearing can be close to catastrophic failure. At the very least, a bearing in this state is generating metal which can cause damage to other components within the gearbox.

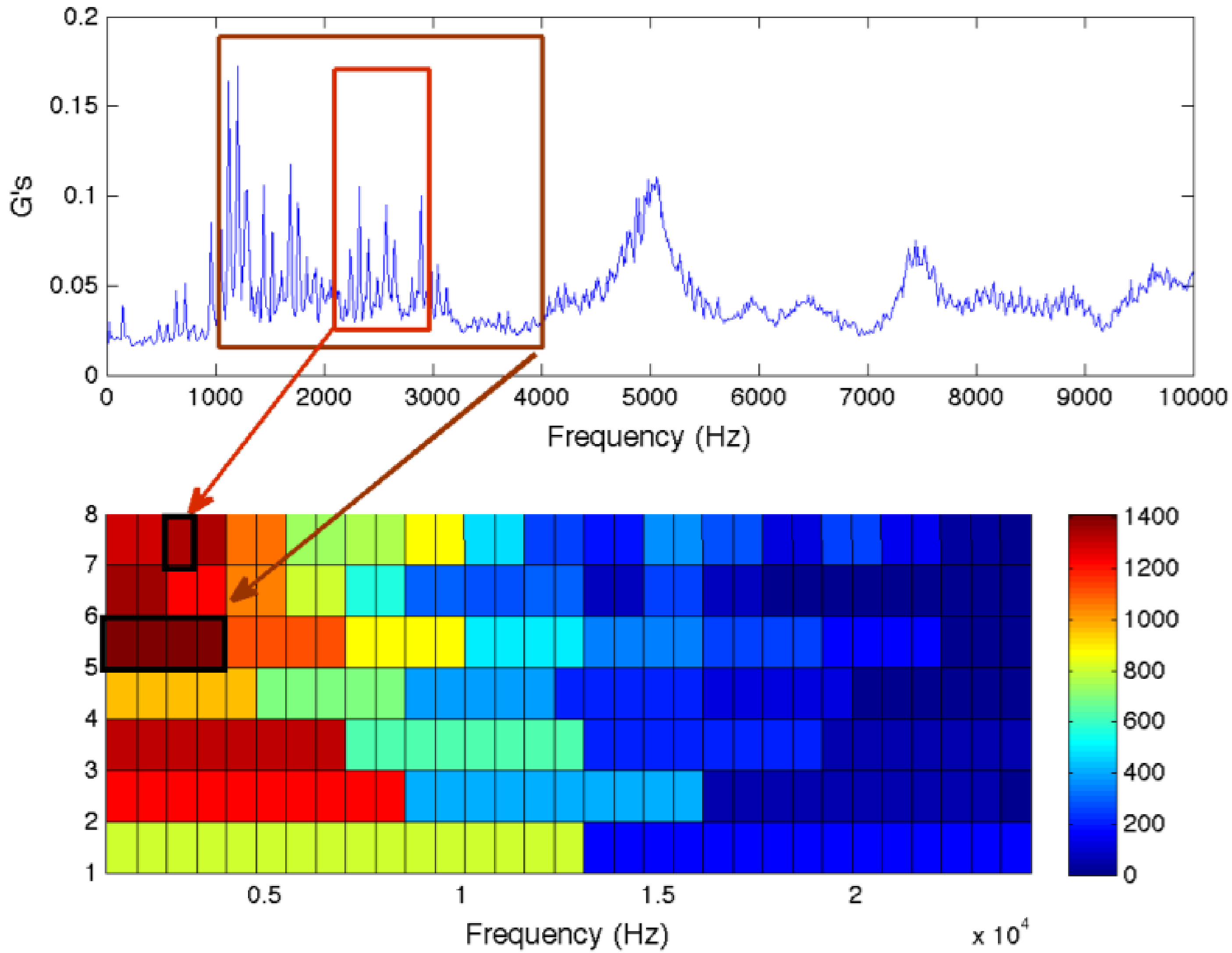
Ultrasonic emission can detect bearing inner and outer race roughness (a “Stage 3” fault), but the remaining useful life of a bearing at this stage is relatively long compared to the overall life of the bearing. Bearing envelope analysis (BEA) can typically detect bearing faults 100s if not 1000s of hours prior to when it is appropriate to do maintenance. It is for this reason that many condition monitoring systems manufacturers are using envelope analysis techniques.

THE BEARING ENVELOPE ANALYSIS: BEA is based on demodulation of high frequency resonance associated with bearing element impacts. For rolling element bearings, when the rolling elements strike a local fault on the inner or outer race, or a fault on a rolling element strikes the inner or outer race, an impact is produced. These impacts modulate a signal at the associated bearing pass frequencies, such as: Cage Pass Frequency (CPF), Ball Pass Frequency Outer Race (BPFO), Ball Pass Frequency Inner Race (BPFI), and Ball Fault Frequency (BFF). Figure (1) is an Outer Race Fault, where the BPFO is approximately 80

Hz. Note that the modulation rate, T1, is approximately 0.125 seconds (e.g. 1/80 Hz). The time T2, the period of the resonance, is approximately 1.12e-4 seconds, or about 9000 Hz.




[7]


0.2
0.15
0.1
0.05
0
G's
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
Frequency (Hz)
8
7
6
5
4
3
2
1
0.5
1
1.5
2
Frequency (Hz)
x 10^4
1400
1200
1000
800
600
400
200
0

[7]

The faults cause certain fault frequencies determined by the bearing geometry and the rotational speed of the shaft. Basically, four types of them are distinguished: bearing pass frequency of outer race (BPFO), bearing pass frequency of inner race (BPFI), fundamental train frequency (FTF), ball spin frequency (BSF) which can be calculated by numerical way.

$$BPFO = \frac{n \cdot f_r}{2}\left\{1 - \frac{d}{D} cos\Phi\right\}$$

$$BPFI = \frac{n \cdot f_r}{2}\left\{1 + \frac{d}{D} cos\Phi\right\}$$

$$FTF = \frac{f_r}{2}\left\{1 - \frac{d}{D} cos\Phi\right\}$$

$$BSF = \frac{D \cdot f_r}{2d}\left\{1 - \left(\frac{d}{D} cos\Phi\right)^2\right\}$$

where $f_r$ is the speed of the shaft, n is the number of rolling elements, φ is the contact angle, d is the ball diameter, D is the pitch diameter.

# 7th week – Questions

### *Question*

What is the cepstrum of a digital signal and what are the main fields of application (3 examples)?

### *Answer*

Cepstrum is the result of following sequence of mathematical operations:

transformation of a signal from the time domain to the frequency domain, computation of the logarithm of the spectral amplitude, transformation to quefrency domain, where the final independent variable, the quefrency, has a time scale.

Power cepstrum: The logarithm is taken from the "power spectrum"

Complex cepstrum: The logarithm is taken from the spectrum, which is calculated via Fourier analysis .

The "cepstrum" was originally defined as power cepstrum by the following relationship

$$C_p = |F^{-1}\{log(|F\{f(t)\}|^2)\}|^2$$

The power cepstrum has main applications in analysis of sound and vibration signals. It is a complementary tool to spectral analysis

$$C_p = |F\ \{log(|F\{f(t)\}|^2)\}|^2$$

Applications:

- Speech analysis and recognition
- Medical applications in analysis of electroencephalogram (EEG) and brain waves
- Machine vibration analysis based on harmonic patterns (gearbox faults, turbine blade failures.

### *Question*

What is the bearing envelope analysis and its role in machine fault diagnosis?

### *Answer*

Bearing envelope analysis (BEA) can typically detect bearing faults 100s if not 1000s of hours prior to when it is appropriate to do maintenance. It is for this reason that many condition monitoring systems manufacturers are using envelope analysis techniques.

THE BEARING ENVELOPE ANALYSIS: BEA is based on demodulation of high frequency resonance associated with bearing element impacts. For rolling element bearings, when the rolling elements strike a local fault on the inner or outer race, or a fault on a rolling element strikes the inner or outer race, an impact is produced. These impacts modulate a signal at the associated bearing pass frequencies, such as: Cage Pass Frequency (CPF), Ball Pass Frequency Outer Race (BPFO), Ball Pass Frequency Inner Race (BPFI), and Ball Fault Frequency (BFF).

## *Question*

What are the bearing fault frequencies and what is the connection between these frequencies and the cepstrum analysis?

## *Answer*

Bearing fault frequencies determined by the bearing geometry and the rotational speed of the shaft. Basically, four types of them are distinguished: bearing pass frequency of outer race (BPFO), bearing pass frequency of inner race (BPFI), fundamental train frequency (FTF), ball spin frequency (BSF) which can be calculated by numerical way.

$$BPFO = \frac{n \cdot f_r}{2}\left\{1 - \frac{d}{D} cos\Phi\right\}$$

$$BPFI = \frac{n \cdot f_r}{2}\left\{1 + \frac{d}{D} cos\Phi\right\}$$

$$FTF = \frac{f_r}{2}\left\{1 - \frac{d}{D} cos\Phi\right\}$$

$$BSF = \frac{D \cdot f_r}{2d}\left\{1 - \left(\frac{d}{D} cos\Phi\right)^2\right\}$$

where $f_r$ is the speed of the shaft, n is the number of rolling elements, φ is the contact angle, d is the ball diameter, D is the pitch diameter.

Using the cepstrum analysis, fault frequencies are identified in the spectrum and the engineer or the technical operator is able to determine the actual faults of the machine.

# 7th week – Exercises

### *Exercise*

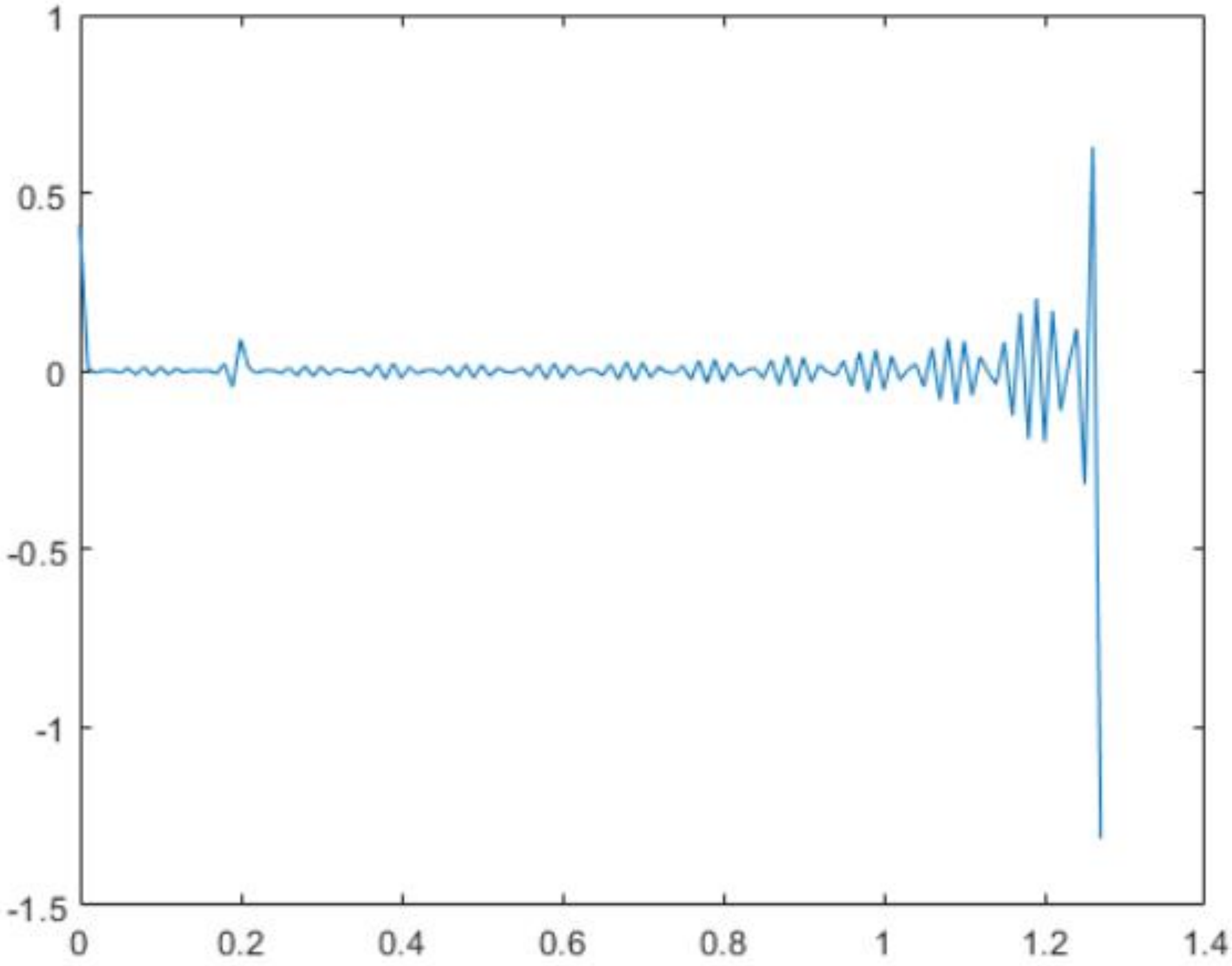
Use the echo detection application in MATLAB. Create a 45 Hz sine wave sampled at 100 Hz. Add an echo of the signal, with half the amplitude, 0.2 seconds after the beginning of the signal.

Compute and plot the complex cepstrum of the new signal.

### *Solution*

```
t = 0:0.01:1.27;

s1 = sin(2*pi*45*t);

s2 = s1 + 0.5*[zeros(1,20) s1(1:108)];


c = cceps(s2);
plot(t,c)
```

The complex cepstrum shows a peak at 0.2 seconds, indicating the echo.


1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4

***Exercise***

Load the speech signal. The recording is of a woman saying "MATLAB". The sampling frequency is 7418 Hz.

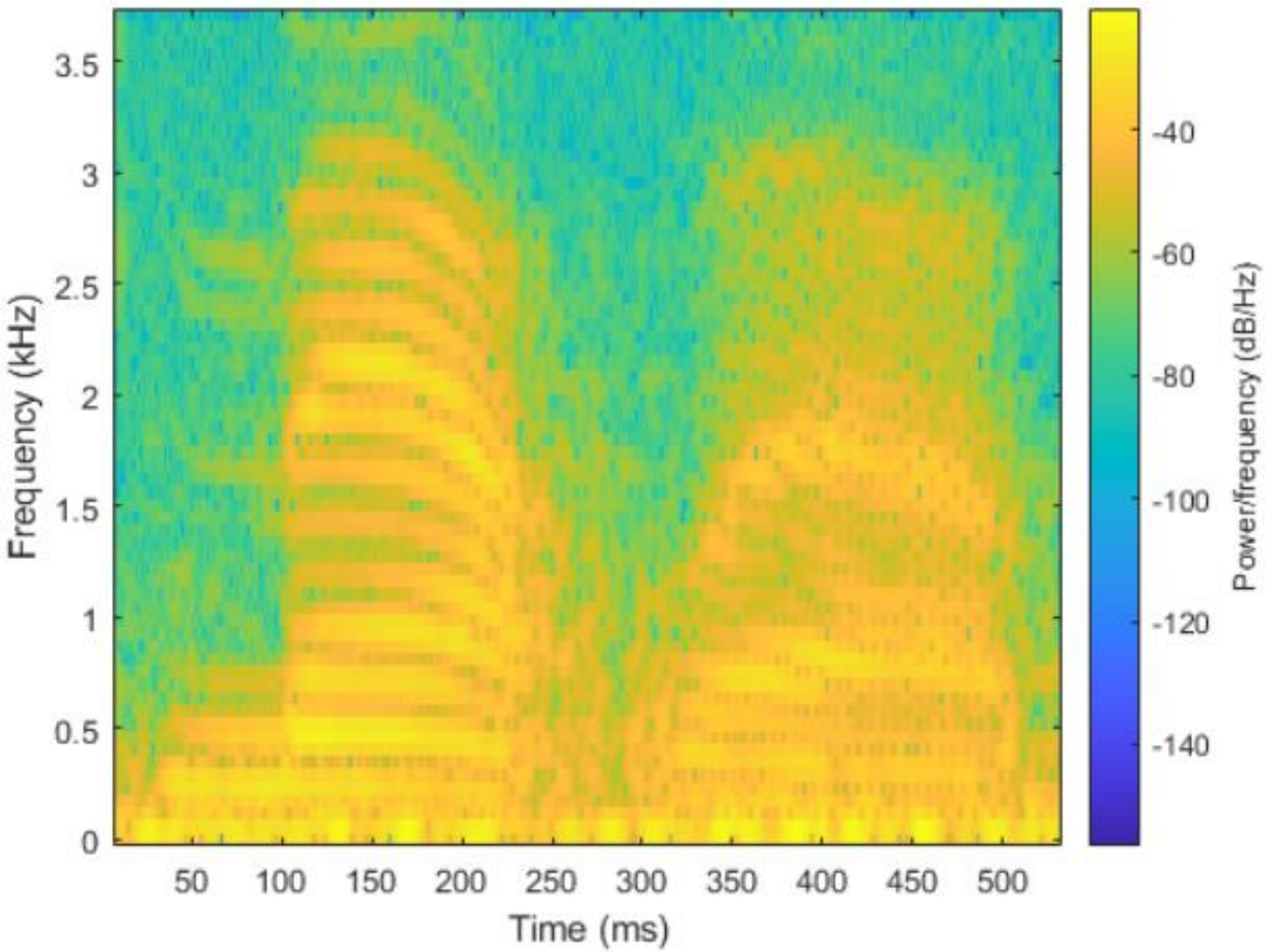
Extract the segment from 0.1 to 0.25 seconds for analysis. Plot the cepstrum in the selected time range and overlay the peak.

***Solution***

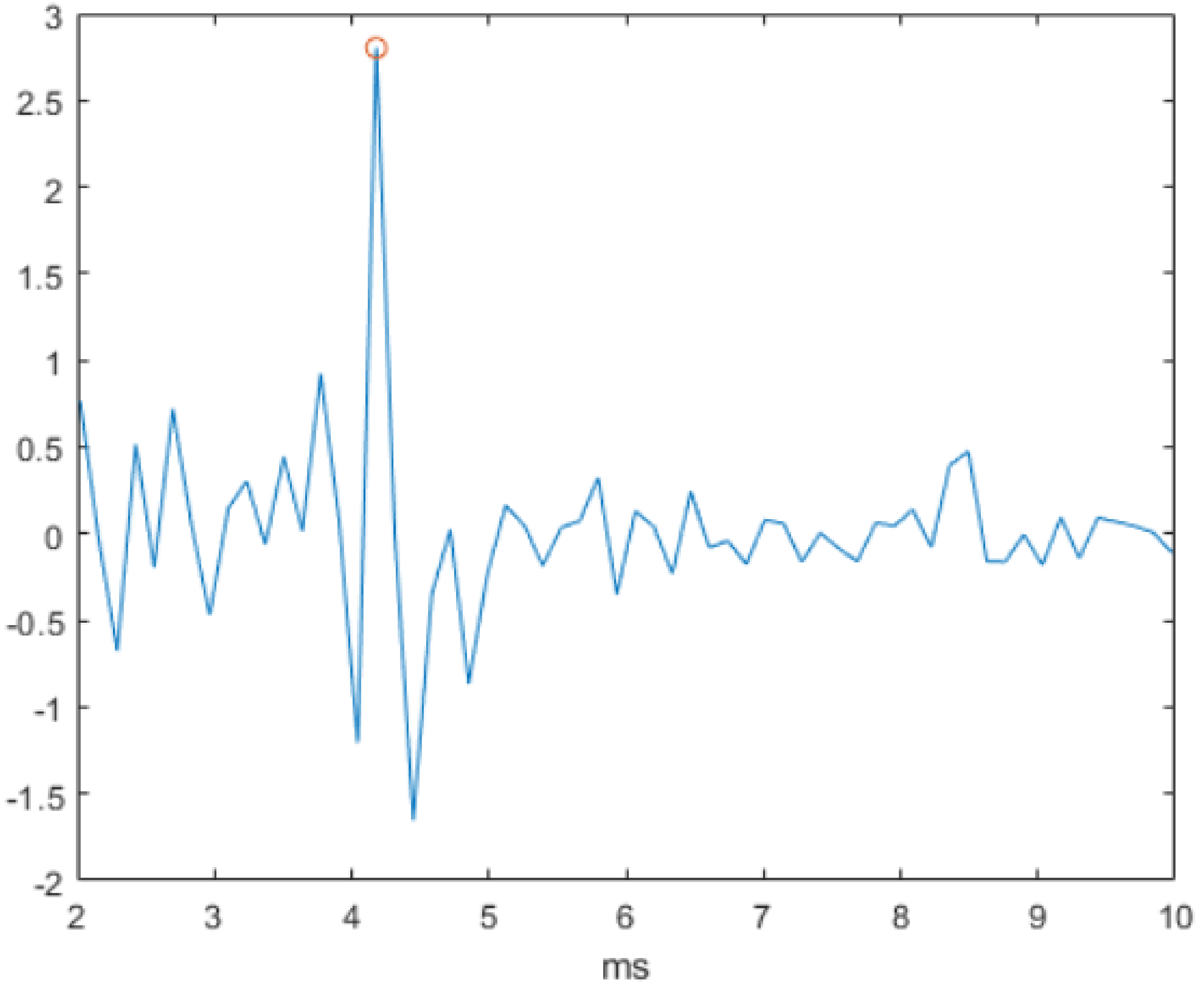
```
load mtlb
```

Use the spectrogram to identify a voiced segment for analysis

```
segmentlen = 100;
noverlap = 90;
NFFT = 128;

spectrogram(mtlb,segmentlen,noverlap,NFFT,Fs,'yaxis')
```


3.5
3
2.5
2
1.5
1
0.5
0
Frequency (kHz)
50
100
150
200
250
300
350
400
450
500
Time (ms)
-40
-60
-80
-100
-120
-140
Power/frequency (dB/Hz)

```
c = cceps(x);
t = 0:dt:length(x)*dt-dt;
trng = t(t>=2e-3 & t<=10e-3);
crng = c(t>=2e-3 & t<=10e-3);
[~,I] = max(crng);
fprintf('Complex cepstrum F0 estimate is %3.2f Hz.\n',1/trng(I))
plot(trng*1e3,crng)
xlabel('ms')
hold on
plot(trng(I)*1e3,crng(I),'o')
hold off
```


3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2
3
4
5
6
7
8
9
10
ms

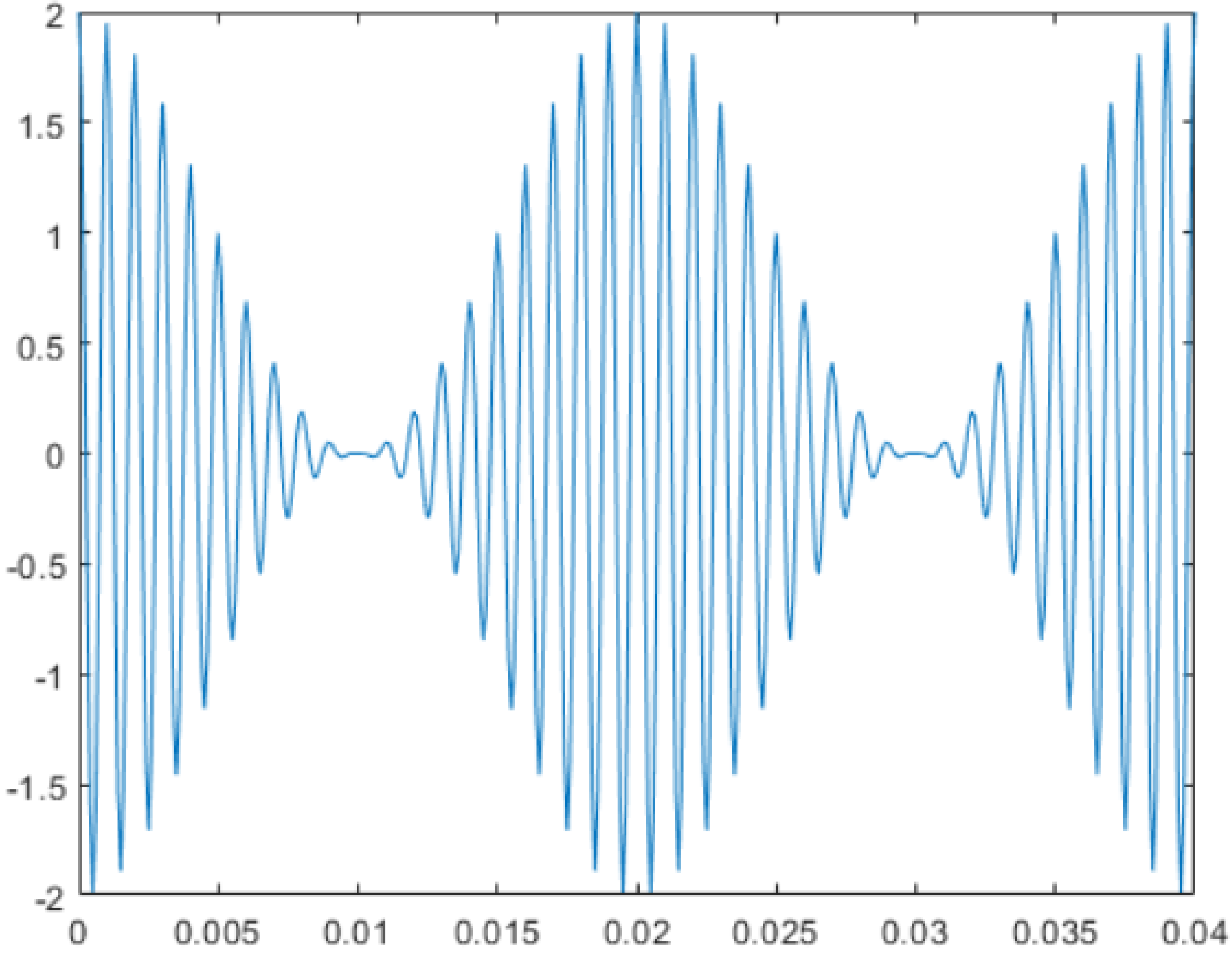
### *Exercise*

Create a double sideband amplitude-modulated signal. The carrier frequency is 1 kHz. The modulation frequency is 50 Hz. The modulation depth is 100%. The sample rate is 10 kHz.

Extract the envelope using the hilbert function. Plot the envelope along with the original signal.

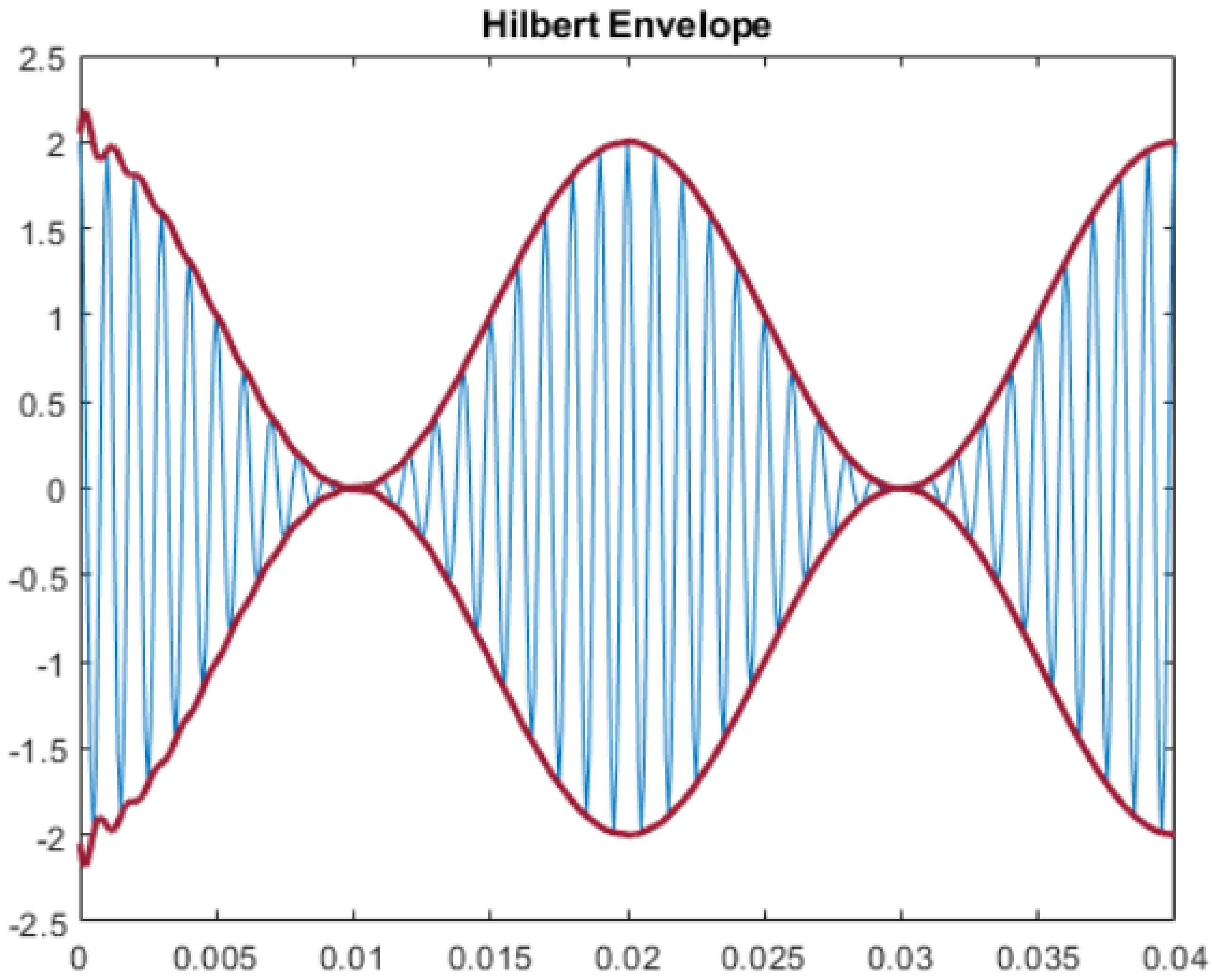
### *Solution*

```
t = 0:1e-4:0.1;
x = (1+cos(2*pi*50*t)).*cos(2*pi*1000*t);


plot(t,x)
xlim([0 0.04])
```




```
y = hilbert(x);
env = abs(y);
plot_param = {'Color', [0.6 0.1 0.2],'Linewidth',2};
```

```
plot(t,x)
hold on
plot(t,[-1;1]*env,plot_param{:})
hold off
xlim([0 0.04])
title('Hilbert Envelope')
```


Hilbert Envelope
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04

# 8th week

# 8 Continuous and Discrete Wavelet Transform

## Wavelet transform

Unlike the Fourier transform, which expresses a signal as the sum of a series of single-frequency sine and cosine functions, the wavelet transform decomposes a signal into a set of basis functions.

These basis functions are obtained from a single base wavelet function by a two-step operation: scaling (through dilation and contraction of the base wavelet along the time axis), and time shift (i.e., translation along the time axis). Essentially, the wavelet transform process measures the ‘similarity’ between the signal being analysed and the base wavelet.

Through variations of the scales and time shifts of the base wavelet function, features hidden within the signal can be extracted, without requiring the signal to have a dominant frequency band. It can be concluded that the wavelet transform provides a powerful mathematical tool for the analysis, characterization, and classification of non-stationary signals typically seen in manufacturing.

The adaptive, multiresolution capability of the wavelet transform makes it well suited for decomposing signals of varying time and frequency resolutions that are characteristic of the underlying defect mechanisms associated with a machine, a dynamical structure, or a manufacturing process. Such capability makes the wavelet transform an enabling tool for advancing the science base of signal processing in manufacturing.

Wavelets are enabling the analysis on several timescales of the local properties of complex signals that can present nonstationary zones. A wavelet is a function oscillating as a wave but quickly damped. Being well localized simultaneously in time and frequency it makes it possible to define a family of analysing functions by translation in time and dilation in scale. Wavelets constitute a mathematical 'zoom' making it possible to simultaneously describe the properties of a signal on several timescales.

A function $\psi$ will be called a wavelet if it verifies the admissibility condition

$$K_\psi = \int_0^{+\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega = \int_{-\infty}^{0} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < +\infty$$

where $\hat{\psi}$ indicates the Fourier transform of $\psi$. The admissibility condition involves, that the wavelet integrates to zero, that is, $\int_{\mathbb{R}} \psi(x)\, dx = 0$. It is often reinforced by requiring that the wavelet has $m$ vanishing moments, i.e.

$$\int_{\mathbb{R}} x^k \cdot \psi(x)\, dx = 0, \quad \text{for } k = 0, \ldots, m$$

The oscillation of a wavelet is measured by the number of vanishing moments and its localization is evaluated by the interval where it takes values significantly different from zero.

Using translation and dilation a family of functions $\{\psi_{a,b}\}$ is defined by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right)$$

for any scale $a \in \mathbb{R}^+$ and any position $b \in \mathbb{R}$. If $\psi$ has norm 1 then all the functions $\psi_{a,b}$ have norm of 1.

A signal $f$ of finite energy can be analysed by its wavelet coefficients

$$C_f(a,b) = \int_{\mathbb{R}} f(x)\cdot\psi^*_{a,b}(x)\,dx, \quad a \in \mathbb{R}^+, b \in \mathbb{R}.$$

The calculation of wavelet coefficients $C_f$ is called continuous wavelet transform.

The $C_f(a,b)$ measures the fluctuation of function $f$ at scale $a$, it depends on the values of $f$ in a neighbourhood of $b$ with a length proportional to $a$. Large values of $C_f(a,b)$ provide information on the local irregularity of $f$ around position $b$ and at scale $a$.

The squared magnitude of wavelet coefficients

$$\left|C_f(a,b)\right|^2 = \left|\int_{-\infty}^{\infty} f(t)\cdot\psi^*_{a,b}(x)\right|^2$$

constitutes the so-called scalogram, which is an important tool in evaluations based on wavelet transforms.

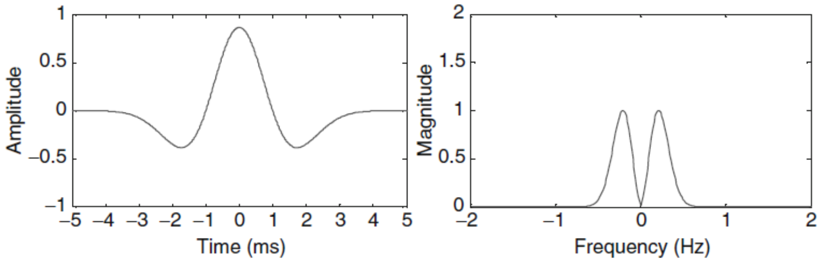
The coloured representation of scalogram in the time-frequency plane is a useful source of visual information e.g. in the identification of transients in vibration signal caused by surface defects of bearings.

Large number of wavelets are defined in the literature and many of them are used in vibration diagnostics as well. Three examples are the Mexican Hat wavelet, the Morlet Wavelet and the Gaussian wavelet.

The Mexican hat wavelet is a normalized, second derivative of a Gaussian function, which is mathematically defined as

$$\psi(t) = \frac{1}{\sqrt{2\pi} \cdot \sigma^3} \cdot \left(1 - \frac{\sigma^2}{t^2}\right) \cdot e^{-\frac{t^2}{2\sigma^2}}$$

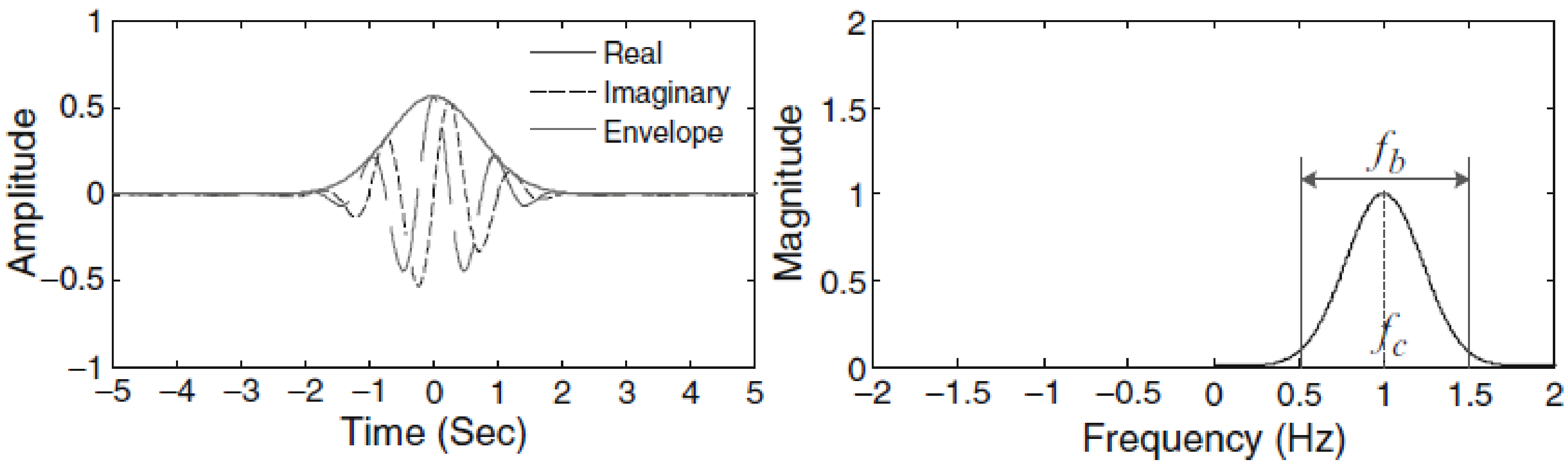
Figure shows the Mexican hat wavelet and its associated magnitude spectrum.




The Morlet wavelet is defined as

$$\psi_M(t) = \frac{1}{\sqrt{\pi \cdot f_b}} \cdot e^{j \cdot 2\pi f_c \cdot t} \cdot e^{-\frac{t^2}{f_b}}$$

where $f_b$ is the bandwidth parameter and $f_c$ denotes the wavelet center frequency. Figure shows the Morlet wavelet and its associated magnitude spectrum.




The Gaussian function is expressed as

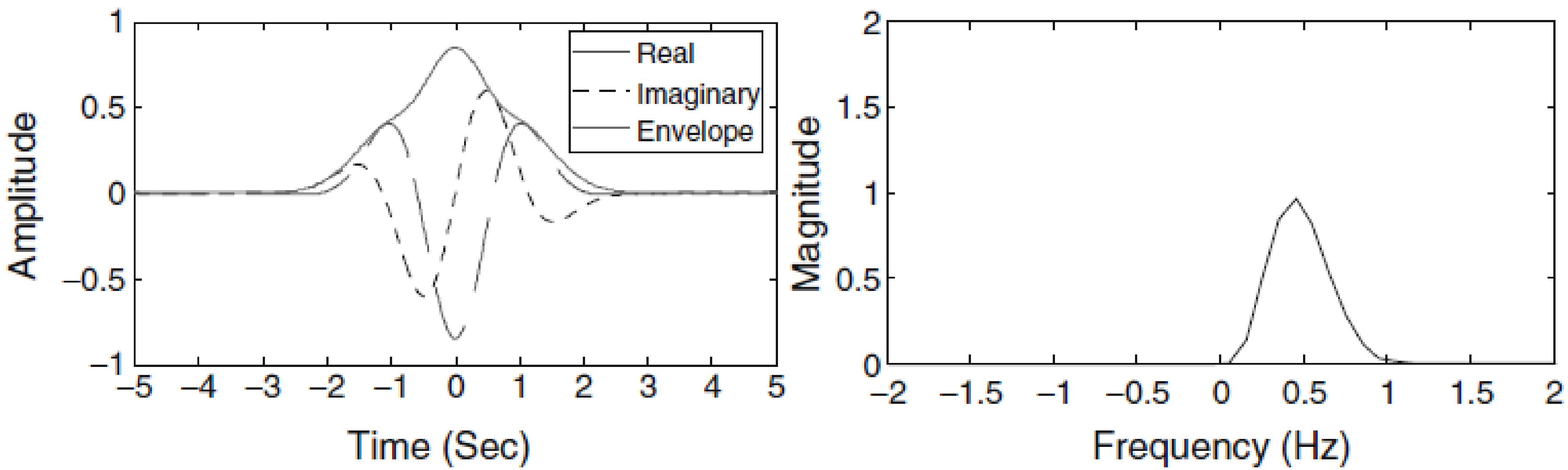
$$f(t) = e^{-j\cdot t} \cdot e^{-t^2}$$

Taking the $N$-th derivative of this function yields the Gaussian wavelet as

$$\psi_G(t) = c_N \cdot \frac{d^{(N)} f(t)}{dt^N}$$

where $N \geq 1$ is an integer parameter and denotes the order of the wavelet, and $c_N$ is a constant introduced to ensure that $\left\| f^{(N)}(t) \right\|^2 = 1$.

Figure shows the Gaussian wavelet and its magnitude spectrum ($N = 2$).




In numerical analysis and functional analysis, a discrete wavelet transform (DWT) is any wavelet transform for which the wavelets are discretely sampled. As with other wavelet transforms, a key advantage it has over Fourier transforms is temporal resolution: it captures both frequency and location information (location in time).

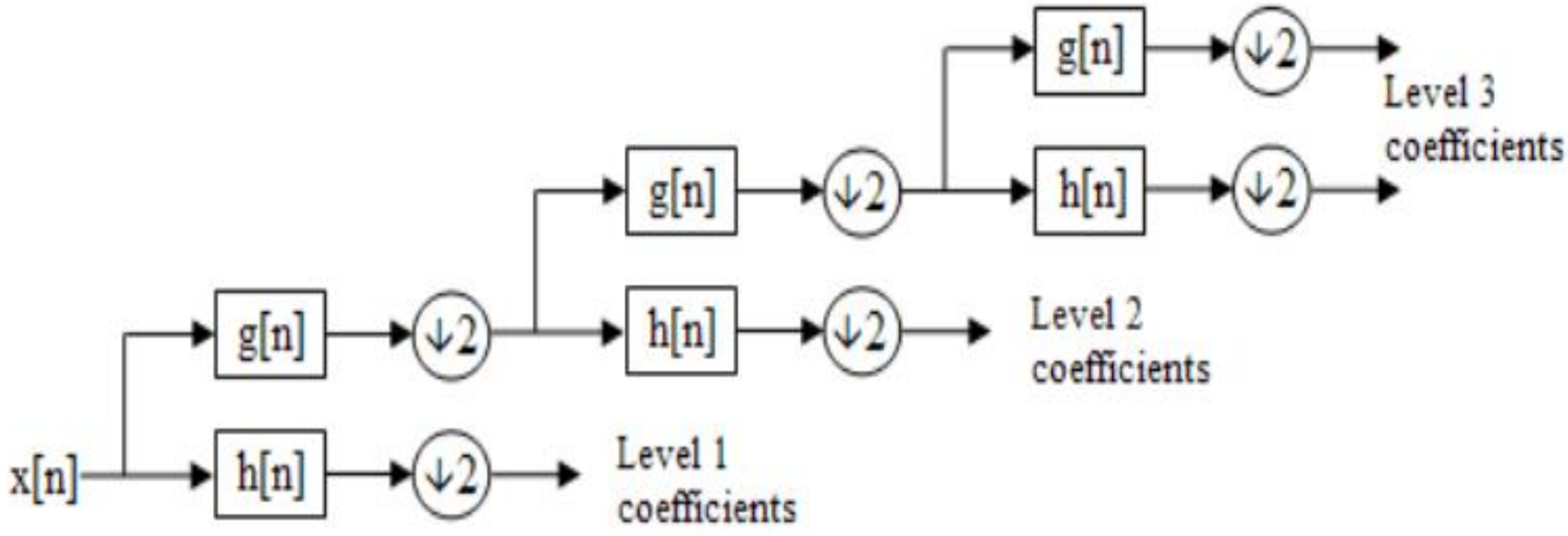
The DWT of a signal $x$ is calculated by passing it through a series of filters. First the samples are passed through a low pass filter with impulse response g resulting in a convolution of the two:

$$y[n] = (x \cdot g)[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-k]$$

However, since half the frequencies of the signal have now been removed, half the samples can be discarded according to Nyquist's rule. The filter output of the low-pass filter g in the diagram above is then subsampled by 2 and further processed by passing it again through a new low- pass filter g and a high- pass filter h with half the cut off frequency of the previous:

$$y_{low}[n] = \sum_{k=-\infty}^{\infty} x[k]g[2n-k]$$

$$y_{high}[n] = \sum_{k=-\infty}^{\infty} x[k]g[2n-k]$$



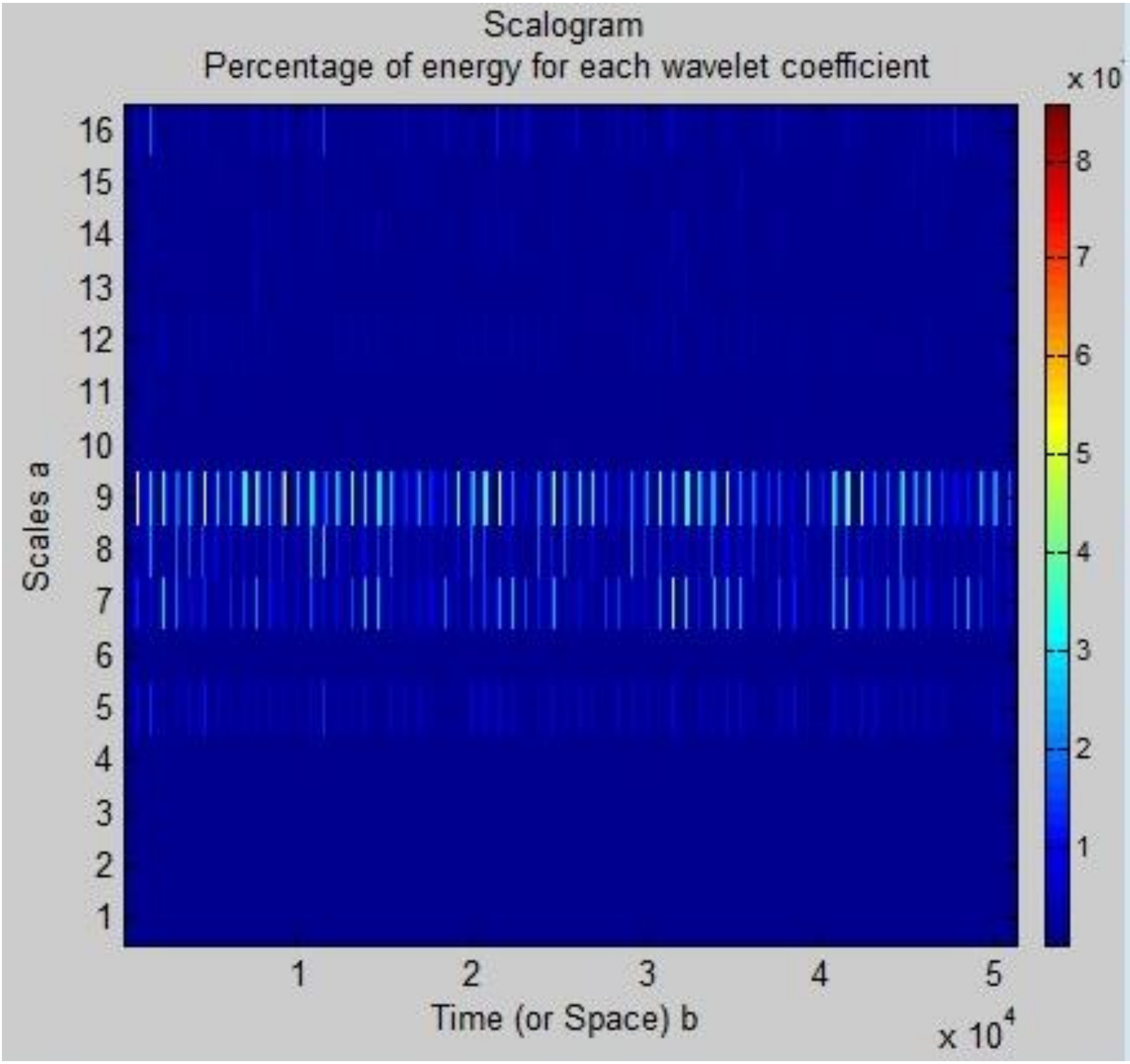

Levels of DWT composition (using MRA analysis)

$$\psi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t - k2^j}{2^j}\right)$$

where $j$ is the scale parameter and $k$ is the shift parameter, both which are integers.

Vibrations signals in engineering diagnostics are non-stationary which means frequency-domain representation (frequency spectrum) changes over time. The detection of low-energy transients in the signal requires information about frequencies and also on the time when a particular frequency component present. Several time-frequency analysis methods are known which are able to provide both types of data, for instance the Windowed Fourier Transforms and the Wavelet Transforms (WT). Because of their flexibility, the wavelet transforms (and the related multiresolution analysis) can be used more effectively in the condition monitoring of machine elements when short time transient signals occur.

Wavelets enable to analyse several timescales of the local properties of complex signals that can present non-stationary zones. A wavelet is a function oscillating as a wave but quickly damped. Being well localized simultaneously in time and frequency it makes it possible to define a family of analysing functions by translation in time and dilation in scale. Wavelets constitute a mathematical "zoom" making it possible to simultaneously describe the properties of a signal on several timescales.




Wavelet scalogram for representation

For instance, a ranking of wavelets can be generated with respect to their efficiency in bearing fault detection using the so-called Energy-to-Shannon entropy criteria using the scalograms. Values in the scalogram are related to the energy content of signal components.

The representation of scalogram in the time-frequency plane is useful source of visual information e.g. in the identification of transients in vibration signal caused by surface defects of bearings.

Supposing certain stronger properties than merely the admissibility condition we limit ourselves to the values

$$a = 2^j, b = k \cdot 2^j, \qquad j, k \in Z$$

This idea leads to the discrete wavelet transform which is closely related to the so-called multi-resolution analysis (MRA).

STFT is similar to the wavelet transform but it is the time-frequency analysis of the signal.

# 8th week – Questions

***Question***

What is the basic concept of the wavelet transform?

***Answer***

These basis functions are obtained from a single base wavelet function by a two-step operation: scaling (through dilation and contraction of the base wavelet along the time axis), and time shift (i.e., translation along the time axis). Essentially, the wavelet transform process measures the 'similarity' between the signal being analysed and the base wavelet.

Through variations of the scales and time shifts of the base wavelet function, features hidden within the signal can be extracted, without requiring the signal to have a dominant frequency band. It can be concluded that the wavelet transform provides a powerful mathematical tool for the analysis, characterization, and classification of non-stationary signals typically seen in manufacturing.

The adaptive, multiresolution capability of the wavelet transform makes it well suited for decomposing signals of varying time and frequency resolutions that are characteristic of the underlying defect mechanisms associated with a machine, a dynamical structure, or a manufacturing process.

Using translation and dilation a family of functions $\{\psi_{a,b}\}$ is defined by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right)$$

for any scale $a \in \mathbb{R}^+$ and any position $b \in \mathbb{R}$. If $\psi$ has norm 1 then all the functions $\psi_{a,b}$ have norm of 1.

### *Question*

Which transform is better to recognize short time transients in the machine fault signals?

### *Answer*

Vibrations signals in engineering diagnostics are non-stationary which means frequency-domain representation (frequency spectrum) changes over time. The detection of low-energy transients in the signal requires information about frequencies and also on the time when a particular frequency component present. Several time-frequency analysis methods are known which are able to provide both types of data, for instance the Windowed Fourier Transforms and the Wavelet Transforms (WT). Because of their flexibility, the wavelet transforms (and the related multiresolution analysis) can be used more effectively in the condition monitoring of machine elements when short time transient signals occur.

## Question

What conditions should be satisfied considering wavelet transform?

## Answer

A function $\psi$ will be called a wavelet if it verifies the admissibility condition

$$K_{\psi} = \int_{0}^{+\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega = \int_{-\infty}^{0} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < +\infty$$

where $\hat{\psi}$ indicates the Fourier transform of $\psi$. The admissibility condition involves, that the wavelet integrates to zero, that is, $\int_{\mathbb{R}} \psi(x)\, dx = 0$. It is often reinforced by requiring that the wavelet has $m$ vanishing moments, i.e.

$$\int_{\mathbb{R}} x^k \cdot \psi(x)\, dx = 0, \quad \text{for } k = 0, \dots, m$$

# 8th week – Exercises

***Exercise***

Create the MATLAB code for edge detection of images with wavelet transform!

***Solution***

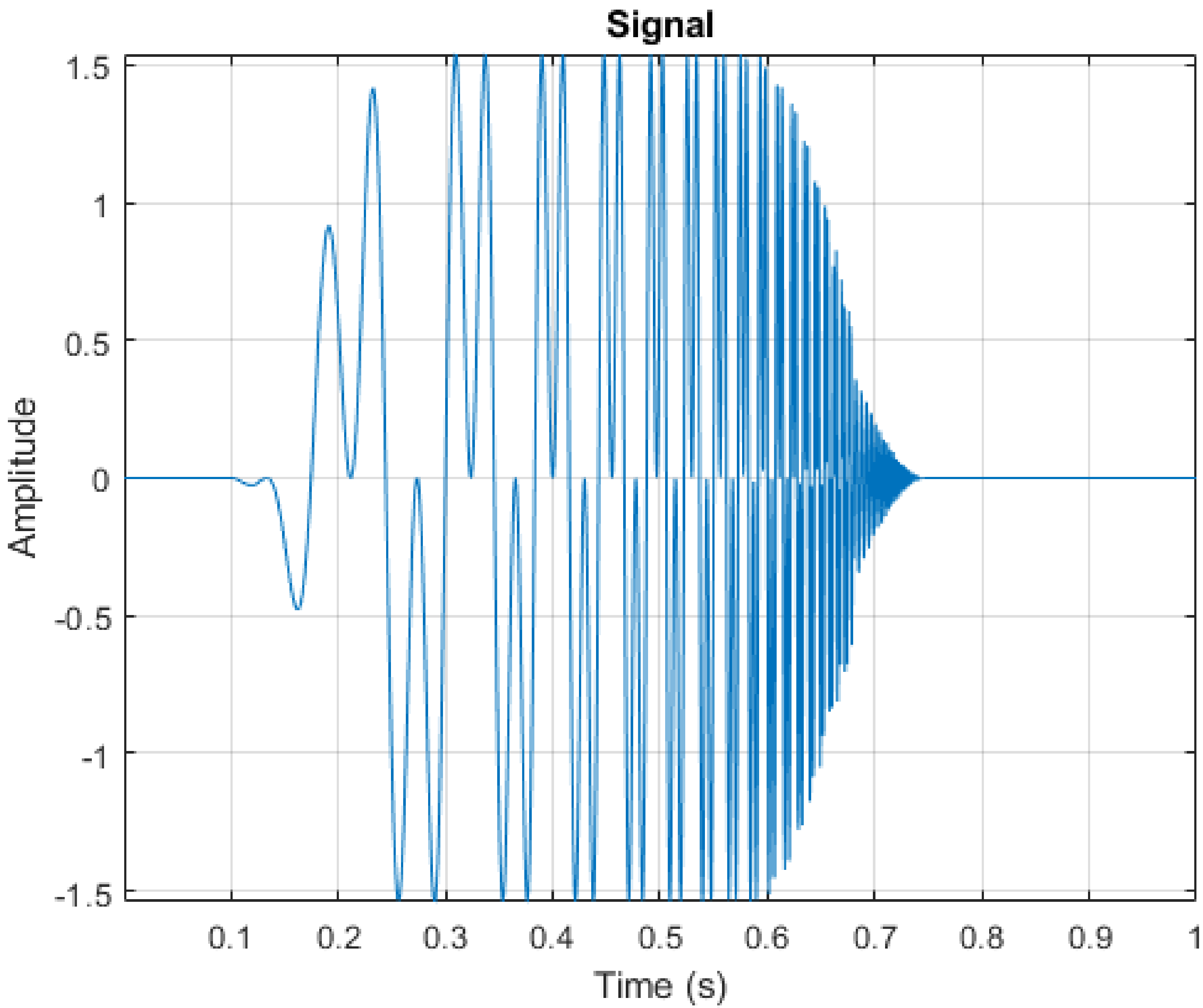
```
wname='bior4.4';

[ca1,ch1,cv1,cd1] = dwt2(imagein,wname);

thr=4;

a0=1;

n=7;

edge_map = local_max_mode_new(cv1,ch1,thr,a0,n);
```

## Exercise

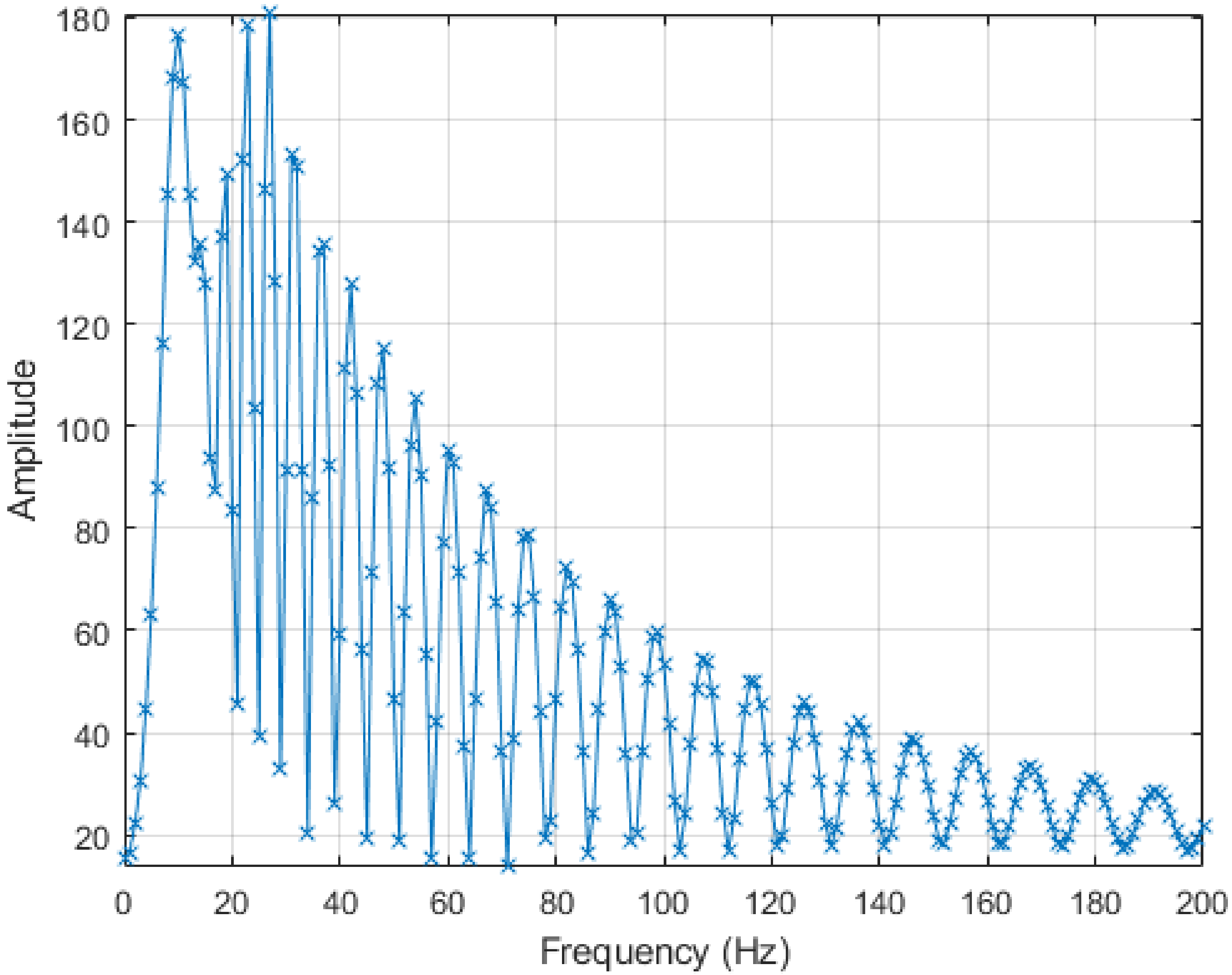
Analyze a chirp signal with wavelet transform in Matlab! At first generate the image, then make its FFT analysis, finally the wavelet analysis!

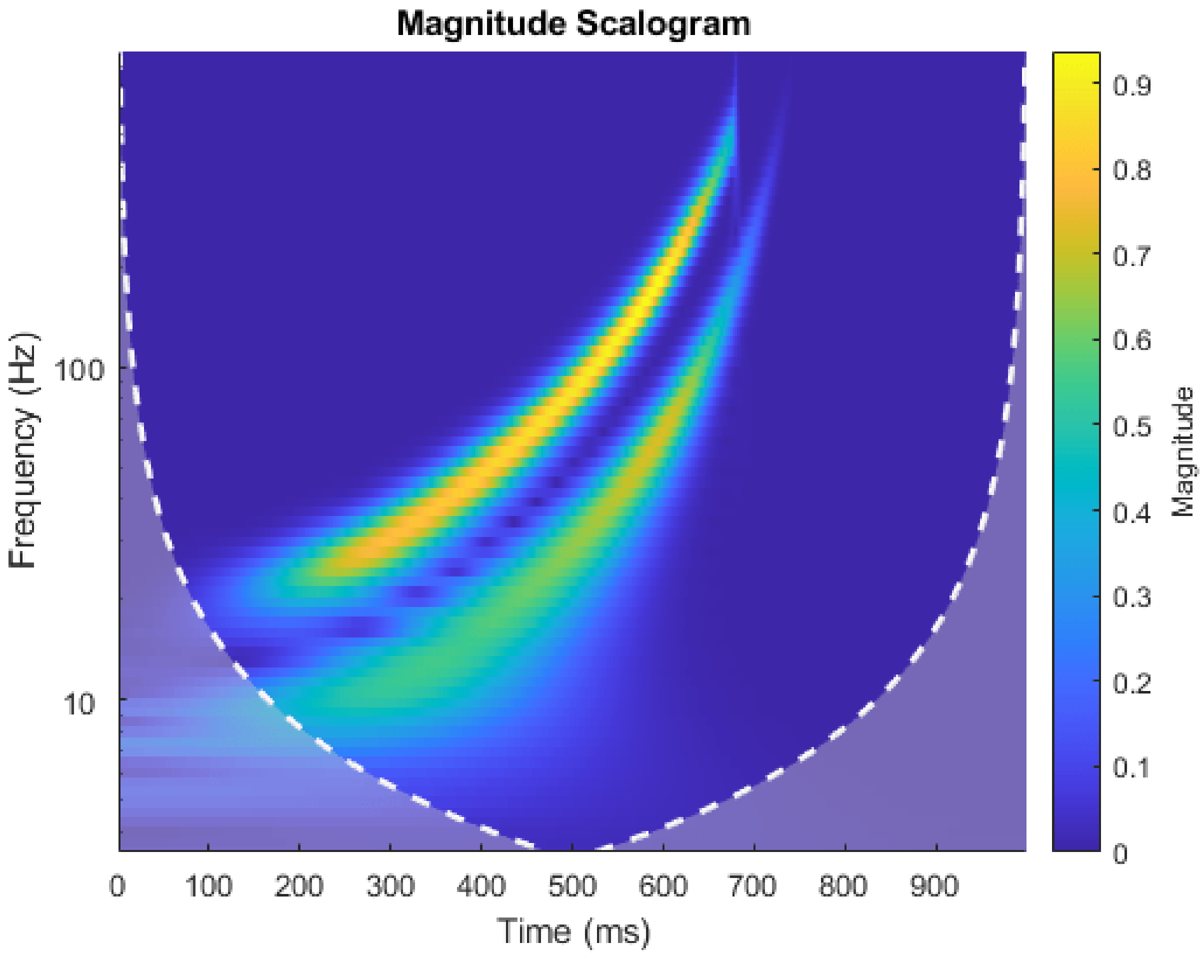
## Solution

```
load hychirp

plot(t,hychirp)

grid on

title('Signal')

axis tight

xlabel('Time (s)')

ylabel('Amplitude')
```


Signal
Amplitude
1.5
1
0.5
0
-0.5
-1
-1.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time (s)

```
sigLen = numel(hychirp);
fchirp = fft(hychirp);
fr = Fs*(0:1/Fs:1-1/Fs);
plot(fr(1:sigLen/2),abs(fchirp(1:sigLen/2)),'x-')
xlabel('Frequency (Hz)')
ylabel('Amplitude')
axis tight
grid on
xlim([0 200])
cwt(hychirp,Fs)
```


Amplitude
Frequency (Hz)
180
160
140
120
100
80
60
40
20
0
20
40
60
80
100
120
140
160
180
200


Magnitude Scalogram
Frequency (Hz)
100
10
0
100
200
300
400
500
600
700
800
900
Time (ms)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Magnitude

## Exercise

Create the time-frequency (STFT) figure of the above chirp function!

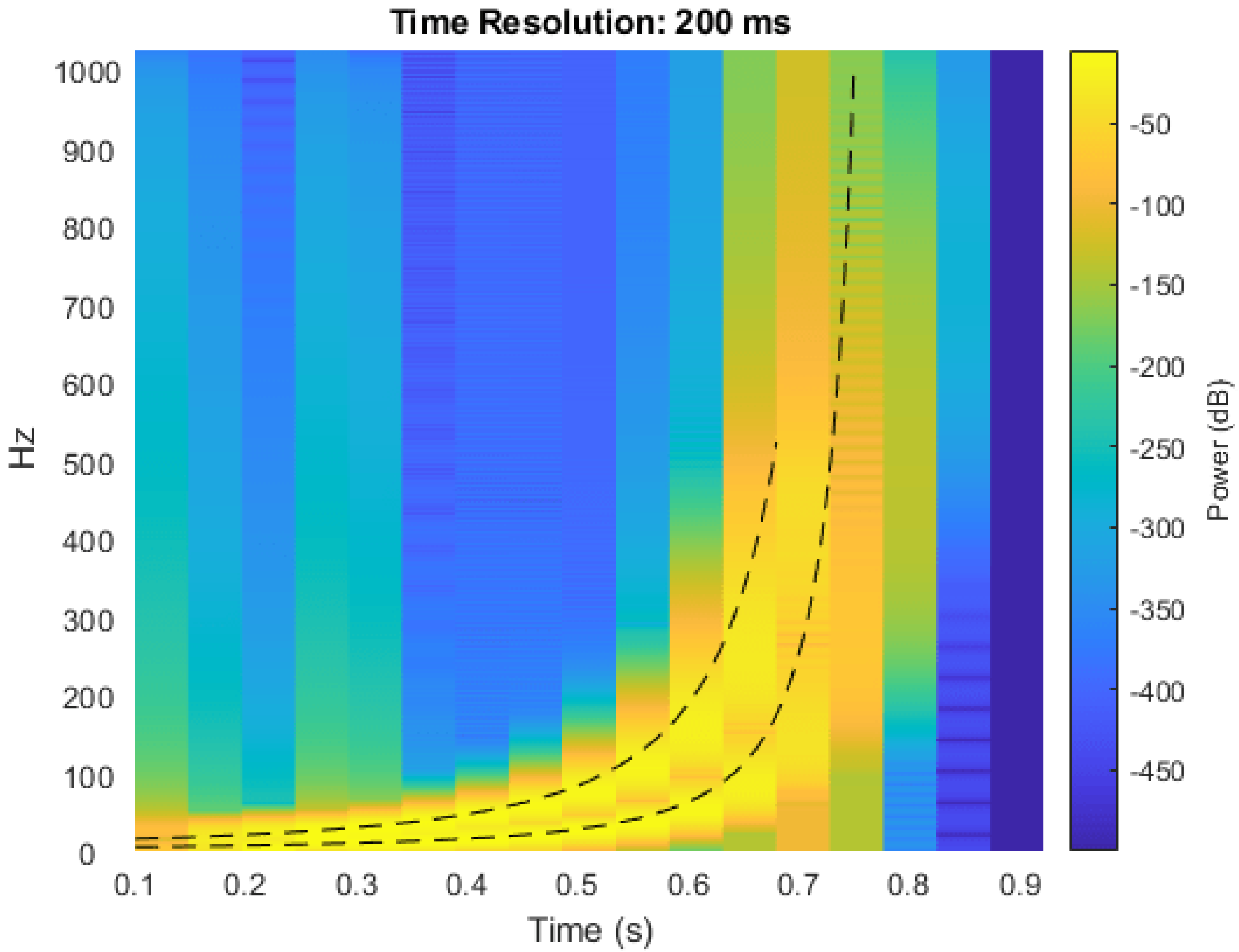
## Solution

```
helperPlotSpectrogram(hychirp,t,Fs,200)
```

The STFT provides some information on both the timing and the frequencies at which a signal event occurs. However, choosing a window (segment) size is key. For time-frequency analysis using the STFT, choosing a shorter window size helps obtain good time resolution at the expense of frequency resolution. Conversely, choosing a larger window helps obtain good frequency resolution at the expense of time resolution.

Once you pick a window size, it remains fixed for the entire analysis. If you can estimate the frequency components you are expecting in your signal, then you can use that information to pick a window size for the analysis.

The instantaneous frequencies of the two chirps at their initial time points are approximately 5 Hz and 15 Hz. Use the helper function helperPlotSpectrogram to plot the spectrogram of the signal with a time window size of 200 milliseconds.


Time Resolution: 200 ms
Hz
1000
900
800
700
600
500
400
300
200
100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Time (s)
-50
-100
-150
-200
-250
-300
-350
-400
-450
Power (dB)

# 9th week

# 9 MRA, Scalogram

A multiresolution analysis (MRA) or multiscale approximation (MSA) is the design method of most of the practically relevant discrete wavelet transforms (DWT) and the justification for the algorithm of the fast wavelet transform (FWT).

It was introduced in this context in 1988/89 by Stephane Mallat and Yves Meyer and has predecessors in the microlocal analysis in the theory of differential equations (the ironing method) and the pyramid methods of image processing.

Signals often consist of multiple physically meaningful components. Quite often, you want to study one or more of these components in isolation on the same time scale as the original data. Multiresolution analysis refers to breaking up a signal into components, which produce the original signal exactly when added back together. To be useful for data analysis, how the signal is decomposed is important. The components ideally decompose the variability of the data into physically meaningful and interpretable parts. The term multiresolution analysis is often associated with wavelets or wavelet packets, but there are non-wavelet techniques which also produce useful MRAs.

Supposing certain stronger properties than merely the admissibility condition we can limit ourselves to the values

$$a = 2^j, b = k \cdot 2^j, \quad j, k \in \mathbb{Z}.$$

This idea leads to the discrete wavelet transform which is closely related to the so-called multi-resolution analysis.

For signals $f$ the following three operations

$$(\tau_a f)(x) = f(x+a), \quad a, x \in \mathbb{R}$$

$$(\delta_s f)(x) = f(s \cdot x)\,, \quad s, x \in \mathbb{R}, s > 0$$

$$(E_1 f)(x) = \sum_{k \in \mathbb{Z}} f(x+k) = \sum_{k \in \mathbb{Z}} (\tau_k f)(x)\,, \quad x \in \mathbb{R}$$

will be used.

The concept of multi-resolution is based on translation-invariant Riesz bases. A system $\{\varphi_k\}_{k \in \mathbb{Z}} \subset X$ is a Riesz basis of $X$ if for all $x \in X$ there exists a unique $(c_k) \in \ell_2$ such that $x = \sum_{k \in \mathbb{Z}} c_k \varphi_k$.

It is important to note that all orthonormal bases are Riesz bases as well.

Let $\{\varphi_k\}_{k \in \mathbb{Z}} = \{\tau_k \varphi\}_{k \in \mathbb{Z}} \subset X$ be a Riesz basis with the generator function $\varphi$. Then for all $n \in \mathbb{Z}$

$$\varphi_k^n(x) = \frac{1}{\sqrt{2^n}} \cdot \varphi\left(\frac{1}{2^n} \cdot x - k\right), \quad x \in \mathbb{R}, k \in \mathbb{Z}$$

is Riesz basis for $X_n$, furthermore, if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is orthonormal, then $\{\varphi_k^n\}_{k\in\mathbb{Z}}$ is orthonormal as well.

A sequence of closed subspaces $\{X_n\}_{n\in\mathbb{Z}} \subset X$ is called multi-resolution analysis of $X$ if

1. $X_n \subset X_{n-1}, n \in \mathbb{Z}$ — *monotonicity property*
2. $\overline{\bigcup_{n\in\mathbb{Z}} X_n} = X$ — *density property*
3. $\bigcap_{n\in\mathbb{Z}} X_n = \{0\}$ — *separability property*
4. $f \in X_n \quad \Leftrightarrow \quad \delta_2 f \in X_{n-1}, n \in \mathbb{Z}$ — *scaling property*
5. $X_0$ is generated by a Riesz-basis given by

$$\varphi_k = \tau_k \varphi, \quad k \in \mathbb{Z}$$

   (The basis is invariant to integer translations).

The *scaling* property characterizes the multi-resolution aspects: spaces $X_j$ are obtained by dyadic dilation or contraction of the functions of the single space $X_0$ through the assumptions

$$\big(x \to f(x)\big) \in X_j \quad \Leftrightarrow \quad \big(x \to f(2\cdot x)\big) \in X_{j-1}, \quad j \in \mathbb{Z}.$$

Property 5 states that there exists $\varphi \in X_0$ such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of $X_0$.

Function $\varphi$ is also called scaling function of the MRA.

Let us denote the subspace generated by Riesz basis $\varphi_k^n$ as

$$X_n^{\varphi} = \{\sum_{k\in\mathbb{Z}} c_k \cdot \varphi_k^n \,|(c_k) \in \ell_2\}$$

Each multi-resolution analysis generates an approximation process. If $P_n: X \to X_n, n \in \mathbb{Z}$ denote the othogonal projection of $f$ onto the closed subspace $X_n$ then $\lim_{n\to\infty} \|P_n f - f\| = 0$, $f \in X$.

Let $\mathcal{M}_0$ be the set of non-negative, even, integrable functions decreasing on [0,1[.

It can be proved that if an MRA is generated by a function $\varphi$ majorated by an element of $\mathcal{M}_0$ then the equality

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1, \quad x \in \mathbb{R}$$

is equivalent with the density property.

Functions with compact support on a topological space X are those whose closed support is a compact subset of X. If X is the real line, or n-dimensional Euclidean space, then a function has compact support if and only if it has bounded support, since a subset of {R} is compact if and only if it is closed and bounded.

***Theorem***: The following three statements are equivalent:

i) $\{\varphi_k^n\}_{k\in\mathbb{Z}}$ is a Riesz basis of $X_n$ for all $n \in \mathbb{Z}$.

ii) $\{\varphi_k^0 = \tau_k\varphi\}_{k\in\mathbb{Z}}$ is a Riesz basis of $X_0$.

iii) There exist real numbers $0 < m \leq M < \infty$ such that $m \leq \sqrt{E_1(|\hat{\varphi}|^2)} \leq M$.

Let $\mathcal{R} = \left\{\varphi \in X \,\middle|\, m \leq \sqrt{E_1(|\hat{\varphi}|^2)} \leq M\right\}$. According to the previous theorem $\mathcal{R}$ contains the functions $\varphi \in X$ for which system $\{\varphi_k^n\}_{k\in\mathbb{Z}}$ is a Riesz basis of $X_n$ for all $n \in \mathbb{Z}$.

***Theorem***: Let $\varphi \in \mathcal{R}$. System $\{\tau_k\varphi\}_{k\in\mathbb{Z}}$ is orthonormal if and only if $E_1(|\hat{\varphi}|^2) = 1$.

Let $\varphi \in \mathcal{R}$ and let $X_n^{\varphi}$ denote the Riesz basis generated by $\{\varphi_k^n\}_{k\in\mathbb{Z}}, n \in \mathbb{Z}$.

Theorem: The monotonicity property $X_n \subset X_{n-1}, n \in \mathbb{Z}$ is equivalent with the scaling equation

$$\hat{\varphi}(2x) = \alpha(x) \cdot \hat{\varphi}(x), \quad x \in \mathbb{R},$$

where $\alpha \in X$ is a suitable 1-periodic function. $\alpha$ is called the low-pass filter belonging to $\varphi$.

Furthermore, the scaling equation is equivalent with the equality

$$\frac{1}{2} \cdot \varphi\left(\frac{1}{2} \cdot x\right) = \sum_{k\in\mathbb{Z}} a_k \cdot \varphi(x+k)\,, \quad x \in \mathbb{R}$$

where $(a_k)$ is the sequence of the Fourier coefficients of $\alpha$.

Let $\mathcal{M}$ denote the set of functions $\varphi \in \mathcal{R}$ having majorant in $\mathcal{M}_0$ and satisfying condition

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t)\,dt = 1.$$

From function $\varphi$ generating MRA on orthonormal wavelet basis $\left\{X_n^{\varphi}\right\}_{n\in\mathbb{Z}}$ of $X$ can be constructed as follows.

In the following we suppose that the normality conditions

$$E_1(|\hat{\varphi}|^2) = 1, \quad E_1\left(\left|\hat{\psi}\right|^2\right) = 1$$

hold, that is, systems $\{\varphi_k^n\}_{k\in\mathbb{Z}}$ and $\{\psi_k^n\}_{k\in\mathbb{Z}}$ are orthonormal for any $n \in \mathbb{Z}$.

It can be proved that there exists a 1-periodic filter $\beta \in X$ such that for $\psi$ defined by

$$\delta_2\left(\hat{\psi}\right) = \beta \cdot \hat{\varphi}$$

we have the orthogonal decomposition $X_0^{\varphi} = X_1^{\varphi} \oplus X_1^{\psi}$.

Consequently, for all $n \in \mathbb{Z}$ we have $X_n^{\varphi} = X_{n+1}^{\varphi} \oplus X_{n+1}^{\psi}$, furthermore

$$X_j^{\varphi} = X_k^{\varphi} \oplus \left( \bigoplus_{i=k}^{j+1} X_i^{\psi} \right), \;\; j < k,$$

$$X_j^{\varphi} = \bigoplus_{i=j+1}^{\infty} X_i^{\psi}$$

$$X = X_j^{\varphi} \oplus \left( \bigoplus_{i=-\infty}^{j} X_i^{\psi} \right)$$

$$X = \bigoplus_{i=-\infty}^{\infty} X_i^{\psi}$$

This result shows that system $\{\psi_k^n\}_{k \in \mathbb{Z}}$, where functions

$$\psi_k^n(x) = \frac{1}{\sqrt{2^n}} \cdot \psi\left(\frac{1}{2^n} \cdot x - k\right), \quad x \in \mathbb{R}, k \in \mathbb{Z}$$

are generated by function $\psi$ satisfying $E_1(|\hat{\varphi}|^2) = 1, E_1\left(\left|\hat{\psi}\right|^2\right) = 1$ and $X_0^{\varphi} = X_1^{\varphi} \oplus X_1^{\psi}$ is orthonormal, that is $\langle \psi_k^n, \psi_l^m \rangle = \delta_{kl} \cdot \delta_{nm}, \quad k, l, n, m \in \mathbb{Z}$. $\psi$ is called the *mother wavelet* of the MRA.

Subspaces $X_j^{\varphi}, k \in \mathbb{Z}$ are called approximation spaces, while spaces $X_j^{\psi}, k \in \mathbb{Z}$ are called detail spaces.

The equality $X_n^{\varphi} = X_{n+1}^{\varphi} \oplus X_{n+1}^{\psi}$ says that an element of the approximation space of level $n$ is decomposed into an approximation at level $n+1$, which is less accurate, and a detail at level $n+1$.

According to the equality $X = \bigoplus_{i=-\infty}^{\infty} X_i^{\psi}$ any signal is the sum of all its details, namely its orthogonal projections onto the spaces $X_j^{\psi}$.

The wavelet coefficients of a signal $f$ are provided by

$$\alpha_{k,n} = C_f(k,n) = \int_{\mathbb{R}} f(x) \cdot \psi_k^n(x)\, dx, \quad k,n \in \mathbb{Z}$$

Considering the decomposition $X = X_j^{\varphi} \oplus \left( \bigoplus_{i=-\infty}^{j} X_i^{\psi} \right)$ the orthonormal system

$$\{\varphi_k^0, \psi_k^n\}_{k,n \in \mathbb{Z}, n<0}$$

is used instead of $\{\psi_k^n\}_{k \in \mathbb{Z}}$.

Theorem: If the continuously differentiable 1-periodic function $\alpha \in X$ satisfies

$$\alpha(0) = 1, \ \ |\alpha(x)|^2 + \left|\alpha(x) + \frac{1}{2}\right|^2 = 1, \qquad x \in \mathbb{R}$$

and

$$|\alpha(x)| > 0, \quad if \quad |x| \leq \frac{1}{4}$$

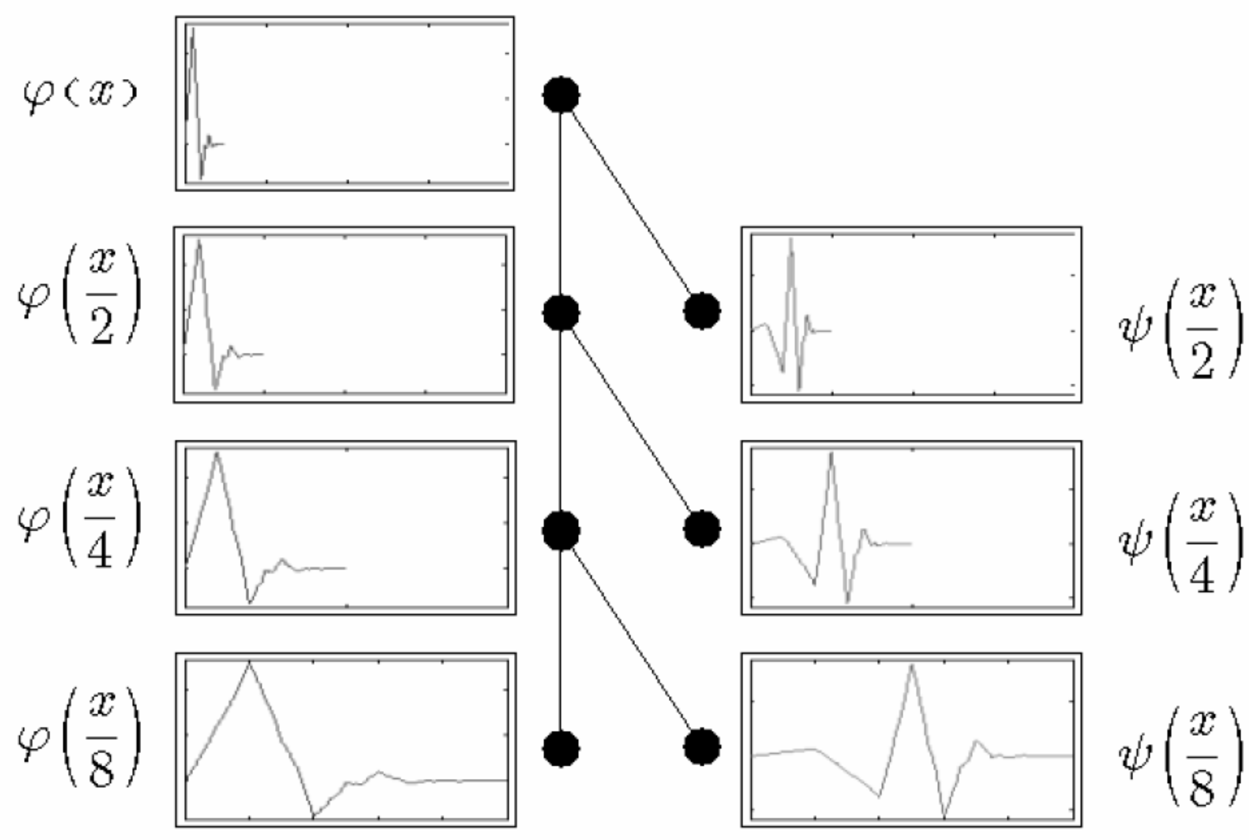
then function $\varphi$ defined by $\hat{\varphi} = A$, where

$$A(x) = \lim_{m \to \infty} \prod_{n=1}^{m} \alpha\left(\frac{1}{2^n} \cdot x\right)$$

generates an MRA of X.

Wavelets are organized using two parameters time and scale. Time k makes it possible to translate the forms for a given level; scale 2^j makes it possible to pass from a level j to the immediately lower level in the underlying tree.

In the first column we find the dyadic dilates of the scaling function φ and in the second column, those of the wavelet ψ.




The functions in the first column are used for calculating the coefficients of approximation

$$\beta_{k,n} = \int_{\mathbb{R}} f(x) \cdot \psi_k^n(x)\, dx,$$

which define local averages of the signal $f(x)$.

The signal

$$A_j(x) = \sum_{k \in \mathbb{Z}} \beta_{j,k} \cdot \varphi_k^n(x)$$

is an approximation.

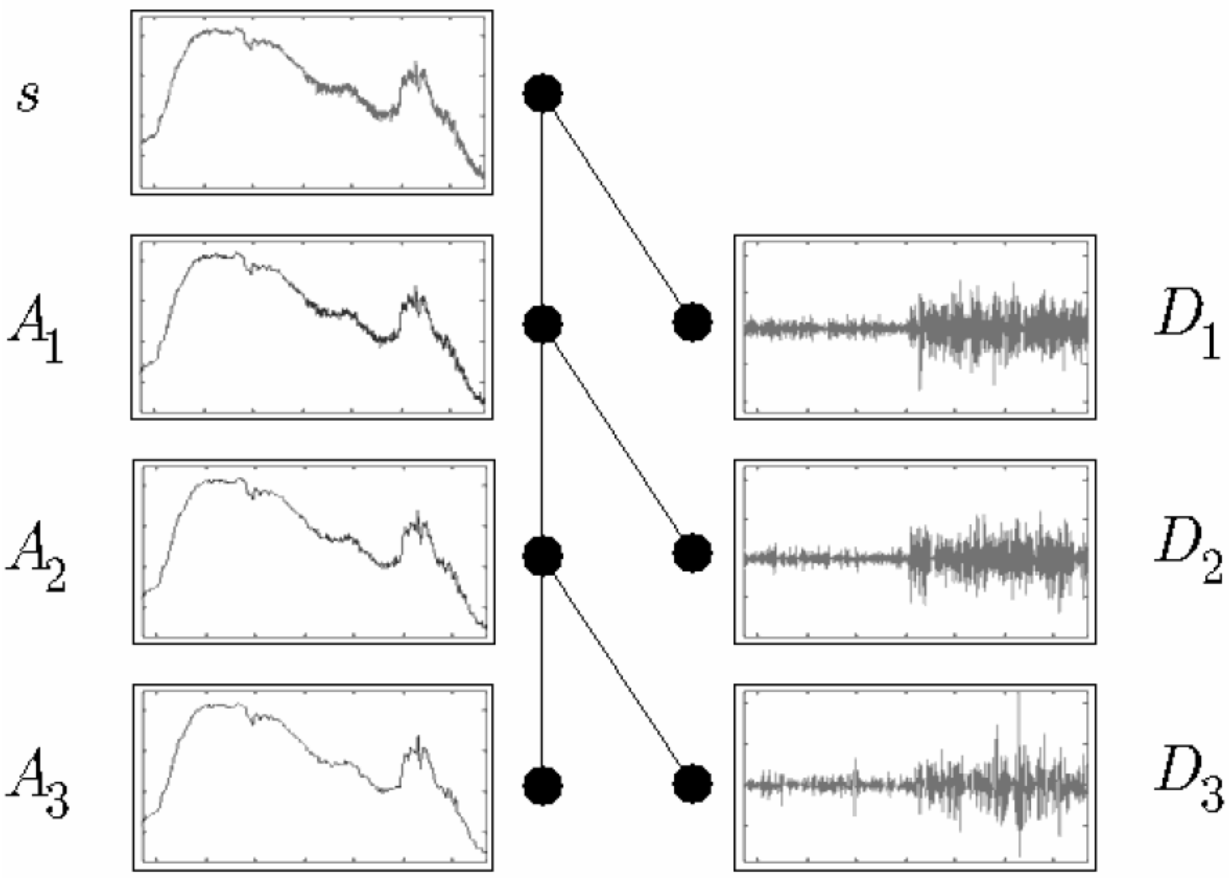
The functions in the second column are associated with the calculation of wavelet coefficients

$$\alpha_{k,n} = \int_{\mathbb{R}} f(t) \cdot \psi_k^n(x)\, dx, \quad k, n \in \mathbb{Z}$$

which relate to the differences between two successive local averages. These are the details of the form:

$$D_j(x) = \sum_{k \in \mathbb{Z}} \alpha_{j,k} \cdot \psi_k^n(x)$$

To present certain approximations and details we use the so-called wavelet tree illustrated in figure for a signal $s$.




At the root of a we find the signal. The first column in the figure yields three approximations, from the finest $A_1$ to the coarsest $A_3$. The differences between two successive approximations are captured in the details denoted $D_1$ to $D_3$. More precisely, we have $D_1 = s - A_1$ , $D_2 = A_1 - A_2$ and, thus, $s = A_2 + D_2 + D_1$.

This representation help us to understand the basic relations between approximations and details, for instance $A_{j-1} = A_j + D_j$, $A_j = \sum_{i>j} D_i$ and $s = A_j + \sum_{i\leq j} D_i$.

The following theorem says that from any scaling function of a MRA a mother wavelet of a complete orthonormal wavelet system can be constructed.

Theorem: Suppose that for $\varphi \in X$ $E_1(|\hat{\varphi}|^2) = 1$ and $\delta_2\hat{\varphi} = \alpha \cdot \hat{\varphi}$ hold, and define functions $\beta$ and $\psi$ as

$$\beta = \epsilon \cdot \tau_{\frac{1}{2}}(\alpha^*), \quad \delta_2\hat{\psi} = \beta \cdot \hat{\varphi}.$$

Then $\left\{X_n^{\varphi}\right\}_{n\in\mathbb{Z}}$ is an MRA of $X$ and $\{\psi_k^n\}_{k,n\in\mathbb{Z}}$ complete orthonormal system.

The Fourier coefficients of the 1-periodic function $\beta \in X$ can be expressed by the Fourier coefficients of $\alpha$, namely if $\alpha = \sum_{k\in\mathbb{Z}} a_k \cdot \epsilon_k$, then $\beta = \sum_{k\in\mathbb{Z}} \alpha_{1-k}^* \cdot \epsilon_k$.

**Wavelets with compact support**

In the applications the MRAs are used first of all whose generator functions has compact support. The filters belonging to these generator functions are trigonometric polynomials [81].

Filter $\alpha$ can be found in the form $\alpha = \left(\frac{1+\epsilon_{-1}}{2}\right)^N \cdot T$, where $T$ is a trigonometric polynomial, and additionally, $\alpha(0) = 1$ and $|\alpha(x)|^2 + \left|\alpha\left(x + \frac{1}{2}\right)\right|^2 = 1, x \in \mathbb{R}$ hold. Then

$$|\alpha(x)|^2 = \cos^{2N}(\pi x) \cdot |T(x)|^2, \quad x \in \mathbb{R}$$

where $|T|^2$ a trigonometric polynomial. We can suppose that $|T|^2$ is an even function. Then

$$|T(x)|^2 = P(\cos 2\pi x), \quad x \in \mathbb{R}$$

where $P$ is an algebraic polynomial, and using $\cos 2\pi x = 1 - 2\sin^2 \pi x\,, x \in \mathbb{R}$ we have

$$|T(x)|^2 = Q(\sin^2 \pi x), \quad x \in \mathbb{R}$$

where $Q(x) = P(1-2x), x \in \mathbb{R}$.

Based on the formulas above we have to find the solutions of equation

$$\alpha(0) = 1, |\alpha(x)|^2 + \left|\alpha\left(x + \frac{1}{2}\right)\right|^2 = 1, x \in \mathbb{R}$$

in the form

$$|\alpha(x)|^2 = \cos^{2N}(\pi x) \cdot Q(\sin^2 \pi x), \quad x \in \mathbb{R}$$

Introducing the notation $y = \sin^2 \pi x$ our equation is equivalent with equation

$$(1-y)^N \cdot Q(y) + y^N \cdot Q(1-y) = 1, \quad 0 \le y \le 1$$

where $Q$ is an algebraic polynomial.

Its solutions are

$$Q_N(y) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} \cdot y^k, \quad y \in \mathbb{R}$$

If $R$ is a polynomial for which $R(y) + R(1-y) = 0$, $y \in \mathbb{R}$ then (non-negative) polynomials

$$Q(y) = Q_N(y) + y^N \cdot R(y), \quad y \in \mathbb{R}$$

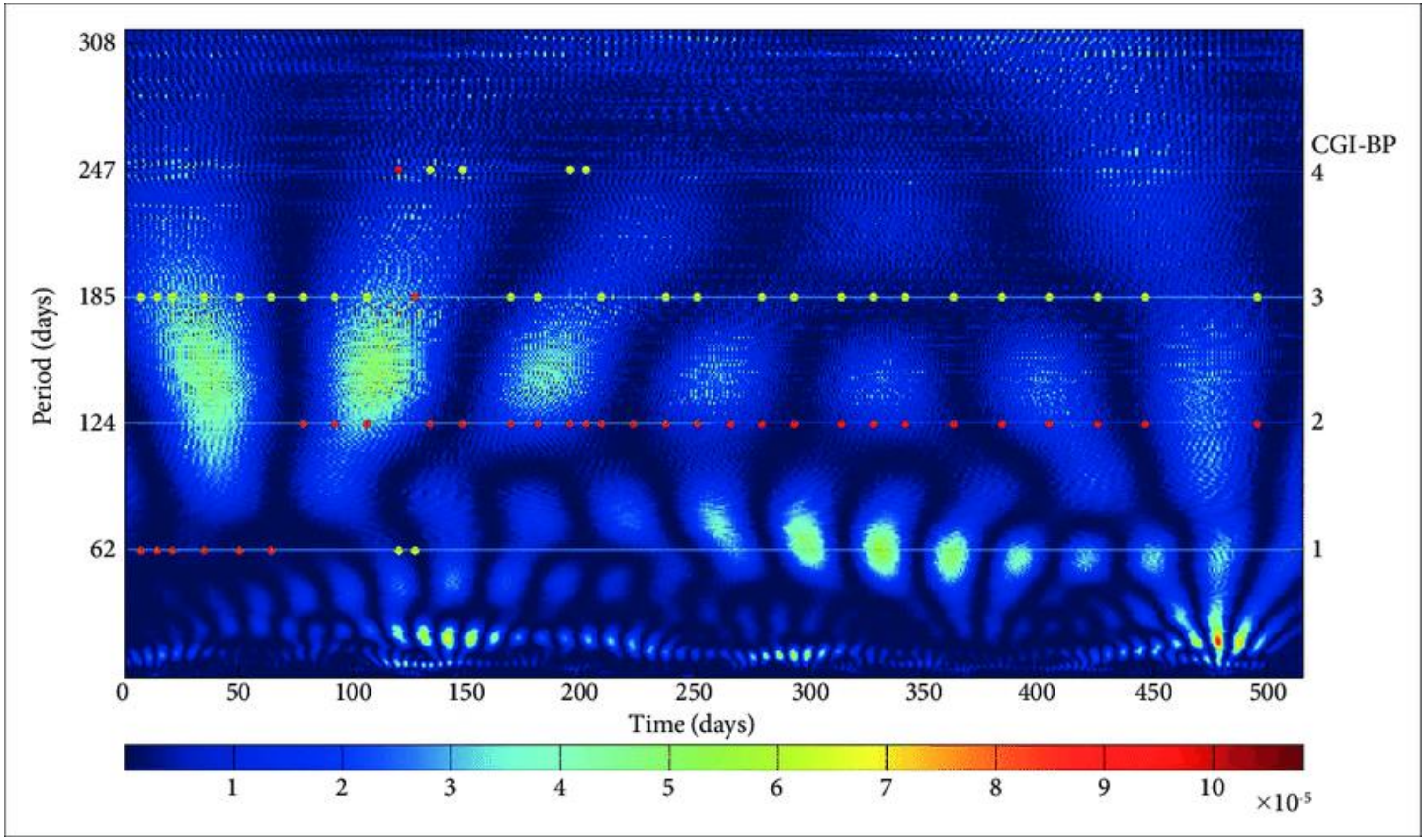
provide functions $\alpha$ in the form

$$|\alpha(x)|^2 = \cos^{2N}(\pi x) \cdot Q(\sin^2 \pi x), \;\; x \in \mathbb{R}.$$

Calculating $T$ we have the filter $\alpha$.

A scalogram is the absolute value of the continuous wavelet transform coefficients of a signal.

The empirical mode decomposition (EMD) is a data-adaptive multiresolution technique. EMD recursively extracts different resolutions from the data without the use of fixed functions or filters. EMD regards a signal as consisting of a fast oscillation superimposed on a slower one. After the fast oscillation is extracted, the process treats the remaining slower component as the new signal and again regards it as a fast oscillation superimposed on a slower one. The process continues until some stopping criterion is reached.




https://www.researchgate.net/figure/Continuous-wavelet-transform-scalogram-on-patients-self-mood-rating-Horizontal-axis_fig1_323455258

# 9th week – Questions

***Question***

What is the multiresolution analysis (MRA)?

***Answer***

A multiresolution analysis (MRA) or multiscale approximation (MSA) is the design method of most of the practically relevant discrete wavelet transforms (DWT) and the justification for the algorithm of the fast wavelet transform (FWT).

It was introduced in this context in 1988/89 by Stephane Mallat and Yves Meyer and has predecessors in the microlocal analysis in the theory of differential equations (the ironing method) and the pyramid methods of image processing.

Wavelets are organized using two parameters time and scale. Time k makes it possible to translate the forms for a given level; scale 2^j makes it possible to pass from a level j to the immediately lower level in the underlying tree.

Signals often consist of multiple physically meaningful components. Quite often, you want to study one or more of these components in isolation on the same time scale as the original data. Multiresolution analysis refers to breaking up a signal into components, which produce the original signal exactly when added back together. To be useful for data analysis, how the signal is decomposed is important. The components ideally decompose the variability of the data into physically meaningful and interpretable parts. The term multiresolution analysis is often associated with wavelets or wavelet packets, but there are non-wavelet techniques which also produce useful MRAs.

***Question***

What is the compact support of a function regarding multiresolution analysis (MRA)?

***Answer***

Functions with compact support on a topological space X are those whose closed support is a compact subset of X. If X is the real line, or n-dimensional Euclidean space, then a function has compact support if and only if it has bounded support, since a subset of {R} is compact if and only if it is closed and bounded.

## *Question*

Describe the properties of the multiresolution analysis!

## *Answer*

1. $X_n \subset X_{n-1}, n \in \mathbb{Z}$ — *monotonicity property*
2. $\overline{\bigcup_{n\in\mathbb{Z}} X_n} = X$ — *density property*
3. $\bigcap_{n\in\mathbb{Z}} X_n = \{0\}$ — *separability property*
4. $f \in X_n \quad \Leftrightarrow \quad \delta_2 f \in X_{n-1}, n \in \mathbb{Z}$ — *scaling property*
5. $X_0$ is generated by a Riesz-basis given by

$$\varphi_k = \tau_k \varphi, \quad k \in \mathbb{Z}$$

(The basis is invariant to integer translations).

# 9th week – Exercises

**Exercise**

Generate a signal in Matlab and create its FFT diagram and its scalogram by wavelet transform!

**Solution**

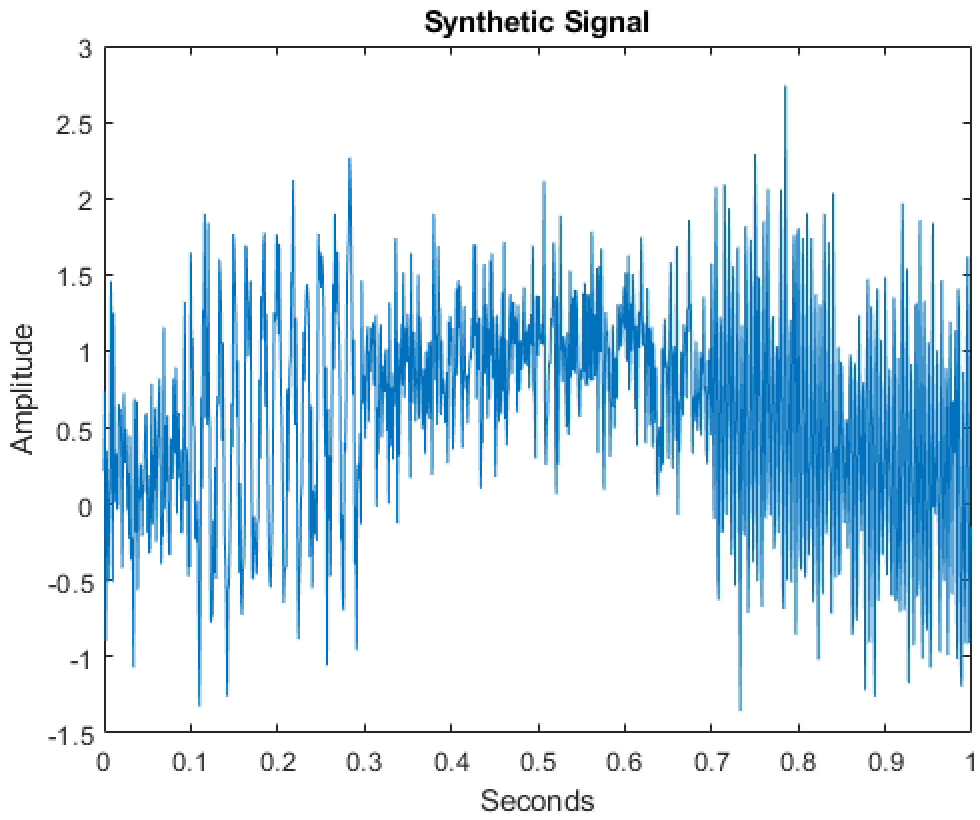
A possible solution is:

As a motivating example of the insights you can gain from an MRA, consider the following synthetic signal. The signal is sampled at 1000 Hz for one second.

Matlab code:

```
Fs = 1e3;

t = 0:1/Fs:1-1/Fs;

comp1 = cos(2*pi*200*t).*(t>0.7);

comp2 = cos(2*pi*60*t).*(t>=0.1 & t<0.3);
```

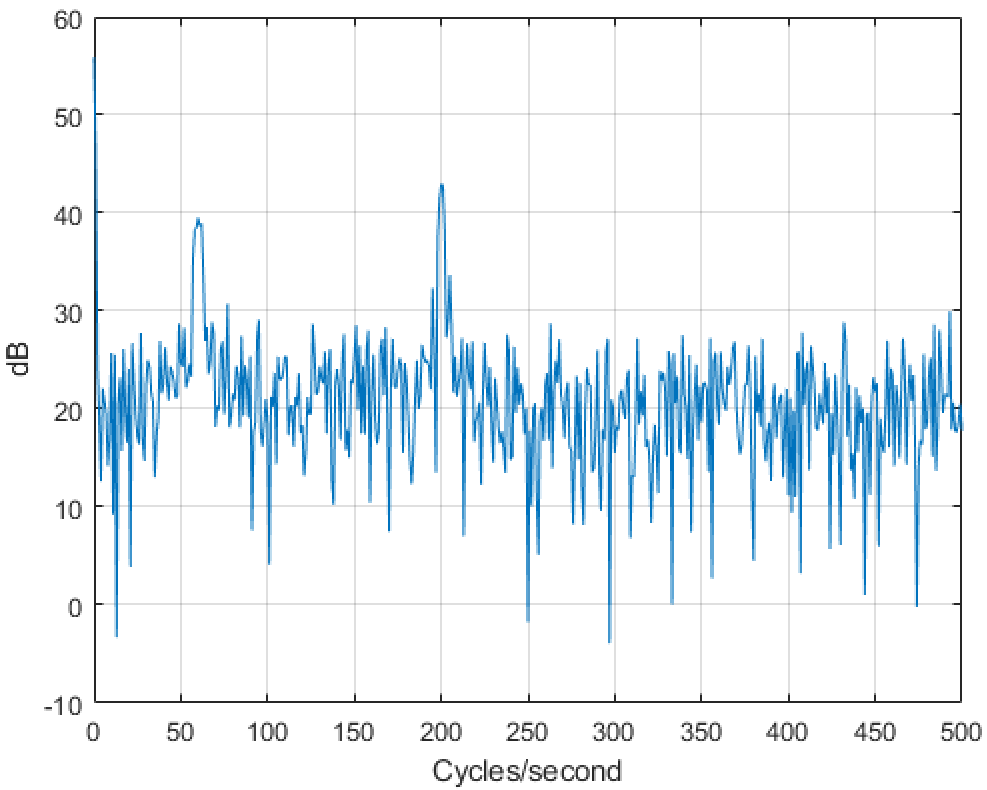
```
trend = sin(2*pi*1/2*t);

rng default

wgnNoise = 0.4*randn(size(t));

x = comp1+comp2+trend+wgnNoise;

plot(t,x)

xlabel('Seconds')

ylabel('Amplitude')

title('Synthetic Signal')
```




The signal is explicitly composed of three main components: a time-localized oscillation with a frequency of 60 cycles/second, a time-localized oscillation with a frequency of 200 cycles/second, and a trend term. The trend term here is also sinusoidal but has a frequency of 1/2 cycle per second, so it completes only 1/2 cycle in the one-second interval. The 60 cycles/second or 60 Hz oscillation occurs between 0.1 and 0.3 seconds, while the 200 Hz oscillation occurs between 0.7 and 1 second.

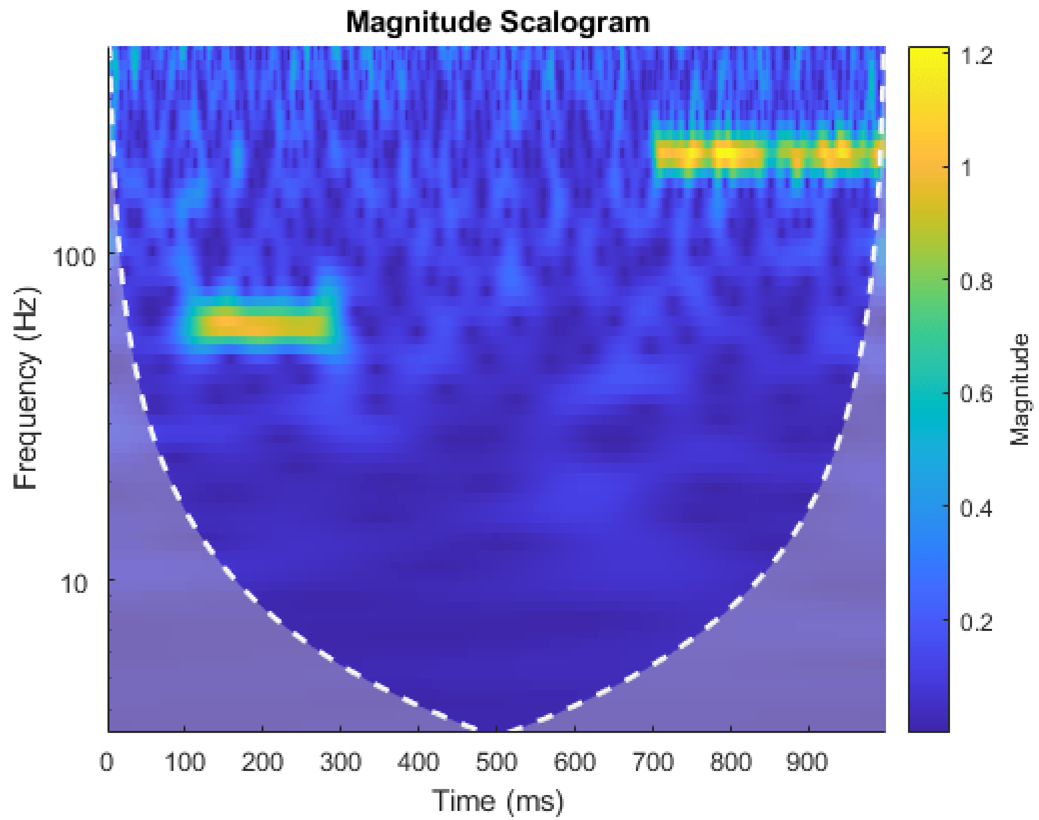
To generate its FFT spectrum the Matlab code is:

```
xdft = fft(x);

N = numel(x);

xdft = xdft(1:numel(xdft)/2+1);

freq = 0:Fs/N:Fs/2;

plot(freq,20*log10(abs(xdft)))

xlabel('Cycles/second')

ylabel('dB')

grid on
```




From the frequency analysis, it is much easier for us to discern the frequencies of the oscillatory components, but we have lost their time-localized nature. It is also difficult to visualize the trend in this view.

To gain some simultaneous time and frequency information, we can use a time-frequency analysis technique like the continuous wavelet transform (cwt).




The time-frequency view provides useful information, but in many situations you would like to separate out components of the signal in time and examine them individually. Ideally, you want this information to be available on the same time scale as the original data.

Multiresolution analysis accomplishes this. In fact, a useful way to think about multiresolution analysis is that it provides a way of avoiding the need for time-frequency analysis while allowing you to work directly in the time domain.

## Exercise

Create the wavelet MRA of the signal above in the previous exercise!

## Solution

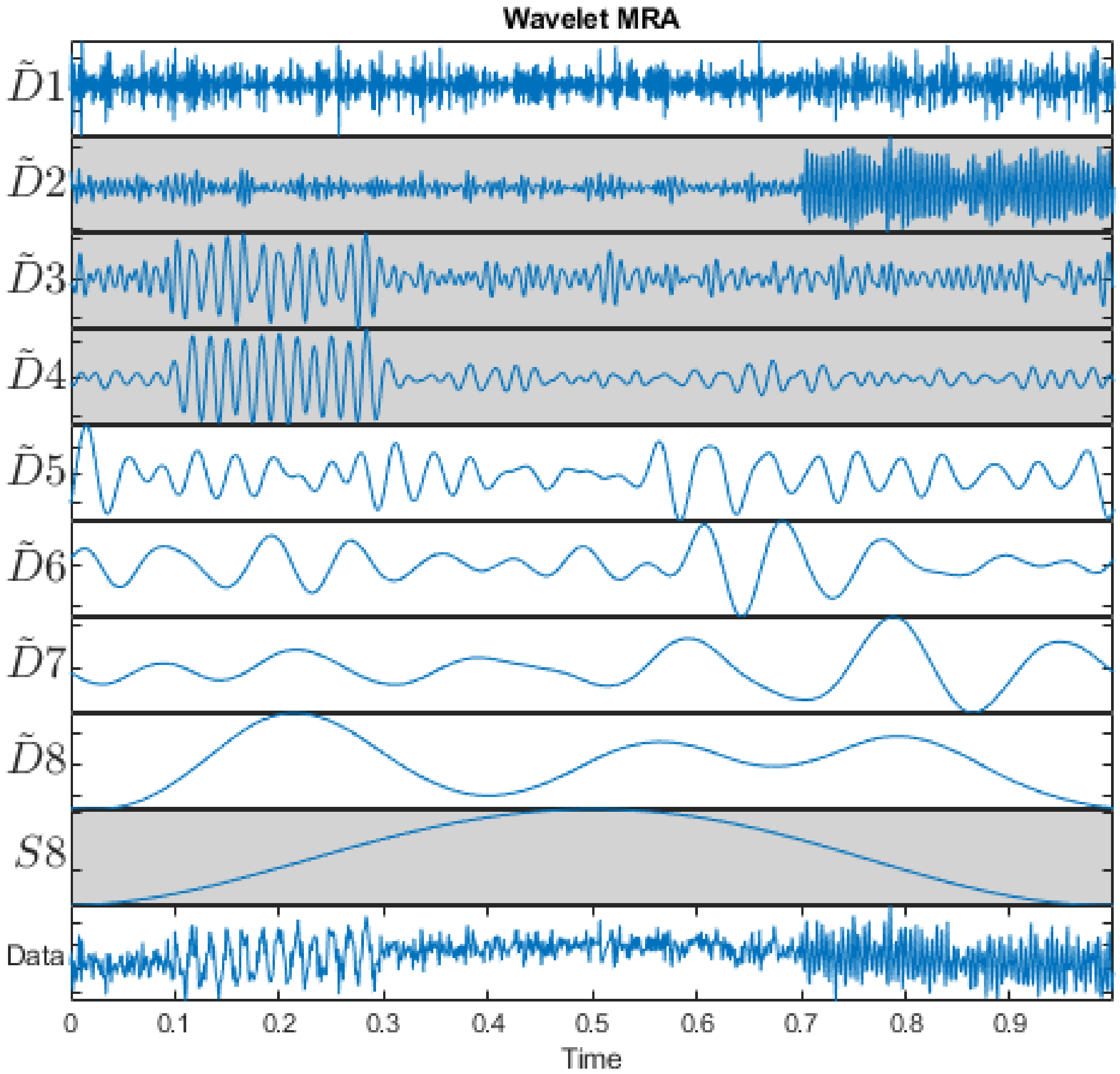
Real-world signals are a mixture of different components. Often you are only interested in a subset of these components. Multiresolution analysis allows you to narrow your analysis by separating the signal into components at different resolutions.

Extracting signal components at different resolutions amounts to decomposing variations in the data on different time scales, or equivalently in different frequency bands (different rates of oscillation). Accordingly, you can visualize signal variability at different scales, or frequency bands simultaneously.

Analyze and plot the synthetic signal using a wavelet MRA. The signal is analyzed at eight resolutions or levels.

Matlab code:

```
mra = modwtmra(modwt(x,8));

helperMRAPlot(x,mra,t,'wavelet','Wavelet MRA',[2 3 4 9])
```


Wavelet MRA
D̃1
D̃2
D̃3
D̃4
D̃5
D̃6
D̃7
D̃8
S8
Data
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Time

## Exercise

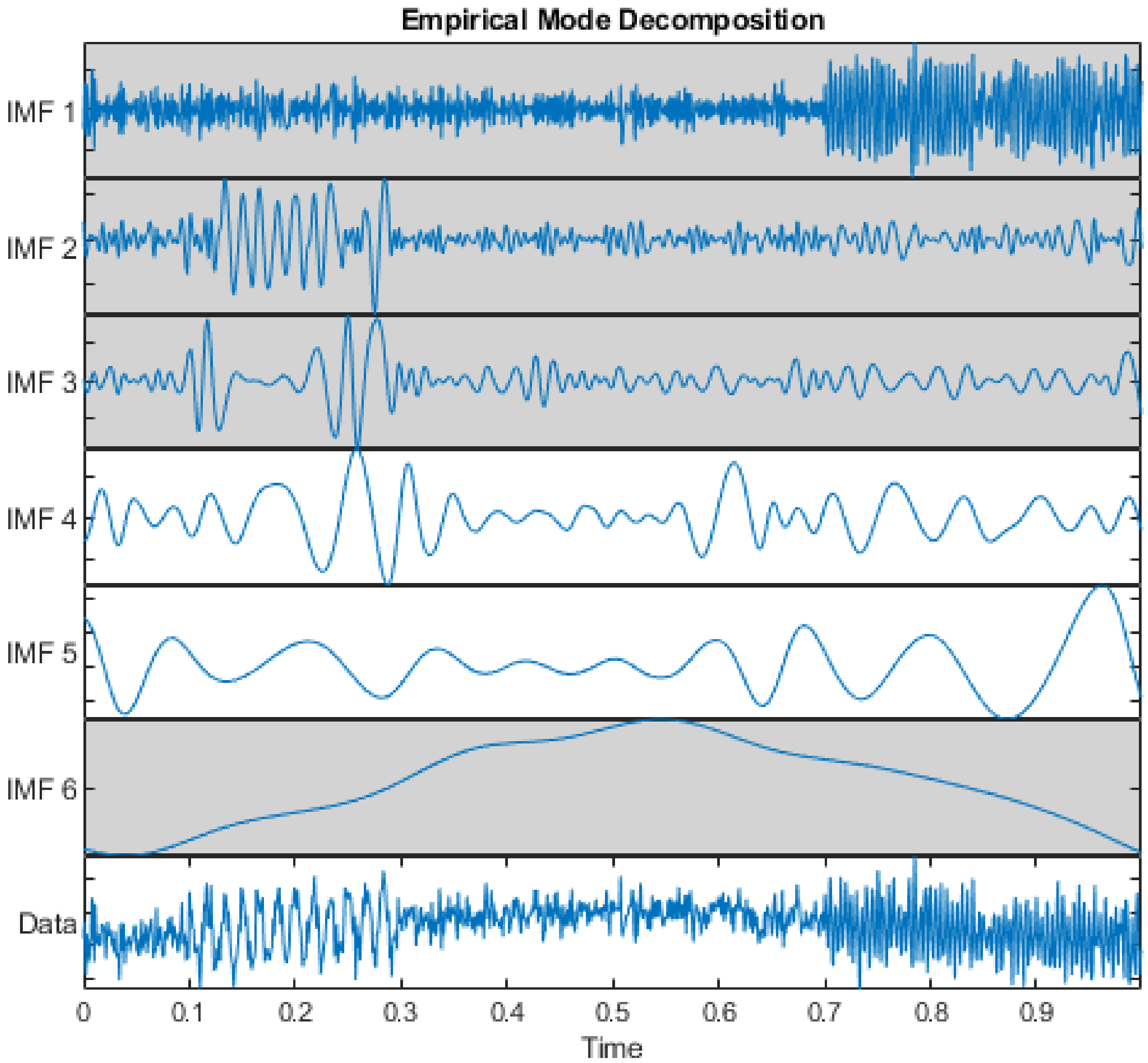
Create the EMD of the signal above! Plot the EMD as well!

## Solution

While EMD does not use fixed functions like wavelets to extract information, the EMD approach is conceptually very similar to the wavelet method of separating the signal into details and approximations and then separating the approximation again into details and an approximation. The MRA components in EMD are referred to as intrinsic mode functions (IMF).

Matlab code:

```
[imf_emd,resid_emd] = emd(x);

helperMRAPlot(x,imf_emd,t,'emd','Empirical Mode Decomposition',[1 2 3 6])
```


Empirical Mode Decomposition
IMF 1
IMF 2
IMF 3
IMF 4
IMF 5
IMF 6
Data
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Time

While the number of MRA components is different, the EMD and wavelet MRAs produce a similar picture of the signal. This is not accidental.

In the EMD decomposition, the high-frequency oscillation is localized to the first intrinsic mode function (IMF 1). The lower frequency oscillation is localized largely to IMF 2, but you can see some effect also in IMF 3. The trend component in IMF 6 is very similar to the trend component extracted by the wavelet technique.

# 10th week

# 10 Wavelet Transforms in Machine Fault Diagnostics

**Wavelet design procedure**

Although, the user-ready wavelets are effective in many cases, the also the design of new wavelets meeting certain criteria is necessary in some cases for the more efficient diagnosis in specific cases. A natural requirement is to find wavelet functions having special shape, for instance being ‘similar’ to a given transient in the analyzed signal.

Some direct calculation algorithms are available in the literature, Chapa and Rao introduce an algorithm for designing a mother wavelet $\psi$ such that it matches a signal of interest and wavelets $\psi_k^n = \frac{1}{\sqrt{2^n}}\psi\left(\frac{1}{2^n}x - k\right)$ form an orthonormal Riesz-basis of $X$. Supposing band-limited spectrum of the scaling and wavelet functions they give the scaling function by discrete matching procedure from the discrete spectrum of the desired transient appearing in the signal of interest.

Suppose that we have a sample

$$\mathbf{W} = \left\{ \left| F\left(k \cdot \frac{2\pi}{2^\ell}\right) \right|^2 \middle| k = \left[\frac{2^\ell}{3}\right], \dots, \left[\frac{2^{\ell+2}}{3}\right] \right\}$$

where $F$ is the spectrum of the signal.

Let us denote the matched wavelet spectrum as $\Psi$. Through a least squares optimization process, Theorem 5 in [28] gives values

$$\mathbf{Y} = \left\{ \left| \Psi\left(k \cdot \frac{2\pi}{2^{\ell}}\right) \right|^2 : k = \left[\frac{2^l}{3}\right], \ldots, \left[\frac{2^{l+2}}{3}\right] \right\}$$

using the error function

$$E(a, \mathbf{Y}) = \frac{(\mathbf{W} - a\mathbf{Y})^T \cdot (\mathbf{W} - a\mathbf{Y})}{\mathbf{W}^T \mathbf{W}}.$$

According to the theorem, the optimal wavelet power spectrum is given by

$$\mathbf{Y} = \frac{1}{a}\mathbf{W} + \mathbf{A}^T (\mathbf{A}\mathbf{A}^T)^{-\mathbf{1}} \left(\mathbf{1} - \frac{1}{a}\mathbf{A}\mathbf{W}\right)$$

where $\mathbf{A}$ is an $L \times 2^l$ matrix and

$$a = \frac{\mathbf{1}^T (\mathbf{A}\mathbf{A}^T)^{-\mathbf{1}} \mathbf{A}\mathbf{W}}{\mathbf{1}^T (\mathbf{A}\mathbf{A}^T)^{-\mathbf{1}} \mathbf{1}}.$$

Theorem 4 in [28] says that, in an orthonormal MRA, values of $|\Phi|$ can be calculated from $|\Psi|$ values using the equality

$$\left| \Phi\left(\frac{k\pi}{2^{\ell}}\right) \right|^2 = \sum_{i=0}^{\ell} \left| \Psi\left(\frac{2k\pi}{2^i}\right) \right|^2 \quad \text{for } k \neq 0.$$

The necessary and sufficient condition on $Y(k) = \left| \Psi\left(k \cdot \frac{2\pi}{2^{\ell}}\right) \right|^2, k \in \mathbb{Z}$ to guarantee that $|\Phi(k)|$, provided by Theorem 4, generates an orthonormal MRA is

$$\sum_{i=0}^{\ell}\sum_{m=-\infty}^{\infty} Y\left(\frac{2^{\ell}}{2^{i}}\cdot(k+2^{\ell+1}\cdot m)\right)=1$$

where

$$\frac{2^{\ell-1}}{3}<\frac{2^{\ell}}{2^{i}}\cdot\left(k+2^{\ell+1}\cdot m\right)<\frac{2^{\ell+2}}{3}$$

To determine the specific set of constraint equations, first expand the summation over $i$.

Condition $\sum_{i=0}^{\ell}\sum_{m=-\infty}^{\infty} Y\left(\frac{2^{\ell}}{2^{i}}\cdot(k+2^{\ell+1}\cdot m)\right)=1$ generates a set of $L$ linear equality constraints in $Y(k)$ of the form

$$\sum_{i=1}^{L}\alpha_{ik}\cdot Y(k)=1 \quad \text{for} \quad k=\left\lceil\frac{2^{\ell}}{3}\right\rceil,\dots,\left\lceil\frac{2^{\ell+2}}{3}\right\rceil$$

where $\alpha_{ik}\in\{0,1,2\}$. The matrix form of the condition is

$$\mathbf{AY}=\mathbf{1}$$

where $\mathbf{A}=(\alpha_{ik})$ is an $L\times 2^{l}$ matrix and $\mathbf{1}$ is a $L\times 1$ vector given by $\mathbf{1}=(1,\dots,1)$.

Let $\theta_{\Phi}(\omega)$, $\theta_{\Psi}(\omega)$, $\theta_{\mathrm{H}}(\omega)$, and $\theta_{\mathrm{F}}(\omega)$ be the phase functions of $\Phi$, $\Psi$, $H$ and $F$, respectively, where $H$ is the spectrum of sequence of coefficients $(h_n)$ in equality

$$\varphi(x) = 2 \cdot \sum_{k=-\infty}^{\infty} h_k \cdot \varphi(2x - k).$$

Let us introduce functions

$$\Gamma_\Phi(\omega) = \Lambda_\Phi(\omega) + \frac{1}{2},\quad \Gamma_\Psi(\omega) = \Lambda_\Psi(\omega) + \frac{1}{2}, \text{and } \Gamma_\mathrm{F}(\omega) = \Lambda_\mathrm{F}(\omega) + \frac{1}{2}$$

where

$$\Lambda_\Phi(\omega) = \frac{\mathrm{d}\theta_\Phi(\omega)}{\mathrm{d}\omega},\quad \Lambda_\Psi(\omega) = \frac{\mathrm{d}\theta_\Psi(\omega)}{\mathrm{d}\omega},\quad \lambda(\omega) = \frac{\mathrm{d}\theta_\mathrm{H}(\omega)}{\mathrm{d}\omega}$$

are the so-called group delays of Φ, Ψ, and H, respectively.

A least squares optimization procedure is presented in [81] for matching $\Gamma_\Psi$ to $\Gamma_\mathrm{F}$ which provides $\Lambda_\Phi$ and $\Lambda_\Psi$ as well. In calculations the periodic function $\lambda$ has a central role, its period $\lambda_T$ is modelled with an $R$-degree polynomial $\lambda_T(\omega) = \sum_{r=0}^{R/2} c_r \cdot \omega^{2r}, \omega \in [-\pi, \pi]$ having only even exponents.

The discrete form for $\lambda$ can now be written in vector notation

$$\boldsymbol{\lambda} = \mathbf{B}\boldsymbol{c}$$

where $\boldsymbol{\lambda}$ is an $N \times 1$ vector, **c** is an $\left(\frac{R}{2} + 1\right) \times 1$ vector, and **B** is an $N \times \left(\frac{R}{2} + 1\right)$ matrix whose elements depend on the parameter settings (sampling time $T$ and sample size $N$) used when sampling $F$. Using this form of $\boldsymbol{\lambda}$ we have

$$\mathbf{\Gamma}_\Psi = \mathbf{D}_\Psi \mathbf{c}$$

where matrix $\mathbf{D}_\Psi$ can be calculated from $\mathbf{B}$.

$\mathbf{\Gamma}_\Psi$ matching $\mathbf{\Gamma}_\mathrm{F}$ can be obtained minimizing the error function

$$\gamma = \sum_{n=-N/2}^{N/2-1} (\mathbf{\Gamma}_\mathrm{F}(n) - \mathbf{\Gamma}_\Psi(n))^2$$

in a least squares sense.

To consider the passband for spectra, the error function needs to be normalized by the weighting function $\Omega(n) = \frac{Y(n)}{\sum Y(n)}$, where $Y(n)$ are the elements of $\mathbf{Y}$ provided by the amplitude matching algorithm:

$$\gamma_\Omega = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \left(\Omega(n) \cdot (\mathbf{\Gamma}_\mathrm{F}(n) - \mathbf{\Gamma}_\Psi(n))\right)^2$$

The vector $\tilde{\boldsymbol{c}}$ minimizing $\gamma$ can be given as

$$\tilde{\boldsymbol{c}} = \left(\overline{\mathbf{D}}_\Psi{}^T \overline{\mathbf{D}}_\Psi\right)^{-1} \overline{\mathbf{D}}_\Psi{}^T \overline{\mathbf{\Gamma}}_\mathrm{F},$$

where the elements of $\overline{\mathbf{\Gamma}}_\mathrm{F}$ are the non-zero values of $\{\Omega(n) \cdot \mathbf{\Gamma}_\mathrm{F}(n)\}$ and the elements of $\overline{\mathbf{D}}_\Psi$ are the corresponding non-zero values of $\{\Omega(n) \cdot d_{n,r}\}$.

Using $\tilde{\boldsymbol{c}}$ functions $\boldsymbol{\lambda}$, $\boldsymbol{\Lambda}_\Psi$ and $\boldsymbol{\Lambda}_\Phi$ and then the discrete phases of Ψ and Φ can be calculated. Combining these discrete phases with the magnitudes we get the estimate of Ψ and Φ which satisfy all conditions for an orthonormal MRA.

The impulse responses, $h$ and $g$, corresponding to the matched wavelet and its scaling function can be found using $\Phi(\omega) = H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right), \Psi(\omega) = G\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right)$ and the inverse Fourier transform.

## Wavelet selection with Energy-to-Shannon Entropy Criteria

The Energy-to-Shannon Entropy Criteria is used to rank wavelets on the basis of scalograms. The Energy-to-Shannon Entropy ratio is a combination of the energy content

$$E(n) = \sum_{i}^{m} \left|C_{n,i}\right|^2$$

and the Shannon entropy

$$S(n) = -\sum_{i=1}^{m} p_i \log_2 p_i$$

related to the wavelet coefficients $C_{n,i}$ , where $m$ is the number of the wavelet coefficients of $n$-th scale and $(p_1, \dots, p_n)$ is the energy distribution of the wavelet coefficients defined by

$$p_i = \left|C_{n,i}\right|^2 / E(n).$$

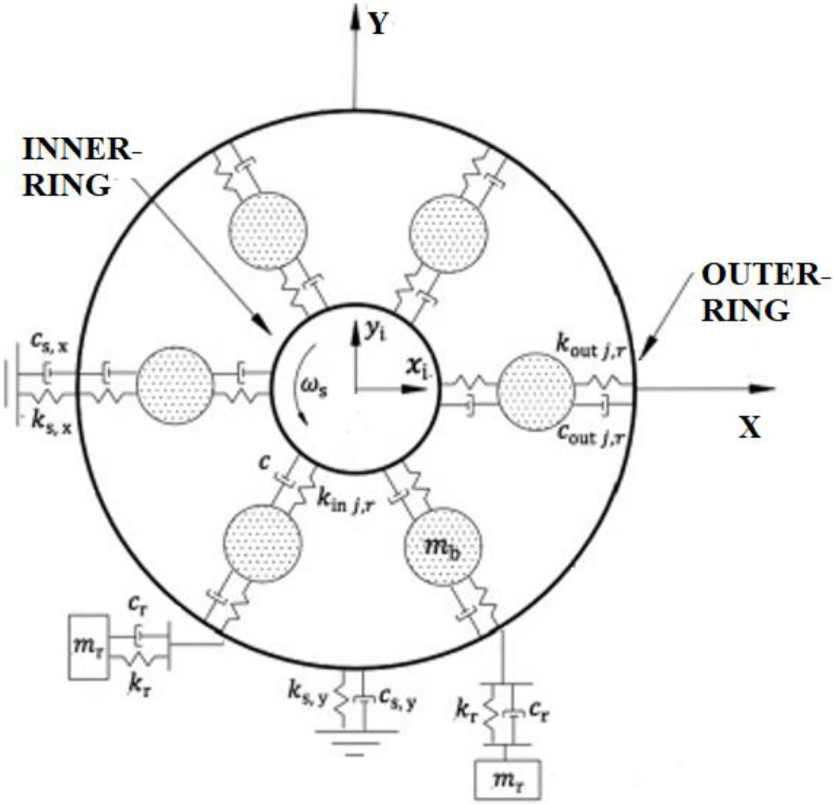
The indicator

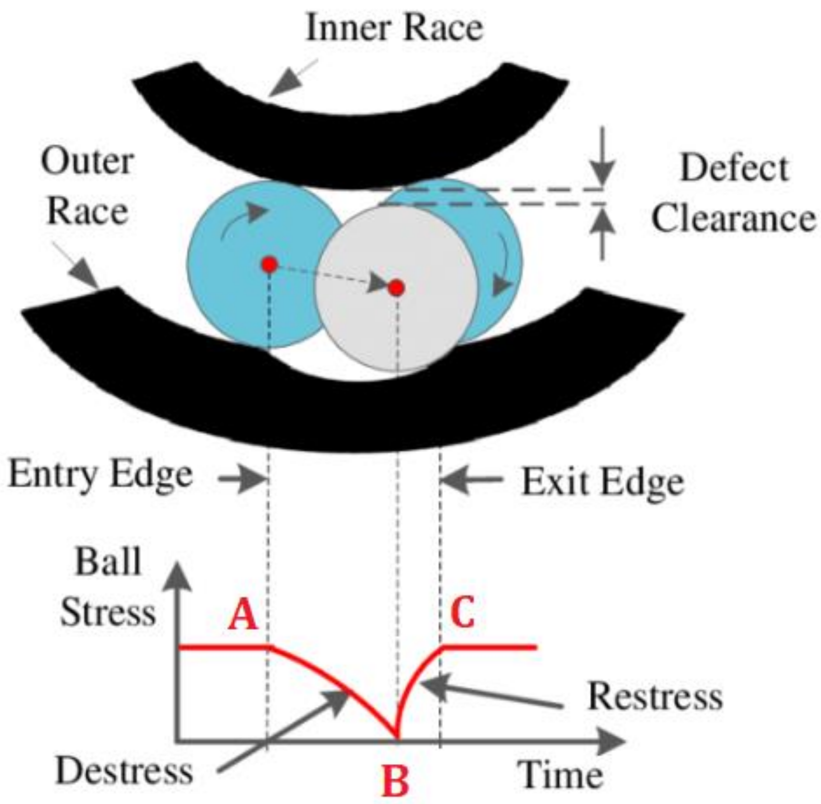
$$\xi(n) = E(n)/S(n)$$

is used to choose the best wavelet for diagnosis of a special fault. [58]

## Fault size estimation

Analysis of the entry point and the exit point are necessary for fault size estimation. Figure below represents the entry and exit events when the roller contacts the edges and the bottom of the fault. Ball stress varies during the process, however in this experiment not the mechanical stress is measured but the vibration acceleration value which is proportional with the stress according the Newton’s second law and Hooke’s law. Linear and isotropic material model is supposed with Poisson values of 0.33 of the 100Cr6 material. Bearing behaves as a mass-spring-damper system with weak damping which creates transient vibration waves when hit the fault.



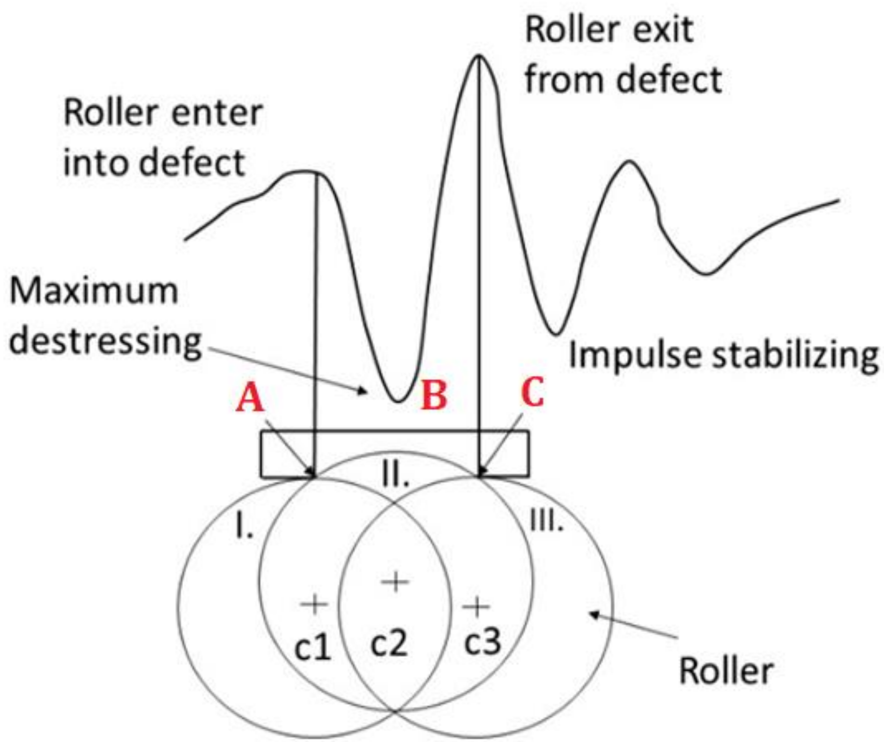




At point *A*, the roller strikes the rectangular shape grinding defect with high impact which results in re-stressing and high impulse in signal. After this event the roller remains in contact with the defect for some time. When the roller comes in contact with the point *B* it again generates high amplitude in the signal and beyond point B progressive decrease in amplitude of signal is observed due to elastic damping of the bearing element.

Fault size is calculated by the time “distance” between points A and C. This method is very useful because the defects width of the bearing can be determined only from the vibration signature. Bearing defects generate transient impulses in the vibration signal when the rollers pass through the defects. The bearing fault frequencies can be calculated by numerical way: bearing pass frequency of outer race (BPFO), bearing pass frequency of inner race (BPFI), fundamental train frequency (FTF), ball spin frequency (BSF). For instance, FTF can be calculated as:

$$FTF = \frac{f_r}{2}\left(1 - \frac{d}{D}cos\Phi\right)$$

where $f_r$ is the rotational frequency of the shaft, $\Phi$ is the contact angle, $d$ is the inner ring diameter, $D$ is the outer ring diameter. The defect can be calculated, where $D_{OI}$ is the outer ring diameter on the raceways, $\Delta t$ is the time duration between the contact points of the bearing elements.




For fault size estimation fault frequencies are calculated which in this experiment are $BPFO = 206.18\,Hz$, $BPFI = 287.15\,Hz$, $FTF = 12.88\,Hz$, $BSF = 89.96\,Hz$ in this experiment at 1800 1/min:

$$L_{OD} = \pi \cdot \Delta t \cdot D_{OI} \cdot FTF = 1713.74 \cdot \Delta t$$

Nine different wavelets are compared to reveal the grinding faults. According to the literature overview it was found that these wavelets were efficient for bearing fault analysis in general bearing fault diagnostics. Values of the Energy-to-Shannon Entropy ratios are in Table.

| E/S | OR1 | OR2 | OR3 | OR4 | Mean |
|---|---|---|---|---|---|
| Sym2 | 59.96 | 80.20 | 100.87 | 109.16 | 87.55 |
| Sym5 | 65.58 | 95.37 | 117.07 | 119.92 | 99.48 |
| Sym8 | 82.81 | 113.77 | 120.16 | 118.48 | 108.81 |
| db02 | 60.91 | 81.09 | 101.12 | 113.46 | 89.14 |
| db06 | 71.74 | 89.02 | 120.42 | 117.40 | 99.65 |
| db10 | 77.76 | 104.69 | 120.34 | 120.45 | 105.81 |
| db14 | 85.02 | 120.03 | 121.37 | 123.78 | 112.55 |
| Meyer | 92.31 | 160.31 | 126.20 | 105.70 | 121.13 |
| Morlet | 113.15 | 194.15 | 142.18 | 138.14 | 146.90 |

It is observed that Morlet wavelet provided the highest value that indicates to be the most efficient wavelet from the nine wavelets for both fault detection and fault size estimation of the special grinding defect.

To determine the defect size MRA is applied by filter banks which is a design method of most of the practically relevant discrete wavelet transforms.

In the case of No. 30205 tapered roller test bearing BPFO is 206.18 $Hz$. Down to 3rd level, where transient impulse is analysed for defect width estimation, wavelet band is 1.25 $kHz$ which is more than 3 times bigger than BPFO.

The raw signal is too noisy to detect entry and exit points of the defect but wavelet decomposition makes it possible to analyse the entry and exit events. Using Energy-to-Shannon Entropy Criteria we obtain the best wavelet to determine the fault size from the vibration signature.

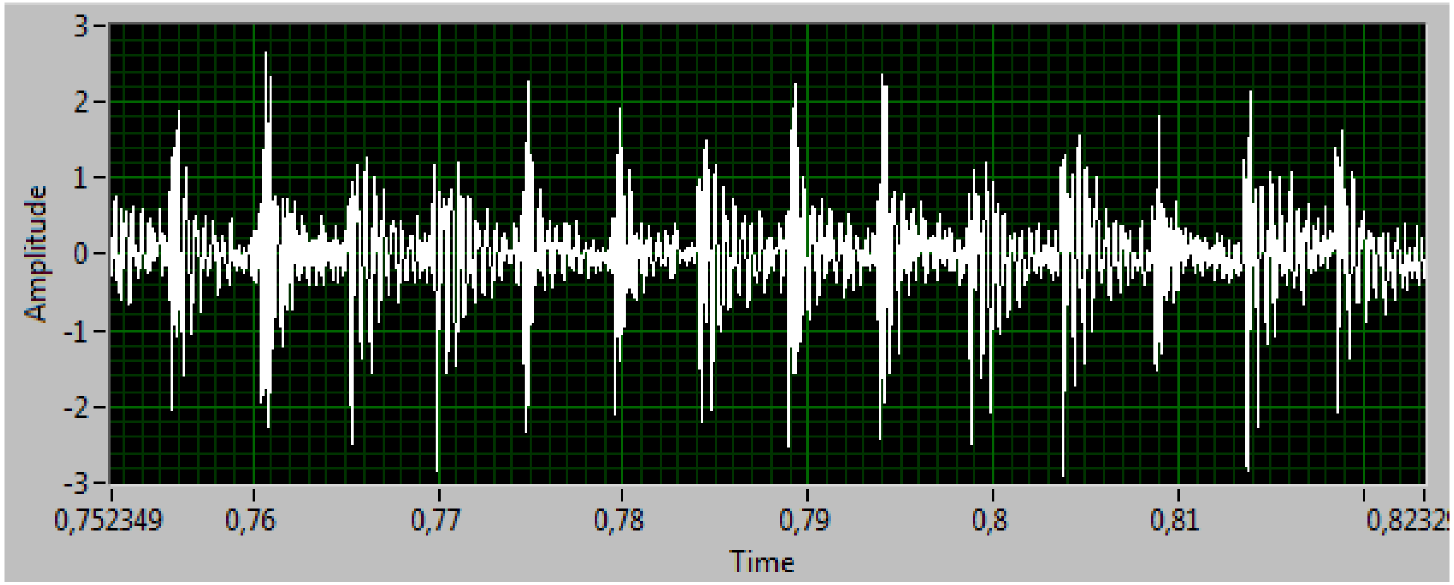
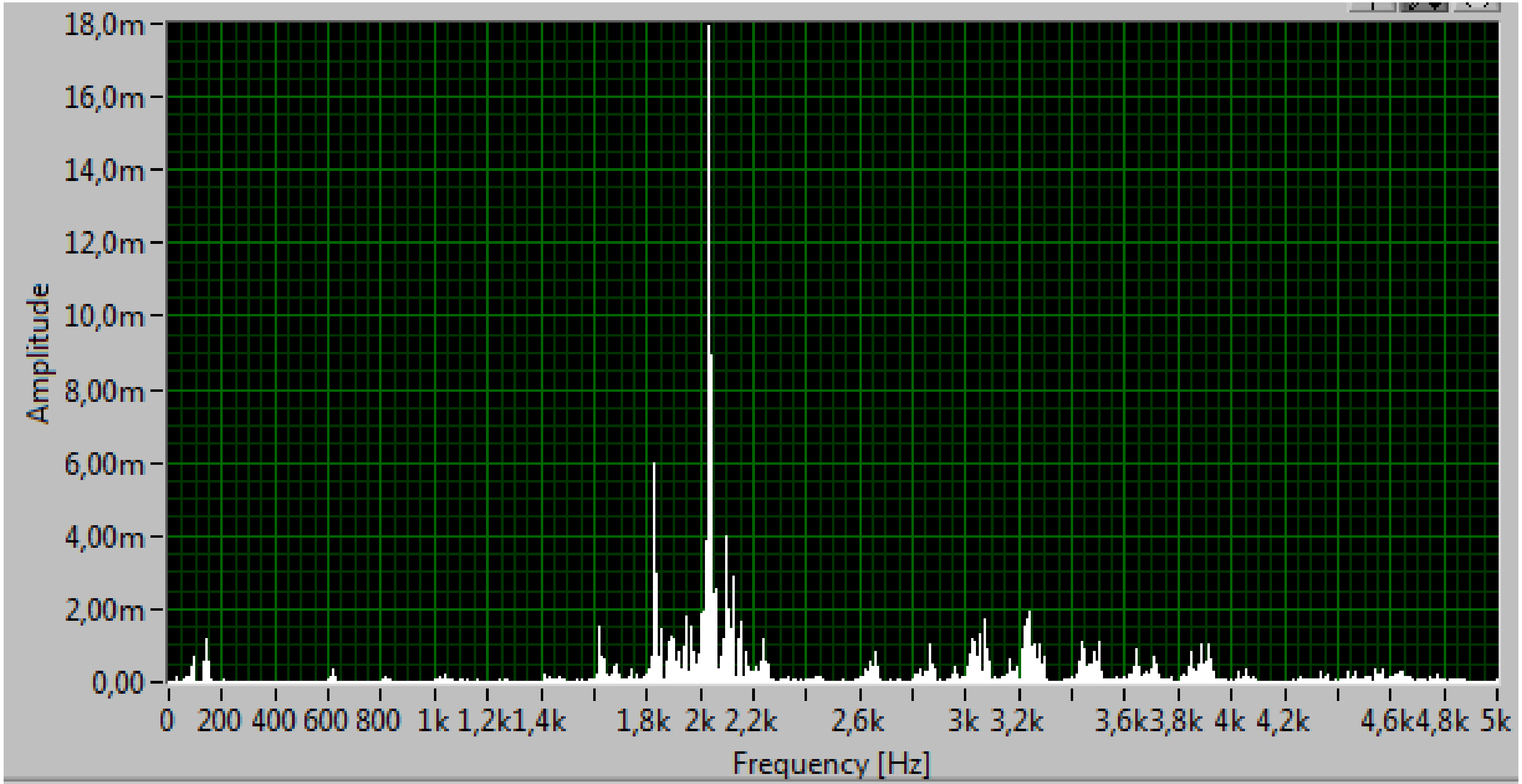


Figure presents the time-domain signal of outer race defect of 0.6311 $mm$. The highest periodic transient impulse related energy content of the burst occurs at 2.09 $kHz$ that causes 5 $ms$ rate of periodicity which is equal to 206.18 $Hz$ BPFO frequency. The spectrum was determined in all outer rings with different fault sizes and they showed similar manner around the peak at 2.09 $kHz$ as it can be seen in Figure 25.

Multiresolution analysis is made in order to obtain precise frequency analysis. Figure 26 presents the wavelet decomposition tree. Higher decomposition is not necessary because it might not reveal any further information of the signal. Regarding the BPFO frequency analysis was made at 3rd detail level (cD3) from 1.25 $kHz$ to 2.5 $kHz$.



18,0m
16,0m
14,0m
12,0m
10,0m
8,00m
6,00m
4,00m
2,00m
0,00
Amplitude
0 200 400 600 800 1k 1,2k 1,4k 1,8k 2k 2,2k 2,6k 3k 3,2k 3,6k 3,8k 4k 4,2k 4,6k 4,8k 5k
Frequency [Hz]

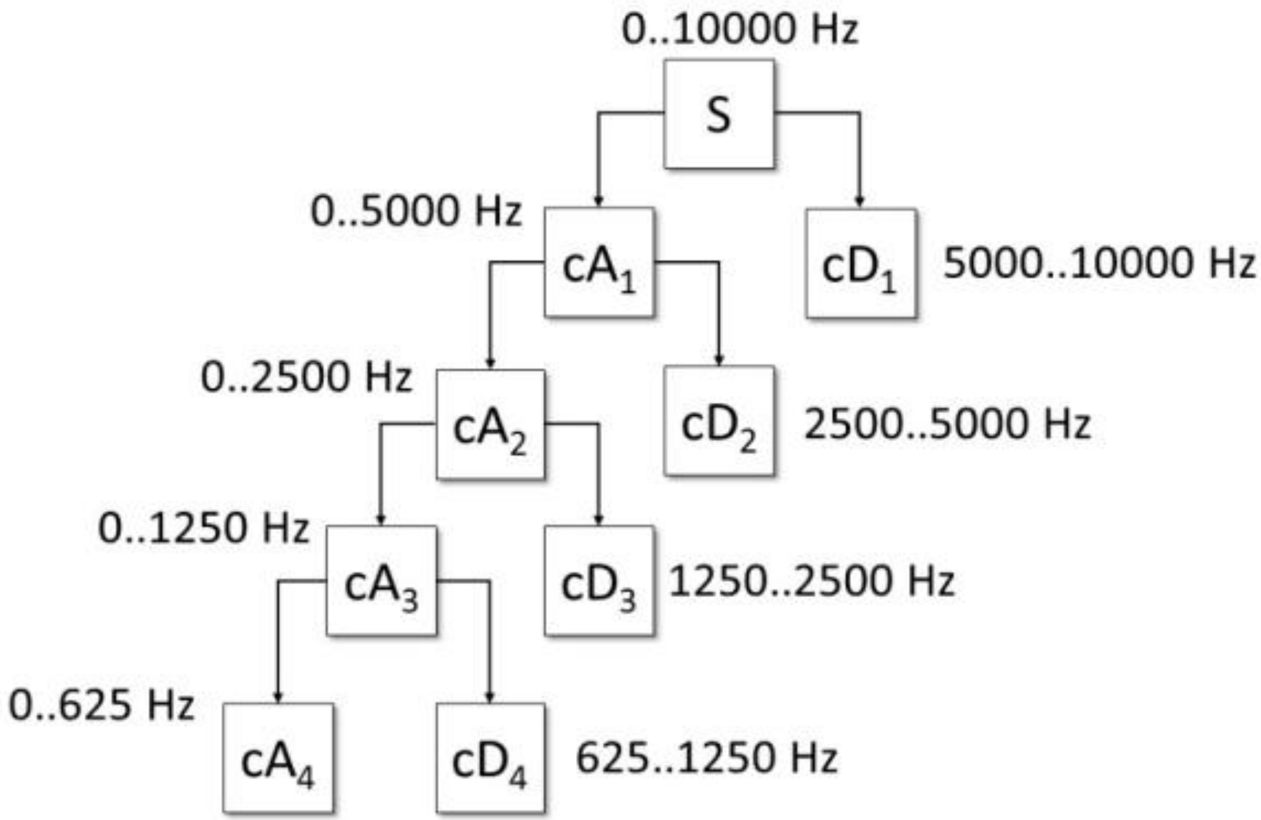
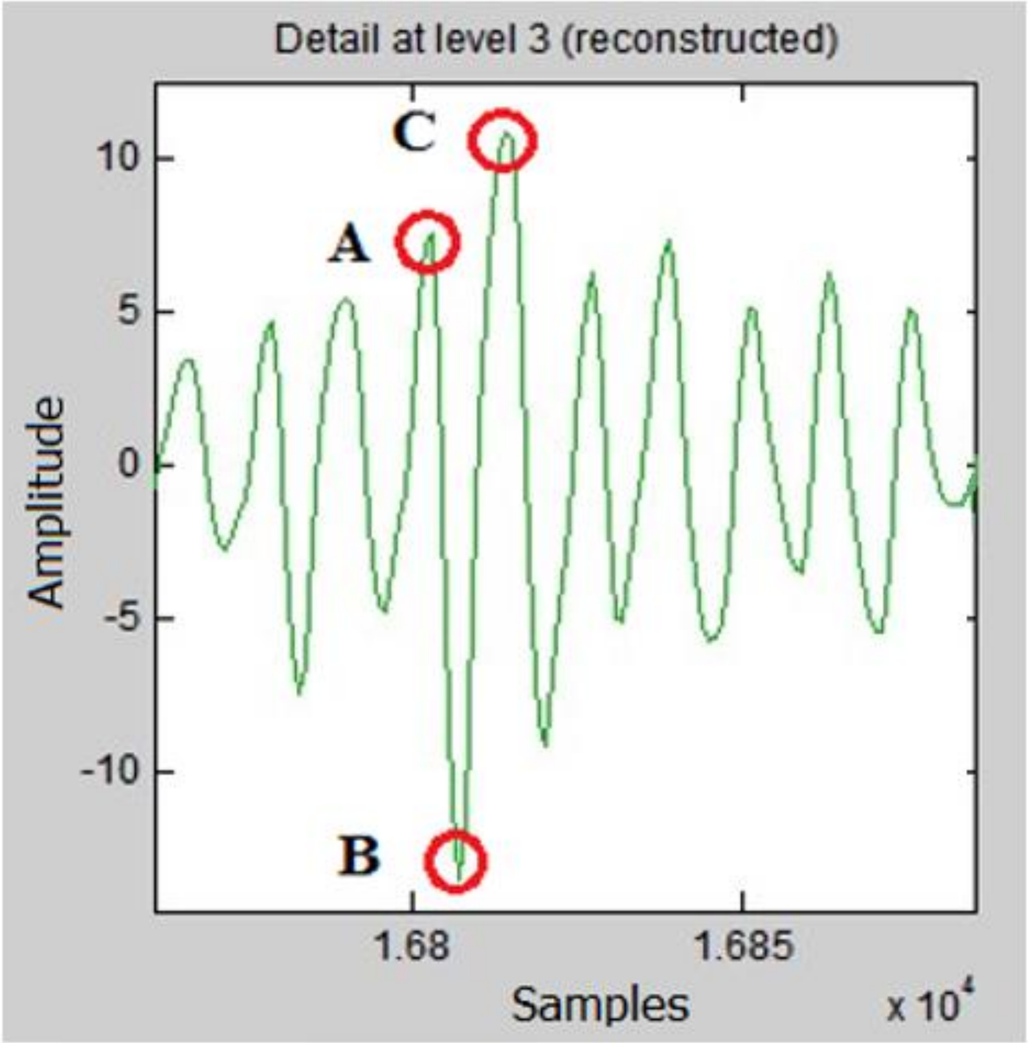
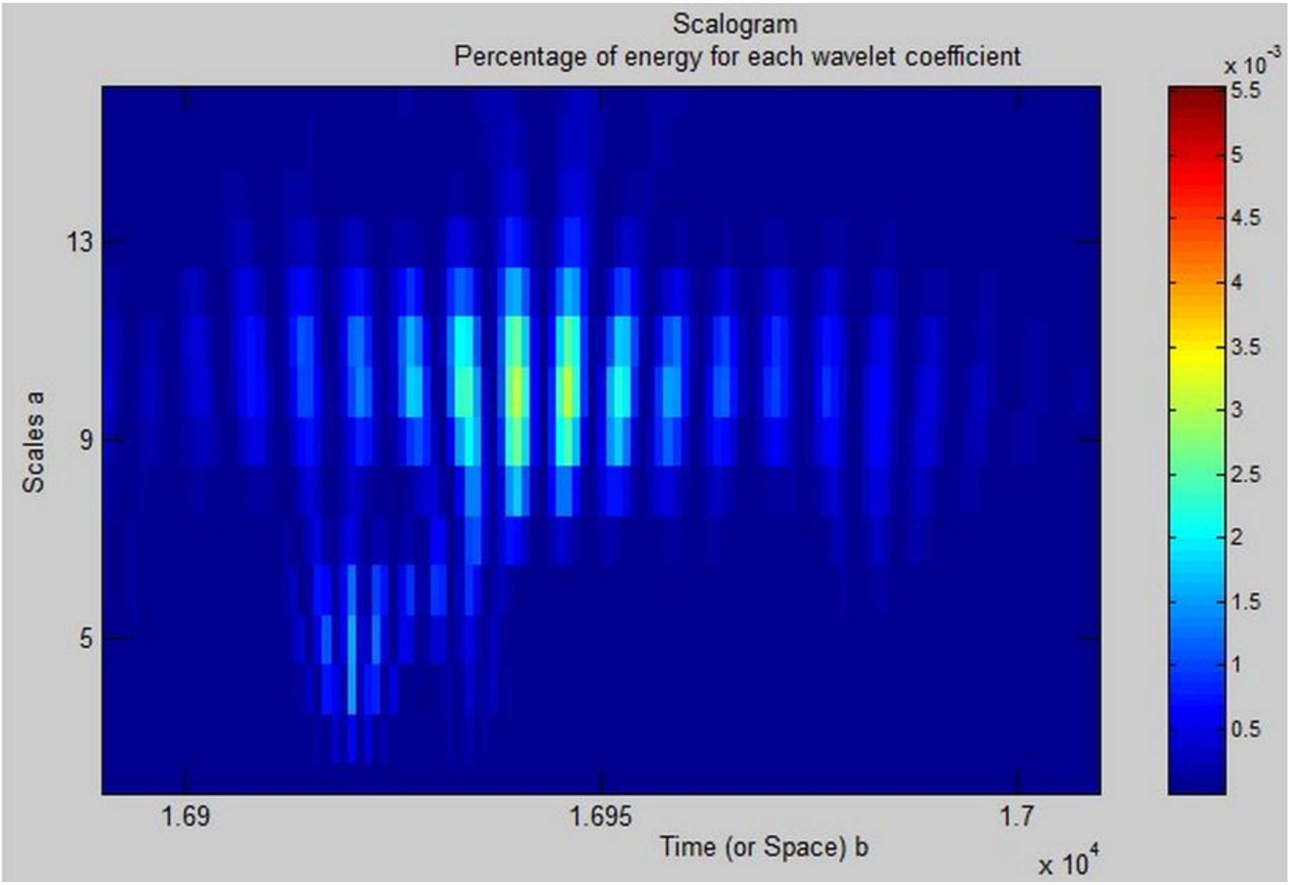



Figure below illustrates the measured transient signal. In the experiment 20 impulses were measured and the average time values were determined between the entry and exit points.


Detail at level 3 (reconstructed)
Amplitude
10
5
0
-5
-10
1.68
1.685
Samples
x 10^4
A
B
C




There is variation in data points as each roller cross over the defect. Average data points are calculated for estimating the time taken by roller to pass over the grinding defect. Figure represents the scalogram by Morlet wavelet which provided the highest wavelet coefficient values.

# 10th week – Questions

***Question***

What is the purpose of the wavelet design in machine fault diagnostics?

***Answer***

Its purpose is to identify the short time transient faults of machine parts such as bearings, gears, shafts etc.

A natural requirement is to find wavelet functions having special shape, for instance being 'similar' to a given transient in the analyzed signal.

Some direct calculation algorithms are available in the literature, for instance in [28] Chapa and Rao introduce an algorithm for designing a mother wavelet $\psi$ such that it matches a signal of interest and wavelets $\psi_k^n = \frac{1}{\sqrt{2^n}}\psi\left(\frac{1}{2^n}x - k\right)$ form an orthonormal Riesz-basis of $X$. Supposing band-limited spectrum of the scaling and wavelet functions they give the scaling function by discrete matching procedure from the discrete spectrum of the desired transient appearing in the
signal of interest.

### *Question*

How you can identify the bearing fault frequencies in the spectrum?

### *Answer*

Using FFT analysis after MRA analysis is a method to produce the spectrum of a machine part then to make a fault analysis.

Here is an example:

The raw signal is too noisy to detect entry and exit points of the defect but wavelet decomposition makes it possible to analyse the entry and exit events. Using Energy-to-Shannon Entropy Criteria we obtain the best wavelet to determine the fault size from the vibration signature.

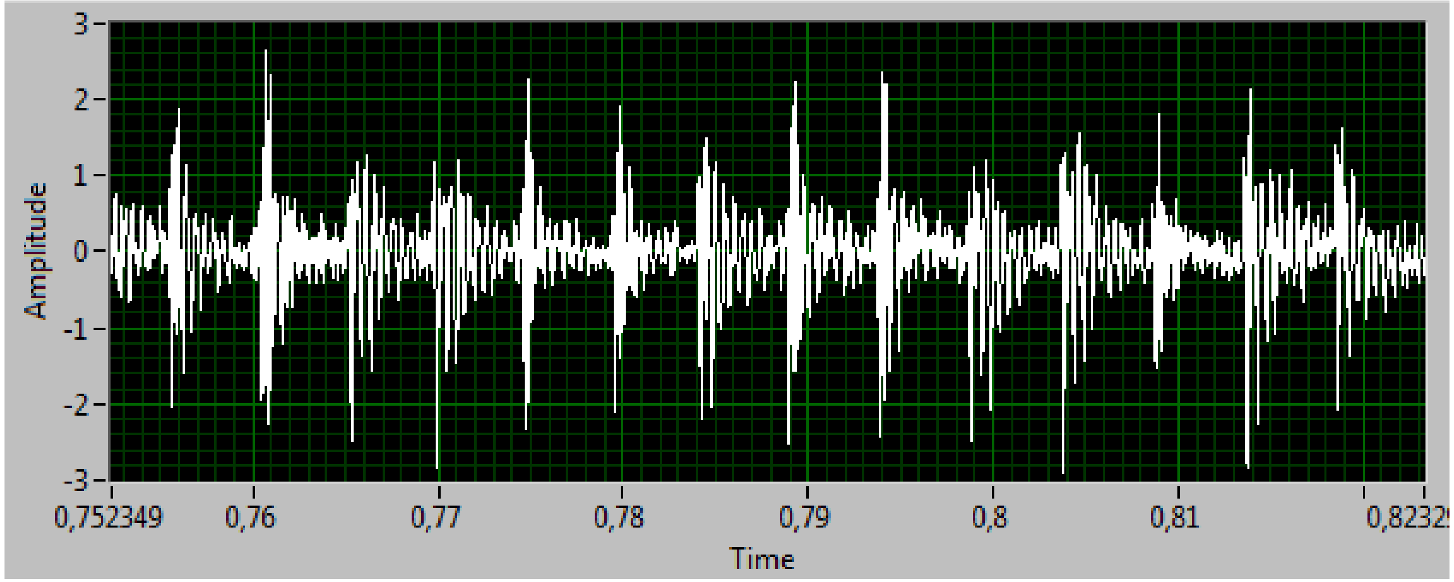
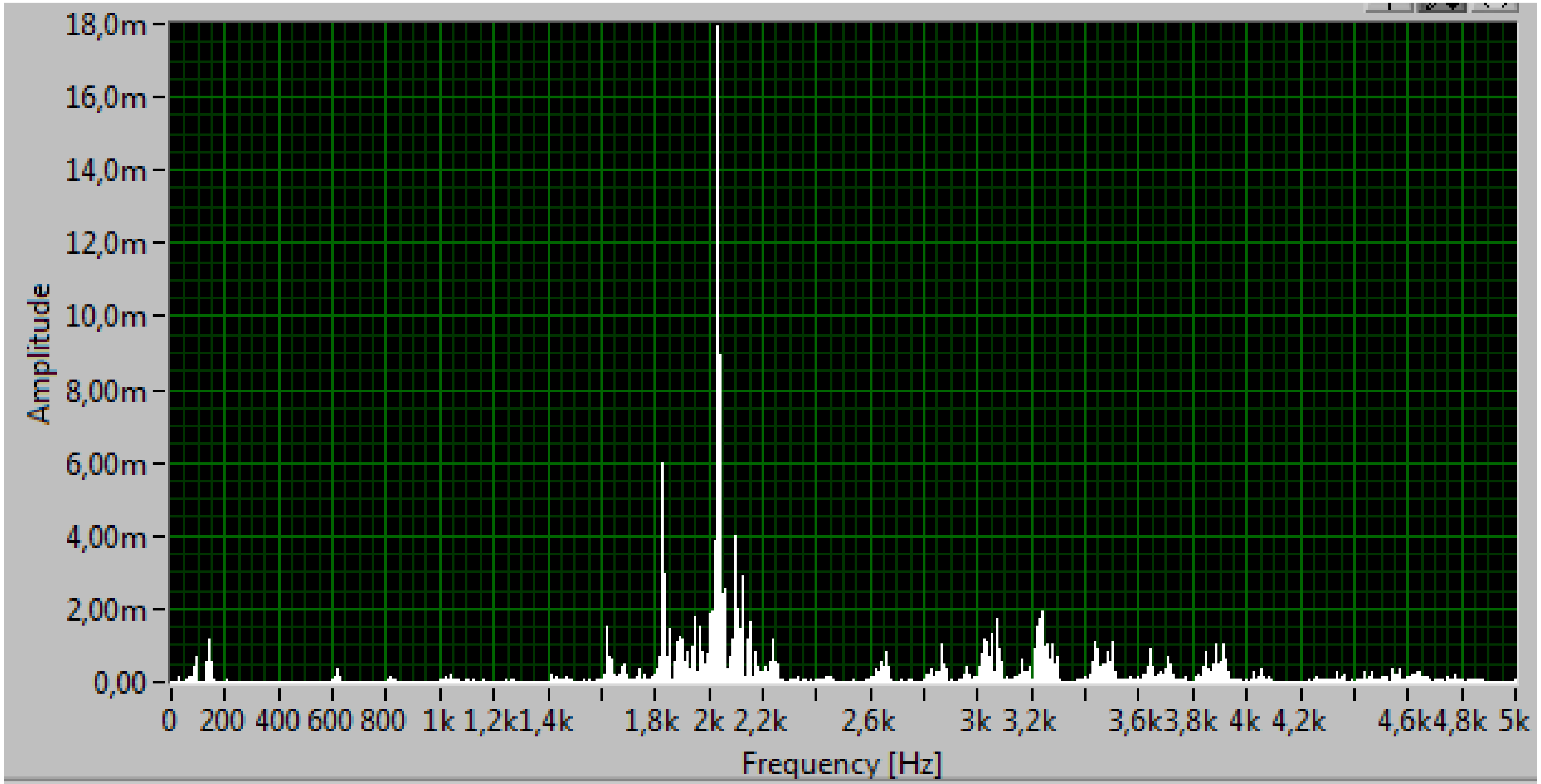



Figure presents the time-domain signal of outer race defect of 0.6311 $mm$. The highest periodic transient impulse related energy content of the burst occurs at 2.09 $kHz$ that causes 5 $ms$ rate of periodicity which is equal to 206.18 $Hz$ BPFO frequency. The spectrum was determined in all outer rings with different fault sizes and they showed similar manner around the peak at 2.09 $kHz$ as it can be seen in Figure 25.

Multiresolution analysis is made in order to obtain precise frequency analysis. Figure 26 presents the wavelet decomposition tree. Higher decomposition is not necessary because it might not reveal any further information of the signal. Regarding the BPFO frequency analysis was made at 3rd detail level (cD3) from 1.25 $kHz$ to 2.5 $kHz$.



18,0m
16,0m
14,0m
12,0m
10,0m
8,00m
6,00m
4,00m
2,00m
0,00
Amplitude
0
200
400
600
800
1k
1,2k
1,4k
1,8k
2k
2,2k
2,6k
3k
3,2k
3,6k
3,8k
4k
4,2k
4,6k
4,8k
5k
Frequency [Hz]

### *Question*

What is the method of selecting the most efficient wavelet for the fault diagnostics?

### *Answer*

The Energy-to-Entropy ratio is an efficient number to find the best wavelet for a certain purpose. The higher the E/S number is, the more efficient the wavelet to make proper feature extraction of the fault signal. The Energy-to-Shannon Entropy Criteria is used to rank wavelets on the basis of scalograms. The Energy-to-Shannon Entropy ratio is a combination of the energy content

$$E(n) = \sum_{i}^{m} \left|C_{n,i}\right|^2$$

$$S(n) = -\sum_{i=1}^{m} p_i \log_2 p_i$$

related to the wavelet coefficients $C_{n,i}$ , where $m$ is the number of the wavelet coefficients of $n$-th scale and $(p_1, \dots, p_n)$ is the energy distribution of the wavelet coefficients defined by

$$p_i = \left|C_{n,i}\right|^2 / E(n).$$

The indicator

$$\xi(n) = E(n)/S(n)$$

is used to choose the best wavelet for diagnosis of a special fault.

# 10th week – Exercises

**Exercise**

Calculate the Shannon entropy of a signal with Matlab code!

**Solution**

E = wentropy(X,T,P) returns the entropy where P is a parameter depending on T.

E = wentropy(X,T,0) is equivalent to E = wentropy(X,T).

Possible solution:

rng default

x = randn(1,200);

Compute the Shannon entropy of x.

e = wentropy(x,'shannon')

e = -224.5551

Compute the log energy entropy of x.

e = wentropy(x,'log energy')

e = -229.5183

Compute the threshold entropy of x with the threshold entropy equal to 0.2.

e = wentropy(x,'threshold',0.2)

e = 168

Compute the Sure entropy of x with the threshold equal to 3.

e = wentropy(x,'sure',3)

e = 35.7962

Compute the norm entropy of x with power equal to 1.1

e = wentropy(x,'norm',1.1)

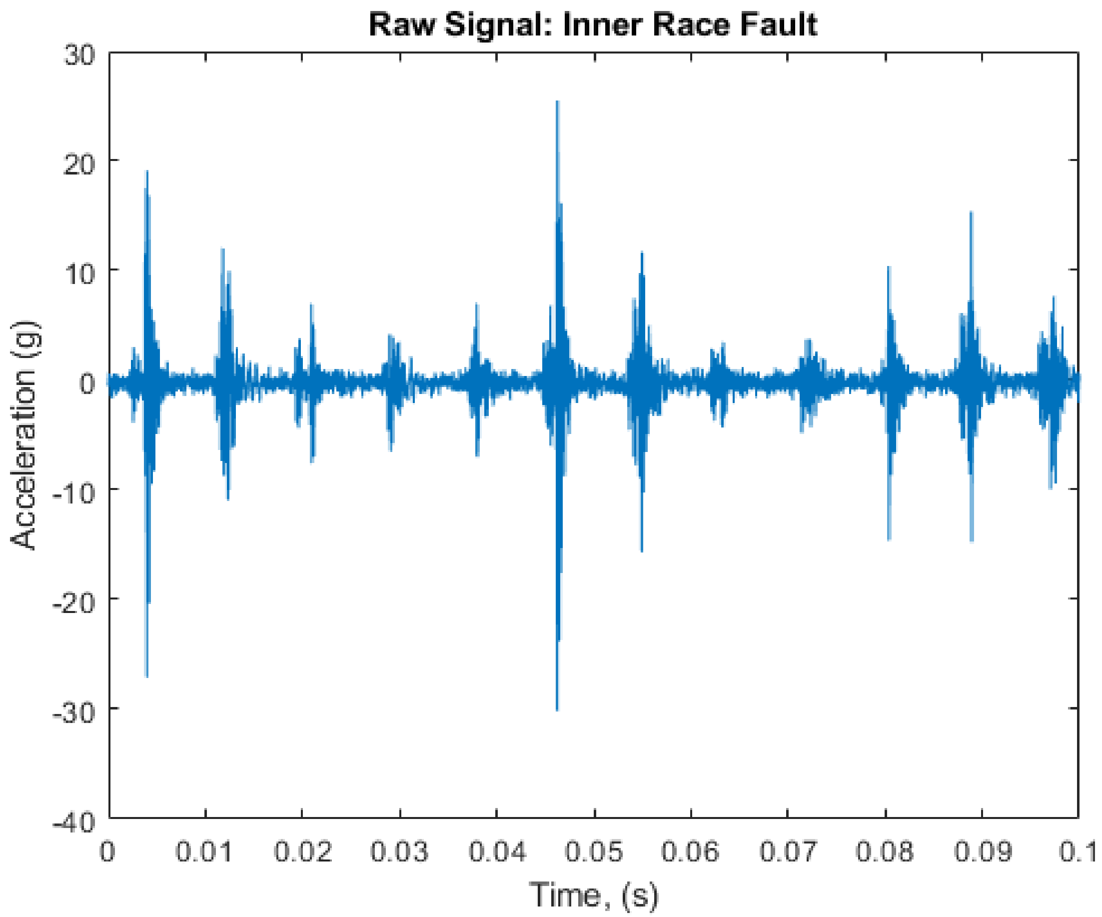
## Exercise

Calculate the time domain of a fault signal in Matlab!

## Solution

```
dataInner = load(fullfile(matlabroot, 'toolbox', 'predmaint', ...
    'predmaintdemos', 'bearingFaultDiagnosis', ...
    'train_data', 'InnerRaceFault_vload_1.mat'));

xInner = dataInner.bearing.gs;
fsInner = dataInner.bearing.sr;
tInner = (0:length(xInner)-1)/fsInner;
figure
plot(tInner, xInner)
xlabel('Time, (s)')
```

```
ylabel('Acceleration (g)')

title('Raw Signal: Inner Race Fault')

xlim([0 0.1])
```

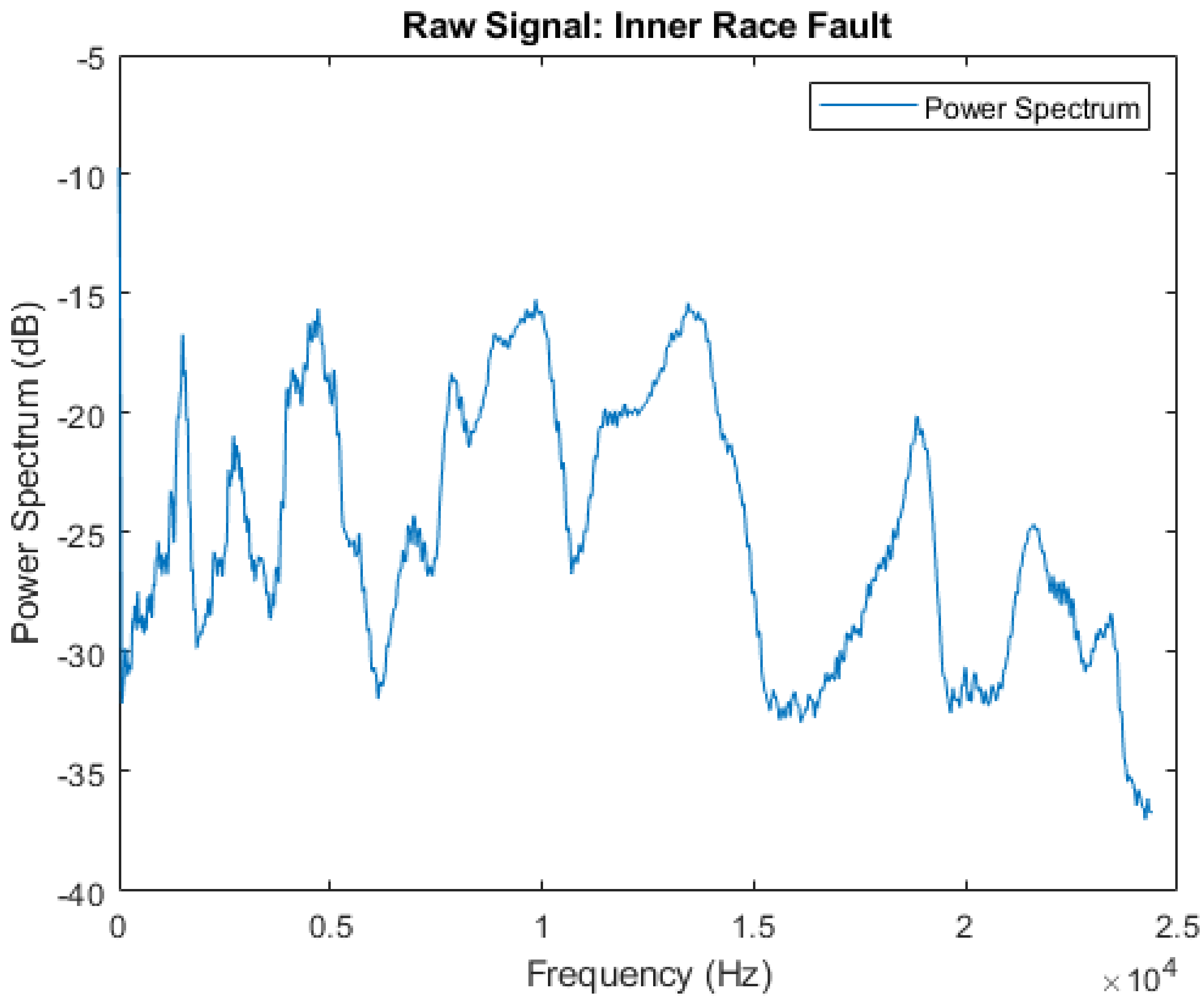


## Exercise

Visualize the raw data in frequency domain!

## Solution

```
figure

[pInner, fpInner] = pspectrum(xInner, fsInner);

pInner = 10*log10(pInner);

plot(fpInner, pInner)

xlabel('Frequency (Hz)')

ylabel('Power Spectrum (dB)')

title('Raw Signal: Inner Race Fault')

legend('Power Spectrum')
```


Raw Signal: Inner Race Fault
Power Spectrum
Power Spectrum (dB)
-5
-10
-15
-20
-25
-30
-35
-40
0
0.5
1
1.5
2
2.5
Frequency (Hz)
×10^4

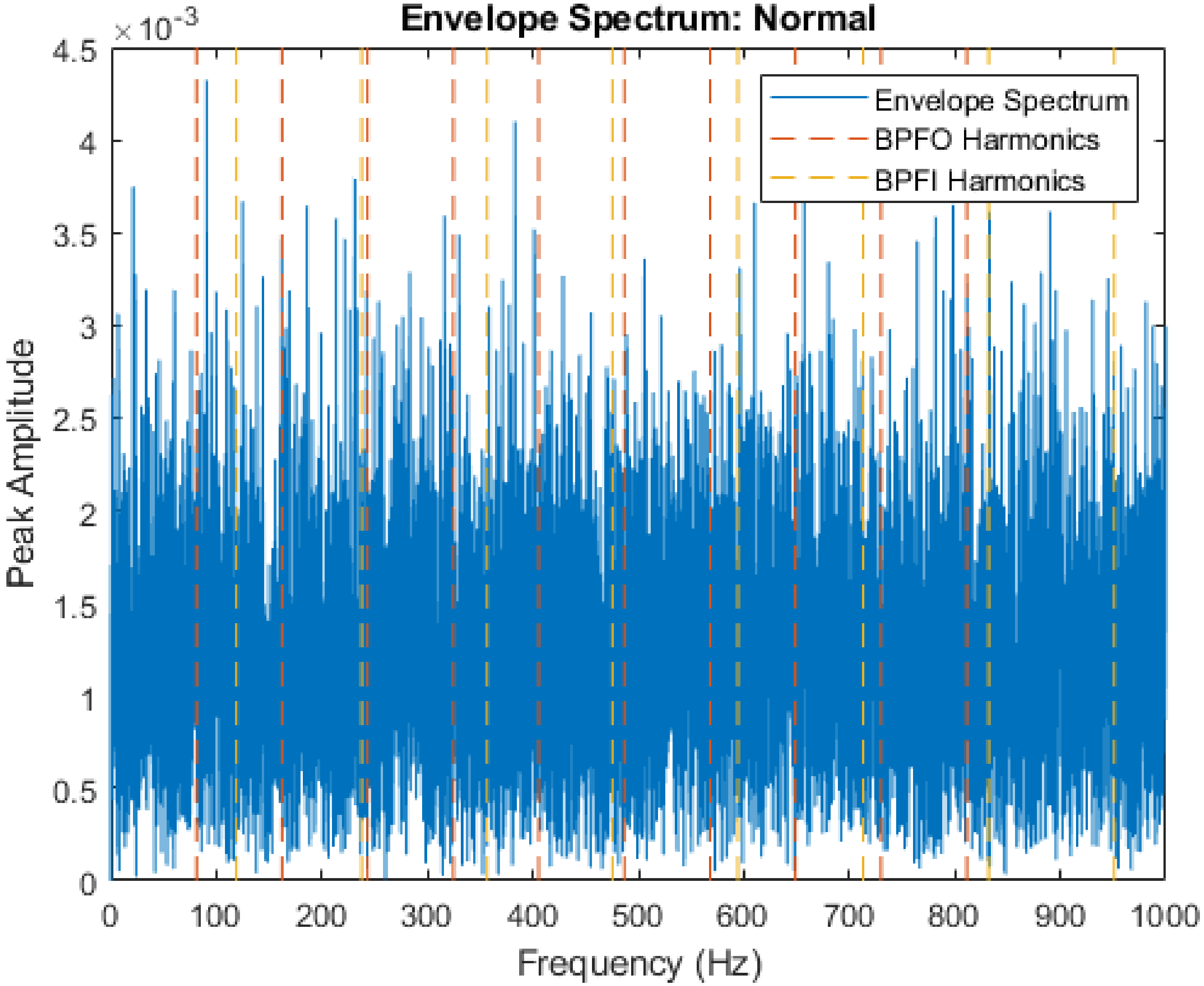
## Exercise

Create the envelope spectrum of the signal above!

## Solution

```
xNormal = dataNormal.bearing.gs;

fsNormal = dataNormal.bearing.sr;

tNormal = (0:length(xNormal)-1)/fsNormal;

[pEnvNormal, fEnvNormal] = envspectrum(xNormal, fsNormal);


figure

plot(fEnvNormal, pEnvNormal)

ncomb = 10;

helperPlotCombs(ncomb, [dataNormal.BPFO dataNormal.BPFI])

xlim([0 1000])
```

```
xlabel('Frequency (Hz)')

ylabel('Peak Amplitude')

title('Envelope Spectrum: Normal')

legend('Envelope Spectrum', 'BPFO Harmonics', 'BPFI Harmonics')
```


Envelope Spectrum: Normal
×10^-3
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Peak Amplitude
0
100
200
300
400
500
600
700
800
900
1000
Frequency (Hz)
Envelope Spectrum
BPFO Harmonics
BPFI Harmonics

# 11th week

# 11 Digital Filters, FIR, IIR

In signal processing, a digital filter is a system that performs mathematical operations on a sampled, discrete-time signal to reduce or enhance certain aspects of that signal. This is in contrast to the other major type of electronic filter, the analog filter, which is typically an electronic circuit operating on continuous-time analog signals.

A digital filter system usually consists of an analog-to-digital converter (ADC) to sample the input signal, followed by a microprocessor and some peripheral components such as memory to store data and filter coefficients etc. Program Instructions (software) running on the microprocessor implement the digital filter by performing the necessary mathematical operations on the numbers received from the ADC. In some high performance applications, an FPGA or ASIC is used instead of a general purpose microprocessor, or a specialized digital signal processor (DSP) with specific paralleled architecture for expediting operations such as filtering.

Digital filters may be more expensive than an equivalent analog filter due to their increased complexity, but they make practical many designs that are impractical or impossible as analog filters. Digital filters can often be made very high order, and are often finite impulse response filters, which allows for linear phase response. When used in the context of real-time analog systems, digital filters sometimes have problematic latency (the difference in time between the input and the response) due to the associated analog-to-digital and digital-to-analog conversions and anti-aliasing filters, or due to other delays in their implementation.

Digital filters are commonplace and an essential element of everyday electronics such as radios, cellphones, and AV receivers.

A digital filter is characterized by its transfer function, or equivalently, its difference equation. Mathematical analysis of the transfer function can describe how it will respond to any input. As such, designing a filter consists of developing specifications appropriate to the problem (for example, a second-order low pass filter with a specific cut-off frequency), and then producing a transfer function which meets the specifications.

The transfer function for a linear, time-invariant, digital filter can be expressed as a transfer function in the Z-domain; if it is causal, then it has the form:[3]

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_M z^{-M}}$$

where the order of the filter is the greater of $N$ or $M$. See Z-transform's LCCD equation for further discussion of this transfer function.

This is the form for a recursive filter, which typically leads to an infinite impulse response (IIR) behaviour, but if the denominator is made equal to unity, i.e. no feedback, then this becomes a finite impulse response (FIR) filter.

## Analysis techniques

A variety of mathematical techniques may be employed to analyze the behavior of a given digital filter. Many of these analysis techniques may also be employed in designs, and often form the basis of a filter specification.

Typically, one characterizes filters by calculating how they will respond to a simple input such as an impulse. One can then extend this information to compute the filter's response to more complex signals.

The impulse response is a characterization of the filter's behaviour. Digital filters are typically considered in two categories: infinite impulse response (IIR) and finite impulse response (FIR). In the case of linear time-invariant FIR filters, the impulse response is exactly equal to the sequence of filter coefficients, and thus:

$$y_n = \sum_{k=0}^{N} b_k x_{n-k} = \sum_{k=0}^{N} h_k x_{n-k}$$

IIR filters on the other hand are recursive, with the output depending on both current and previous inputs as well as previous outputs. The general form of an IIR filter is thus:

$$\sum_{m=0}^{M} a_m y_{n-m} = \sum_{k=0}^{N} b_k x_{n-k}$$

Plotting the impulse response reveals how a filter responds to a sudden, momentary disturbance. An IIR filter is always recursive. While it is possible for a recursive filter to have a finite impulse response, a non-recursive filter always has a finite impulse response. An example is the moving average (MA) filter, which can be implemented both recursively and non recursively.

Digital filters are not subject to the component non-linearities that greatly complicate the design of analog filters. Analog filters consist of imperfect electronic components, whose values are specified to a limit tolerance (e.g. resistor values often have a tolerance of ±5%) and which may also change with temperature and drift with time. As the order of an analog filter increases, and thus its component count, the effect of variable component errors is greatly magnified. In digital filters, the coefficient values are stored in computer memory, making them far more stable and predictable.

Because the coefficients of digital filters are definite, they can be used to achieve much more complex and selective designs – specifically with digital filters, one can achieve a lower passband ripple, faster transition, and higher stopband attenuation than is practical with analog filters. Even if the design could be achieved using analog filters, the engineering cost of designing an equivalent digital filter would likely be much lower. Furthermore, one can readily modify the coefficients of a digital filter to make an adaptive filter or a user-controllable parametric filter. While these techniques are possible in an analog filter, they are again considerably more difficult.

Digital filters can be used in the design of finite impulse response filters. Equivalent analog filters are often more complicated, as these require delay elements.

Digital filters rely less on analog circuitry, potentially allowing for a better signal-to-noise ratio. A digital filter will introduce noise to a signal during analog low pass filtering, analog to digital conversion, digital to analog conversion and may introduce digital noise due to quantization. With analog filters, every component is a source of thermal noise (such as Johnson noise), so as the filter complexity grows, so does the noise.

However, digital filters do introduce a higher fundamental latency to the system. In an analog filter, latency is often negligible; strictly speaking it is the time for an electrical signal to propagate through the filter circuit. In digital systems, latency is introduced by delay elements in the digital signal path, and by analog-to-digital and digital-to-analog converters that enable the system to process analog signals.

In very simple cases, it is more cost effective to use an analog filter. Introducing a digital filter requires considerable overhead circuitry, as previously discussed, including two low pass analog filters.

Another argument for analog filters is low power consumption. Analog filters require substantially less power and are therefore the only solution when power requirements are tight.

When making an electrical circuit on a PCB it is generally easier to use a digital solution, because the processing units are highly optimized over the years. Making the same circuit with analog components would take up a lot more space when using discrete components.

## Types of digital filters

There are various ways to characterize filters; for example:

A linear filter is a linear transformation of input samples; other filters are nonlinear. Linear filters satisfy the superposition principle, i.e. if an input is a weighted linear combination of different signals, the output is a similarly weighted linear combination of the corresponding output signals.

A causal filter uses only previous samples of the input or output signals; while a non-causal filter uses future input samples. A non-causal filter can usually be changed into a causal filter by adding a delay to it.

A time-invariant filter has constant properties over time; other filters such as adaptive filters change in time.

A stable filter produces an output that converges to a constant value with time, or remains bounded within a finite interval. An unstable filter can produce an output that grows without bounds, with bounded or even zero input.

A finite impulse response (FIR) filter uses only the input signals, while an infinite impulse response (IIR) filter uses both the input signal and previous samples of the output signal. FIR filters are always stable, while IIR filters may be unstable.

A filter can be represented by a block diagram, which can then be used to derive a sample processing algorithm to implement the filter with hardware instructions. A filter may also be described as a difference equation, a collection of zeros and poles or an impulse response or step response.

Some digital filters are based on the fast Fourier transform, a mathematical algorithm that quickly extracts the frequency spectrum of a signal, allowing the spectrum to be manipulated (such as to create very high order band-pass filters) before converting the modified spectrum back into a time-series signal with an inverse FFT operation. These filters give O(n log n) computational costs whereas conventional digital filters tend to be O(n2).

Another form of a digital filter is that of a state-space model. A well used state-space filter is the Kalman filter published by Rudolf Kálmán in 1960.

Traditional linear filters are usually based on attenuation. Alternatively nonlinear filters can be designed, including energy transfer filters,which allow the user to move energy in a designed way so that unwanted noise or effects can be moved to new frequency bands either lower or higher in frequency, spread over a range of frequencies, split, or focused. Energy transfer filters complement traditional filter designs and introduce many more degrees of freedom in filter design. Digital energy transfer filters are relatively easy to design and to implement and exploit nonlinear dynamics.

**Infinite impulse response (IIR)** is a property applying to many linear time-invariant systems that are distinguished by having an impulse response h(t) which does not become exactly zero past a certain point, but continues indefinitely. This is in contrast to a finite impulse response (FIR) system in which the impulse response does become exactly zero at times t>T for some finite T, thus being of finite duration. Common examples of linear time-invariant systems are most electronic and digital filters. Systems with this property are known as IIR systems or IIR filters.

In practice, the impulse response, even of IIR systems, usually approaches zero and can be neglected past a certain point. However the physical systems which give rise to IIR or FIR responses are dissimilar, and therein lies the importance of the distinction. For instance, analog electronic filters composed of resistors, capacitors, and/or inductors (and perhaps linear amplifiers) are generally IIR filters. On the other hand, discrete-time filters (usually digital filters) based on a tapped delay line employing no feedback are necessarily FIR filters. The capacitors (or inductors) in the analog filter have a "memory" and their internal state never completely relaxes following an impulse (assuming the classical model of capacitors and inductors where quantum effects are ignored). But in the latter case, after an impulse has reached the end of the tapped delay line, the system has no further memory of that impulse and has returned to its initial state; its impulse response beyond that point is exactly zero.

Although almost all analog electronic filters are IIR, digital filters may be either IIR or FIR. The presence of feedback in the topology of a discrete-time filter (such as the block diagram shown below) generally creates an IIR response. The z domain transfer function of an IIR filter contains a non-trivial denominator, describing those feedback terms. The transfer function of an FIR filter, on the other hand, has only a numerator as expressed in the general form derived below.

The transfer functions pertaining to IIR analog electronic filters have been extensively studied and optimized for their amplitude and phase characteristics. These continuous-time filter functions are described in the Laplace domain. Desired solutions can be transferred to the case of discrete-time filters whose transfer functions are expressed in the z domain, through the use of certain mathematical techniques such as the bilinear transform, impulse invariance, or pole–zero matching method. Thus digital IIR filters can be based on well-known solutions for analog filters such as the Chebyshev filter, Butterworth filter, and elliptic filter, inheriting the characteristics of those solutions.

Digital filters are often described and implemented in terms of the difference equation that defines how the output signal is related to the input signal:

$$y[n] = \frac{1}{a_0}\big(b_0x[n] + b_1x[n-1] + \ldots + b_px[n-P] - a_1y[n-1] \\ - a_2y[n-2] - \ldots - a_Qy[n-Q]\big)$$

where:

- $P$ is the feedforward filter order
- $b_i$ are the feedforward filter coefficients
- $Q$ is the feedback filter order
- $a_i$ are the feedback filter coefficients
- $x[n]$is the input signal
- $y[n]$ is the output signal.

$$y[n] = \frac{1}{a_0}\left(\sum_{i=0}^{P} b_ix[n-i] - \sum_{j=1}^{Q} a_jy[n-j]\right)$$

The main advantage digital IIR filters have over FIR filters is their efficiency in implementation, in order to meet a specification in terms of passband, stopband, ripple, and/or roll-off. Such a set of specifications can be accomplished with a lower order (Q in the above formulae) IIR filter than would be required for an FIR filter meeting the same requirements. If implemented in a signal processor, this implies a correspondingly fewer number of calculations per time step; the computational savings is often of a rather large factor.

On the other hand, FIR filters can be easier to design, for instance, to match a particular frequency response requirement. This is particularly true when the requirement is not one of the usual cases (high-pass, low-pass, notch, etc.) which have been studied and optimized for analog filters. Also FIR filters can be easily made to be linear phase.

Typical IIR filters in practice are Butterworth, Csebisev, Elliptic filters.

In signal processing, a **finite impulse response (FIR) filter** is a filter whose impulse response (or response to any finite length input) is of finite duration, because it settles to zero in finite time. This is in contrast to infinite impulse response (IIR) filters, which may have internal feedback and may continue to respond indefinitely (usually decaying).

FIR filters can be discrete-time or continuous-time, and digital or analog.

For a causal discrete-time FIR filter of order N, each value of the output sequence is a weighted sum of the most recent input values:

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_N x[n-N] = \sum_{i=0}^{N} b_i \cdot x[n-i]$$

where:

- $x[n]$ is the input signal,
- $y[n]$ is the output signal,
- $N$ is the filter order; an $N^{th}$-order filter has $N+1$ terms on the right-hand side
- $b_i$ is the value of the impulse response at the i'th instant for $0 \leq i \leq N$ of an $N^{th}$-order FIR filter. If the filter is a direct form FIR filter then $b_i$ is also a coefficient of the filter.

An FIR filter has a number of useful properties which sometimes make it preferable to an infinite impulse response (IIR) filter. FIR filters:

Require no feedback. This means that any rounding errors are not compounded by summed iterations. The same relative error occurs in each calculation. This also makes implementation simpler.

Can easily be designed to be linear phase by making the coefficient sequence symmetric. This property is sometimes desired for phase-sensitive applications, for example data communications, seismology, crossover filters, and mastering.

The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness or selectivity, especially when low frequency (relative to the sample rate) cutoffs are needed. However, many digital signal processors provide specialized hardware features to make FIR filters approximately as efficient as IIR for many applications.

# 11th week – Questions

***Question***

What is the theory and the application fields of the FIR digital filters?

***Answer***

A **finite impulse response (FIR) filter** is a filter whose impulse response (or response to any finite length input) is of finite duration, because it settles to zero in finite time. This is in contrast to infinite impulse response (IIR) filters, which may have internal feedback and may continue to respond indefinitely (usually decaying).

FIR filters can be discrete-time or continuous-time, and digital or analog.

They should be designed with filter coefficients.

## *Question*

What is the theory and the application fields of the IIR digital filters?

## *Answer*

**Infinite impulse response (IIR)** is a property applying to many linear time-invariant systems that are distinguished by having an impulse response h(t) which does not become exactly zero past a certain point, but continues indefinitely. This is in contrast to a finite impulse response (FIR) system in which the impulse response does become exactly zero at times t>T for some finite T, thus being of finite duration. Common examples of linear time-invariant systems are most electronic and digital filters. Systems with this property are known as IIR systems or IIR filters.

IIR filters are so-called "ready" filters such as Butterworth, Chebisev etc. which can be used directly for applications.

### *Question*

What is the difference between analogue and digital filters? What are the advantages of digital filtering?

### *Answer*

A finite impulse response (FIR) filter uses only the input signals, while an infinite impulse response (IIR) filter uses both the input signal and previous samples of the output signal. FIR filters are always stable, while IIR filters may be unstable.

Advantages:

Flexibility

Easy to modify for certain purposes

Higher filtering cabability

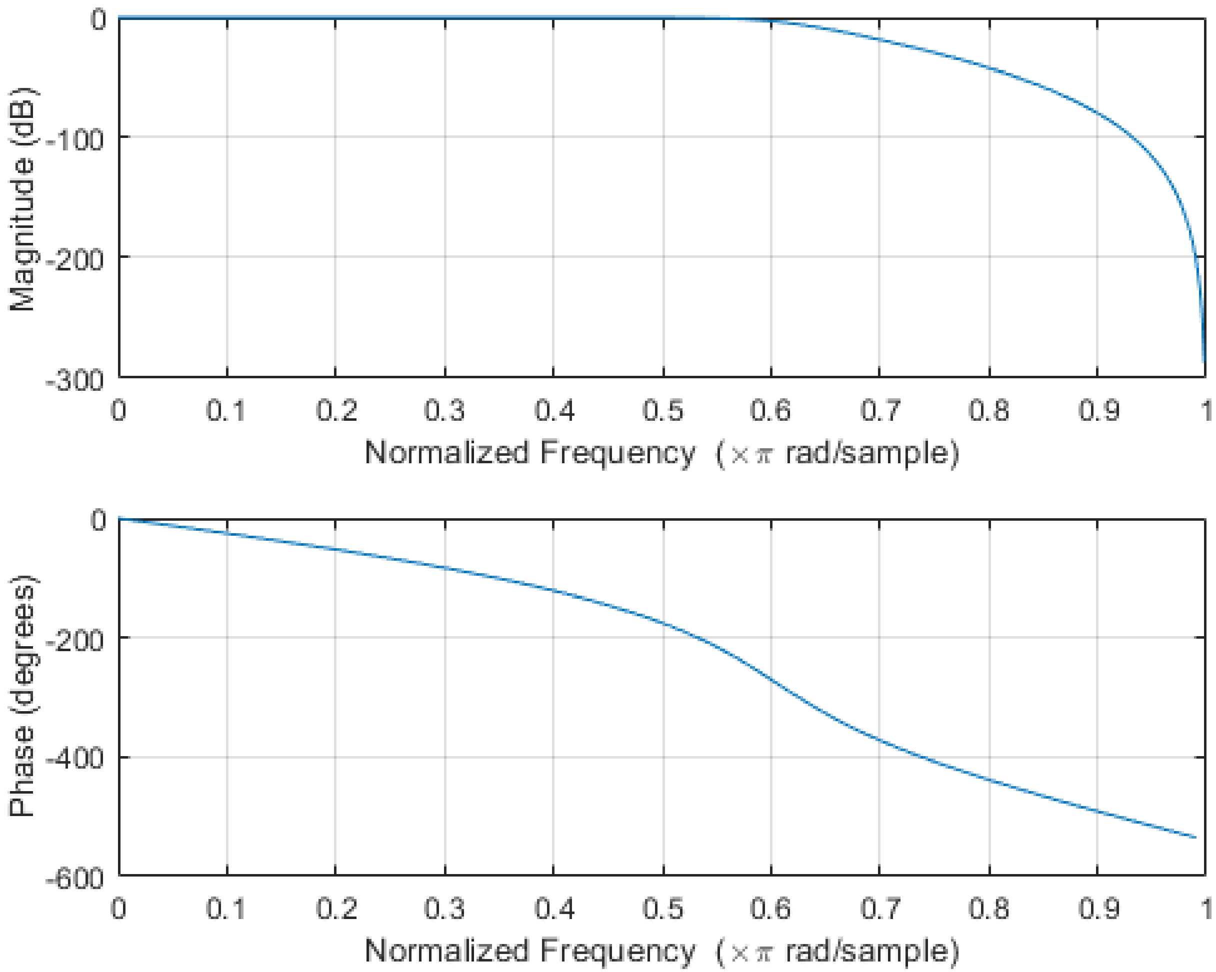
# 11th week – Exercises

**Exercise**

IIR filter of a 6th-order lowpass Butterworth must be designed with a cutoff frequency of 300 Hz, which, for data sampled at 1000 Hz, corresponds to 0.6π rad/sample. Plot its magnitude and phase responses. Use it to filter a 1000-sample random signal.

**Solution**

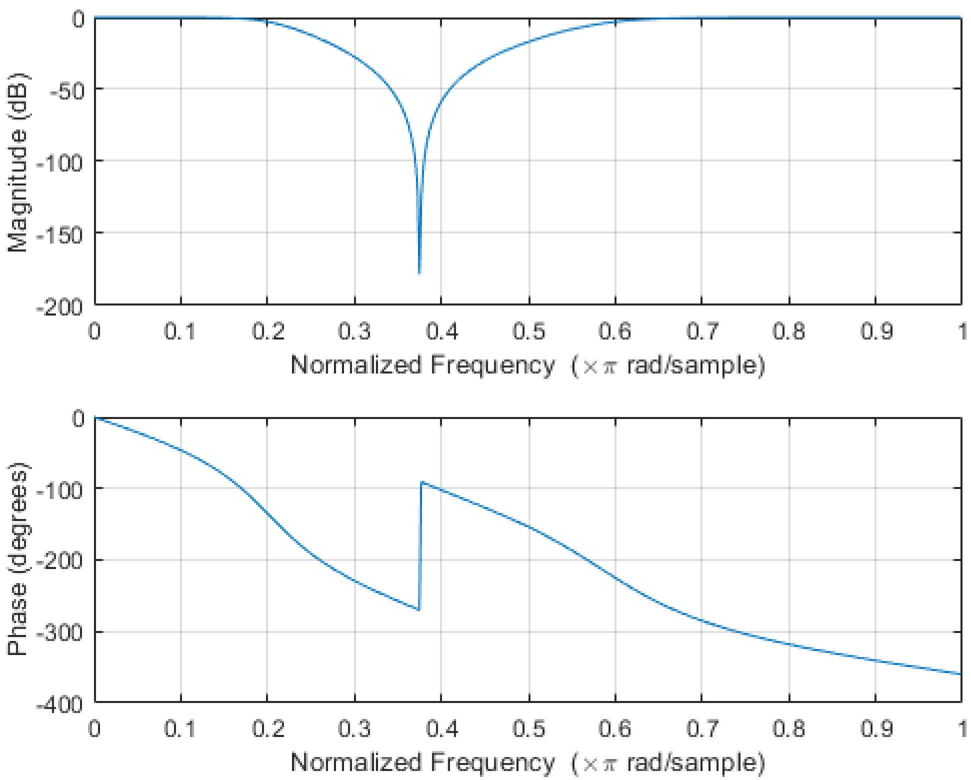
```
fc = 300;

fs = 1000;

[b,a] = butter(6,fc/(fs/2));

freqz(b,a)

dataIn = randn(1000,1);

dataOut = filter(b,a,dataIn);
```


Magnitude (dB)
0
-100
-200
-300
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
Phase (degrees)
0
-200
-400
-600
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)

## Exercise

Design an IIR filter 6th-order Butterworth bandstop filter with normalized edge frequencies of 0.2π and 0.6π rad/sample. Plot its magnitude and phase responses. Use it to filter random data.

## Solution

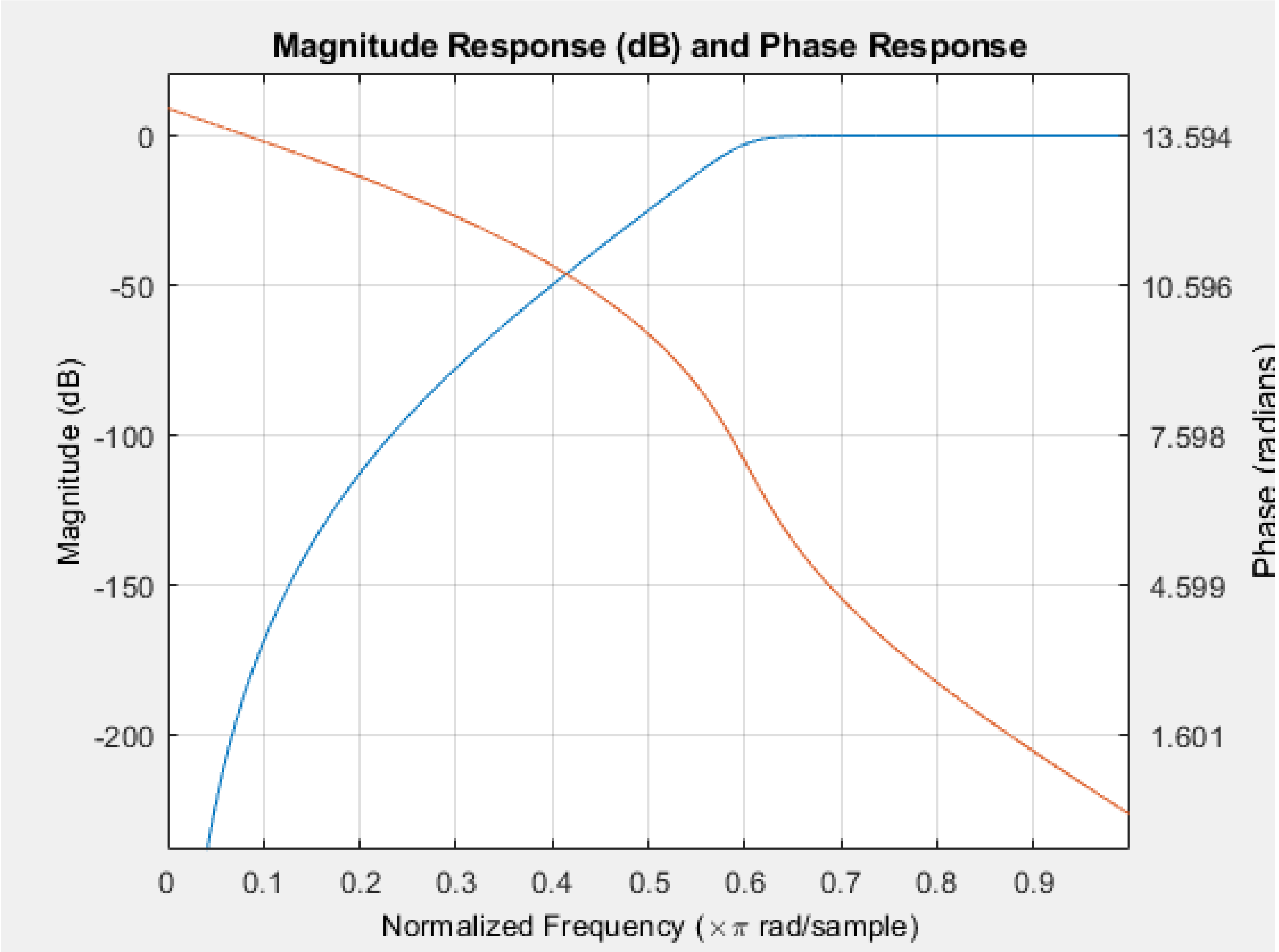


## Exercise

Design a 9th-order highpass Butterworth filter. Specify a cutoff frequency of 300 Hz, which, for data sampled at 1000 Hz, corresponds to 0.6π rad/sample. Plot the magnitude and phase responses. Convert the zeros, poles, and gain to second-order sections for use by fvtool.

## Solution

```
[z,p,k] = butter(9,300/500,'high');

sos = zp2sos(z,p,k);

fvtool(sos,'Analysis','freq')
```


Magnitude Response (dB) and Phase Response
Magnitude (dB)
0
-50
-100
-150
-200
Phase (radians)
13.594
10.596
7.598
4.599
1.601
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Normalized Frequency (×π rad/sample)

# 12th week

# 12 Digital Filter Design

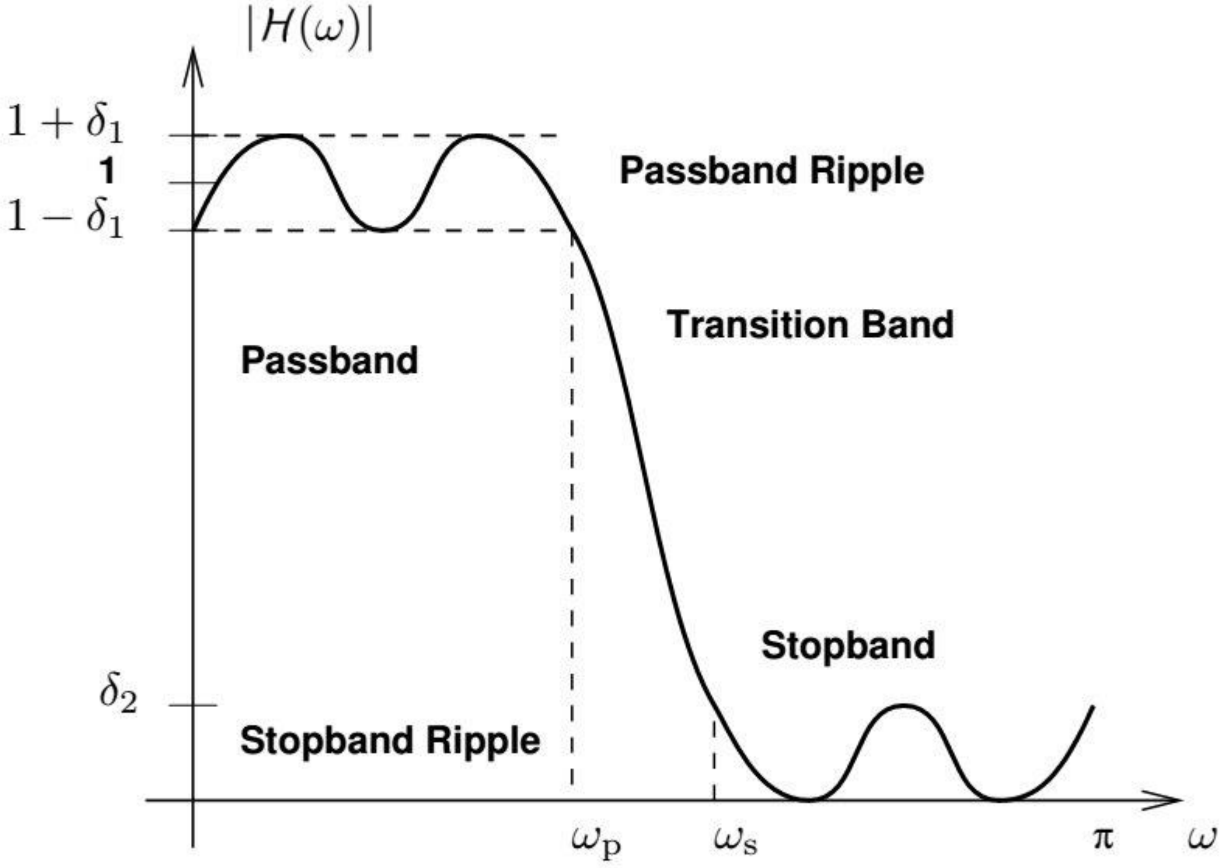
Filters are used in a wide variety of applications. Most of the time, the final goal of using a filter is to achieve a kind of frequency selectivity on the spectrum of the input signal.

As an example, suppose that a 50-Hz noise falls on top of the signal produced by a sensor. The noise component may be strong enough to limit the measurement precision. The output of the sensor is usually converted to a digital signal by an ADC to be processed by a DSP or a microcontroller. Therefore, we can use a digital filter after the ADC to eliminate the noise component. In this particular example, a notch filter centered at 50 Hz can be utilized to suppress the noise.

Although filters are easily understood and calculated, the practical challenges of their design and implementation are significant and are the subject of much advanced research.

There are two categories of digital filter: the recursive filter and the nonrecursive filter. These are often referred to as infinite impulse response (IIR) filters and finite impulse response (FIR) filters, respectively.

An FIR filter is designed by finding the coefficients and filter order that meet certain specifications, which can be in the time domain (e.g. a matched filter) and/or the frequency domain (most common). Matched filters perform a cross-correlation between the input signal and a known pulse shape. The FIR convolution is a cross-correlation between the input signal and a time-reversed copy of the impulse response. Therefore, the matched filter's impulse response is "designed" by sampling the known pulse-shape and using those samples in reverse order as the coefficients of the filter.




When a particular frequency response is desired, several different design methods are common:

Window design method

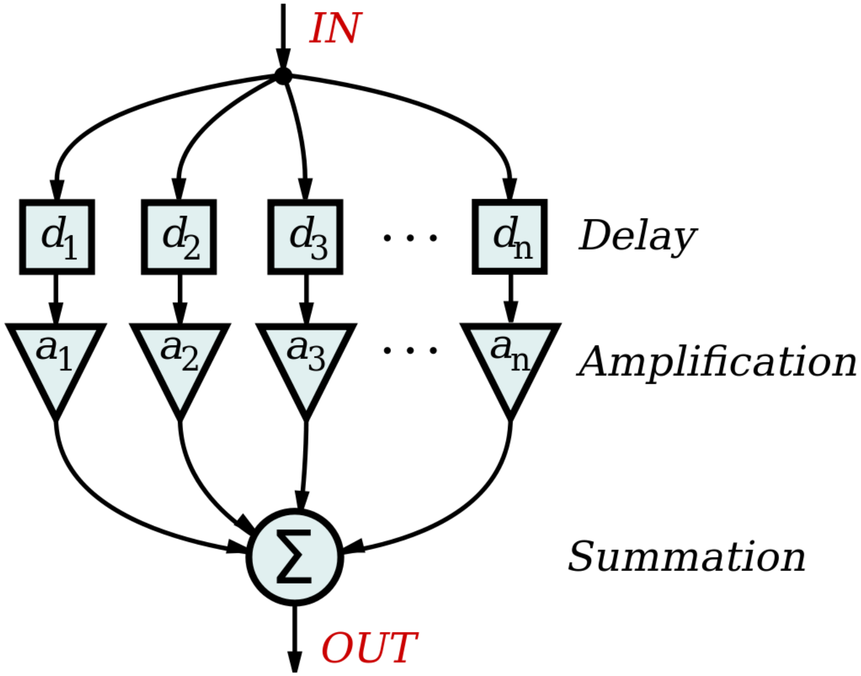
Frequency sampling method

Least MSE (mean square error) method

Parks–McClellan method (also known as the equiripple, optimal, or minimax method). The Remez exchange algorithm is commonly used to find an optimal equiripple set of coefficients. Here the user specifies a desired frequency response, a weighting function for errors from this response, and a filter order N. The algorithm then finds the set of N+1 coefficients that minimize the maximum deviation from the ideal. Intuitively, this finds the filter that is as close as possible to the desired response given that only N+1 coefficients can be used. This method is particularly easy in practice since at least one includes a program that takes the desired filter and N, and returns the optimum coefficients.

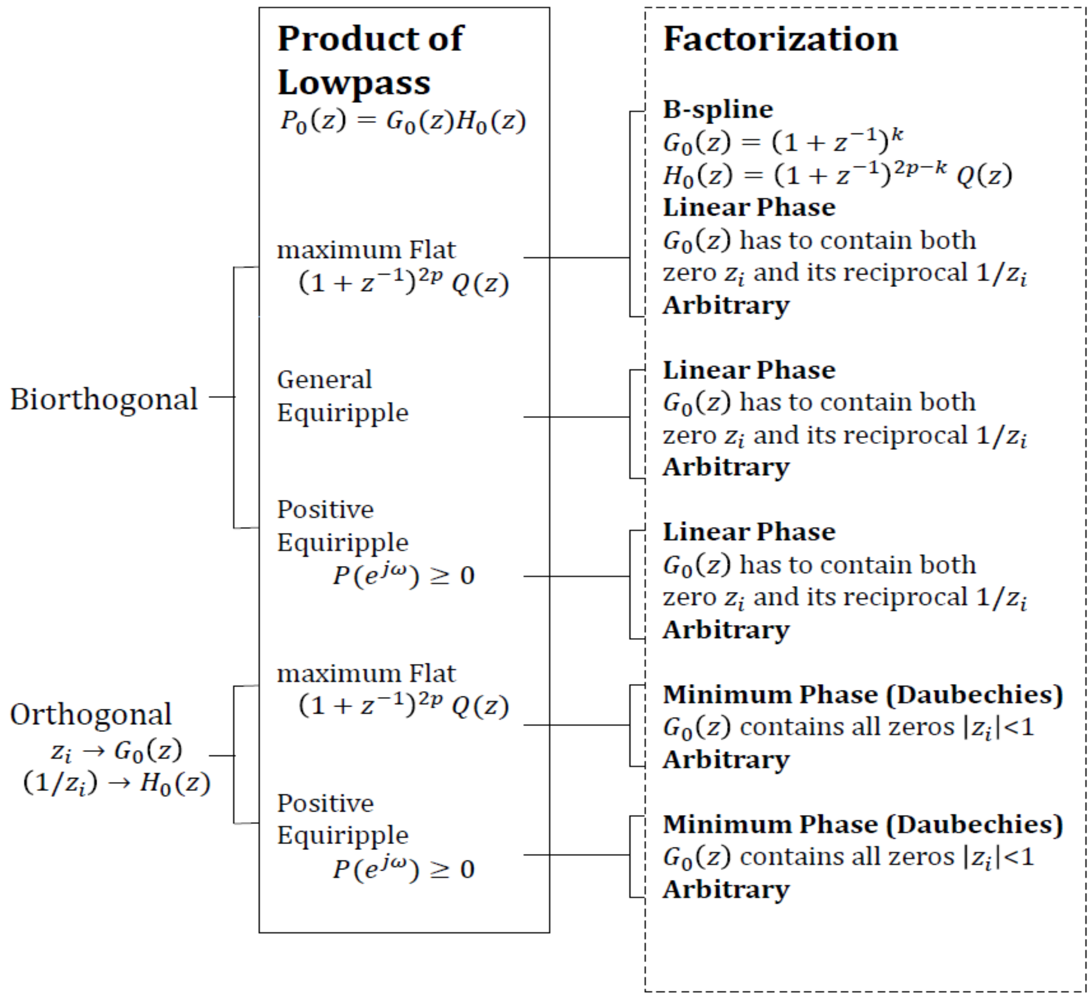
Equiripple FIR filters can be designed using the DFT algorithms as well. The algorithm is iterative in nature. The DFT of an initial filter design is computed using the FFT algorithm (if an initial estimate is not available, h[n]=delta[n] can be used). In the Fourier domain, or DFT domain, the frequency response is corrected according to the desired specs, and the inverse DFT is then computed. In the time-domain, only the first N coefficients are kept (the other coefficients are set to zero). The process is then repeated iteratively: the DFT is computed once again, correction applied in the frequency domain and so on.

Software packages such as MATLAB, GNU Octave, Scilab, and SciPy provide convenient ways to apply these different methods.




Levels of filters

The design procedure by Equiripple method with Labview software consists of three steps:



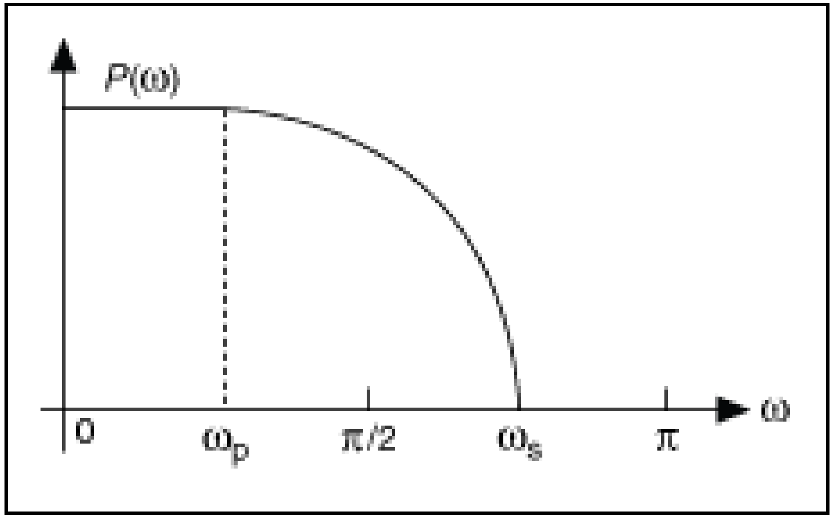

Design procedure for wavelets and filter banks in Labview

Figure above represents the wavelet design configuration. The combination of zeros is not unique. Because all filters act as real-valued FIR filters, the zeros of $P_0(z)$, $G_0(z)$ and $H_0(z)$ are symmetrical in the z-plane. Zeros of $G_0$ and $H_0$ of the new-designed wavelet are chosen in a way to provide the best result for the diagnosis of the grinding problem.

For both orthogonal and biorthogonal wavelets and filter banks, either maximum flat or equiripple filters for the product of lowpass filters $P_0(z)$ can be used. The maximum flat filters have good frequency attenuation, but wider transition band. In the experiment, the positive equiripple is used which is a halfband filter, namely a special case of general equiripple halfband filters. It proved to be more efficient than the maximum flat filter. The Fourier transform of the positive equiripple filter is always non-negative. Positive equiripple halfband filter is appropriate for orthogonal wavelets because the auxiliary function $P_0(z)$ must be non-negative.

Remez exchange algorithm was used as the pert of the Parks-McClellan method to find an optimal equiripple set of coefficients which is an iterative algorithm used to find simple approximations to functions. The algorithm then finds the set of $N + 1$ coefficients that minimize the maximum deviation from the ideal. Intuitively, this finds the filter that is as close as possible to the desired response given that only $N + 1$ coefficients can be used. Parks-McClellan VI generates a set of linear-phase FIR multiband digital filter coefficients using the number of taps, sampling frequency: fs, band parameters and filter type.

Two parameters are associated with equiripple filters number of taps and Passband. Use the number of taps control to define the number of coefficients of $P_0(z)$. Because $P_0(z)$ is a type-I FIR filter, the length of $P_0(z)$ must be odd. Use the Passband control to define the normalized passband frequency, $\omega_p$, of $P_0(z)$. The value of $\omega_p$ must be less than 0.5 and the value of 0.3 was chosen here. Longer filters improve the sharpness of the transition band and the magnitude of the attenuation in the stopband at the expense of extra computation time for implementation. This lowpass filter with 31 taps provided the wavelet with sufficient correlation to the transient under analysis (Figure 31). In a general equiripple halfband filter, halfband refers to a filter in which $\omega_s + \omega_p = \pi$, where $\omega_s$ denotes the stopband frequency and $\omega_p$ denotes the passband frequency, as Figure shows.



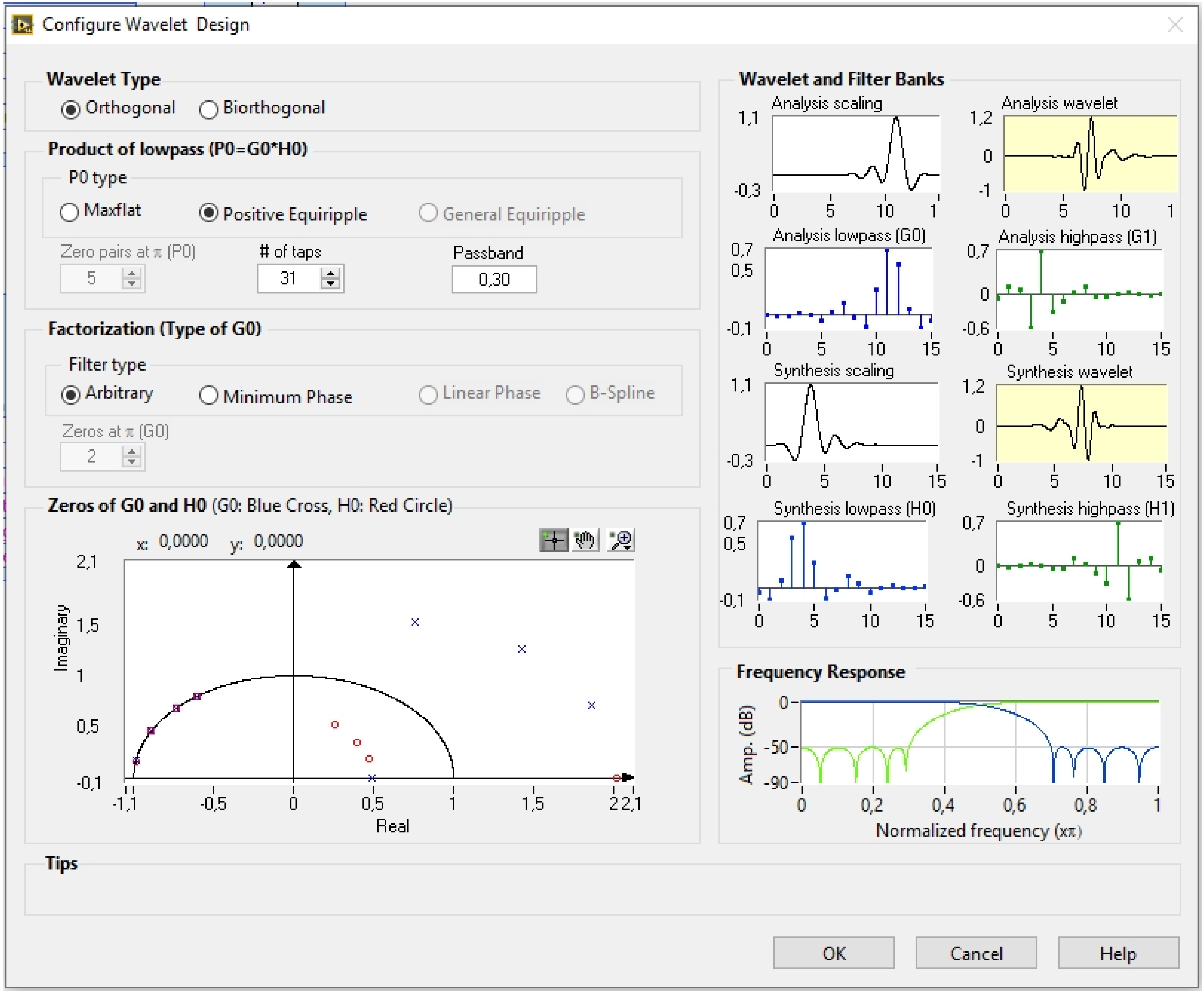

General equiripple halfband filter

The filter has the form

$$P_0(z) = (1 + z^{-1})^{2p} Q(z)$$

and as many zeros are imposed at $\omega = \pi$ as we like. The halfband equiripple filters only can have a pair of zeros at $\omega = \pi$, which gives the equiripple type filters slower convergence rates. However, it is easier to balance the frequency attenuation and transition band for an equiripple filter therefore it is used in this experiment. For a given transition band, the attenuation is proportional to the filter order of $P_0(z)$. The larger the order, the better the attenuation. The zero pairs at $\pi$ specifies the value of the parameter p, which determines the number of zeros placed at $\pi$ on the unit circle. The more the zeros at $\pi$, the smoother the corresponding wavelet. The value of p also affects the transition band of the frequency response. A large value of p results in a narrow transition band. In the time domain, a narrower transition band implies more oscillations in the corresponding wavelet. Once $P_0(z)$ is determined it is necessary factorize it into the lowpass filters, $G_0(z)$) and $H_0(z)$. Factorization filter type was chosen arbitrary.

Using these setting parameters the designed wavelet provided higher $E/S$ ratio than the conventional Symlet and Daubechies wavelets which are frequently used in bearing fault diagnostics.



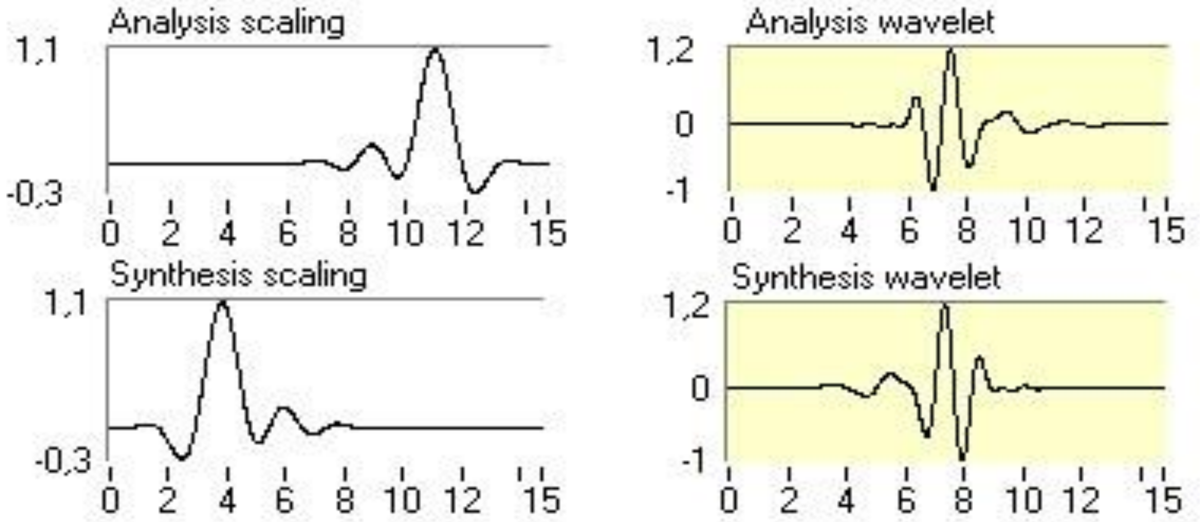

Configuration of the wavelet design

Since the new wavelet basis cannot be given in closed form it is given by the filter coefficients in Table.

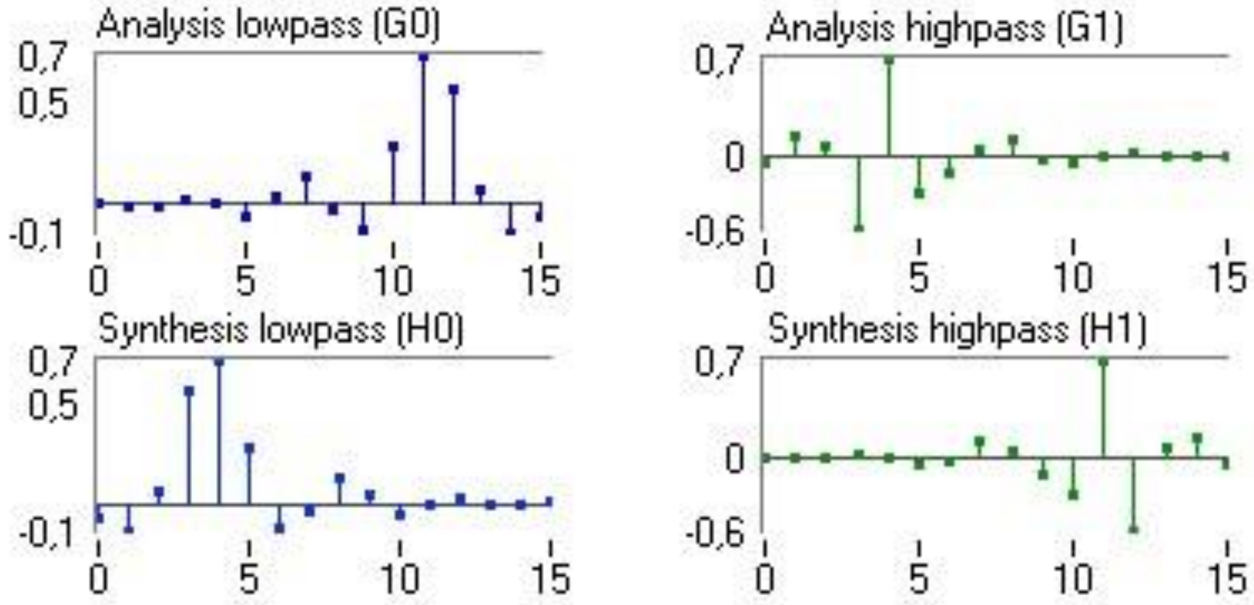
Designed wavelet filter coefficients

| | Synthesis filters | | Analysis filters | |
|---|---|---|---|---|
| No. | Lowpass (H0) | Highpass (H1) | Lowpass (G0) | Highpass (G1) |
| 0 | -0.07 | 0.0032 | 0.00 | -0.0657 |
| 1 | -0.14 | -0.0069 | -0.01 | 0.142 |
| 2 | 0.07 | 0.0025 | 0.00 | 0.0698 |
| 3 | 0.57 | 0.0258 | 0.03 | -0.5680 |
| 4 | 0.73 | 0.0023 | 0.00 | 0.7290 |
| 5 | 0.28 | -0.0529 | -0.05 | -0.2811 |
| 6 | -0.12 | -0.0375 | 0.04 | -0.1241 |
| 7 | -0.03 | 0.0129 | 0.13 | 0.0328 |
| 8 | 0.13 | 0.0330 | -0.03 | 0.1300 |
| 9 | 0.04 | -0.1230 | -0.12 | -0.0373 |
| 10 | -0.05 | -0.2810 | 0.28 | -0.0531 |

| | | | | |
|---|---|---|---|---|
| 11 | 0.00 | 0.7280 | 0.73 | 0.0022 |
| 12 | 0.03 | -0.5670 | 0.57 | 0.0258 |
| 13 | 0.00 | 0.0704 | 0.07 | 0.0026 |
| 14 | -0.01 | 0.1420 | -0.14 | -0.0068 |
| 15 | 0.00 | -0.0658 | -0.07 | -0.0032 |




Analysis and synthesis functions of the new-designed wavelet




Values of the analysis and synthesis filters of the new-designed wavelet

The coefficients of analysis and synthesis filters are in Figures. The filter coefficients were checked with Chapa and Rao's method, where the Meyer wavelet amplitude and phase spectra were matched independently to the signal. Since the two wavelets provided the same efficiency in fault detection and width estimation in practice, the filter coefficients provided by Labview were applied. The new wavelet was compared with five generally used discrete wavelets using the Maximum Energy-to-Shannon Entropy ratio criteria.

# 12th week – Questions

***Question***

What is the Parks–McClellan method and its background?

***Answer***

Parks–McClellan method (also known as the equiripple, optimal, or minimax method). The Remez exchange algorithm is commonly used to find an optimal equiripple set of coefficients. Here the user specifies a desired frequency response, a weighting function for errors from this response, and a filter order N. The algorithm then finds the set of N+1 coefficients that minimize the maximum deviation from the ideal. Intuitively, this finds the filter that is as close as possible to the desired response given that only N+1 coefficients can be used. This method is particularly easy in practice since at least one includes a program that takes the desired filter and N, and returns the optimum coefficients.

### *Question*

Describe the Equiripple FIR filters and its conventional software packages in practice!

### *Answer*

Equiripple FIR filters can be designed using the DFT algorithms as well. The algorithm is iterative in nature. The DFT of an initial filter design is computed using the FFT algorithm (if an initial estimate is not available, h[n]=delta[n] can be used). In the Fourier domain, or DFT domain, the frequency response is corrected according to the desired specs, and the inverse DFT is then computed. In the time-domain, only the first N coefficients are kept (the other coefficients are set to zero). The process is then repeated iteratively: the DFT is computed once again, correction applied in the frequency domain and so on.

Software packages such as MATLAB, GNU Octave, Scilab, and SciPy provide convenient ways to apply these different methods.

### *Question*

What is the point of the Remez exchange algorithm?

### *Answer*

Remez exchange algorithm was used as the pert of the Parks-McClellan method to find an optimal equiripple set of coefficients which is an iterative algorithm used to find simple approximations to functions. The algorithm then finds the set of N+1 coefficients that minimize the maximum deviation from the ideal. Intuitively, this finds the filter that is as close as possible to the desired response given that only N+1 coefficients can be used. Parks-McClellan VI generates a set of linear-phase FIR multiband digital filter coefficients using the number of taps, sampling frequency: fs, band parameters and filter type.

# 12th week – Exercises

**Exercise**

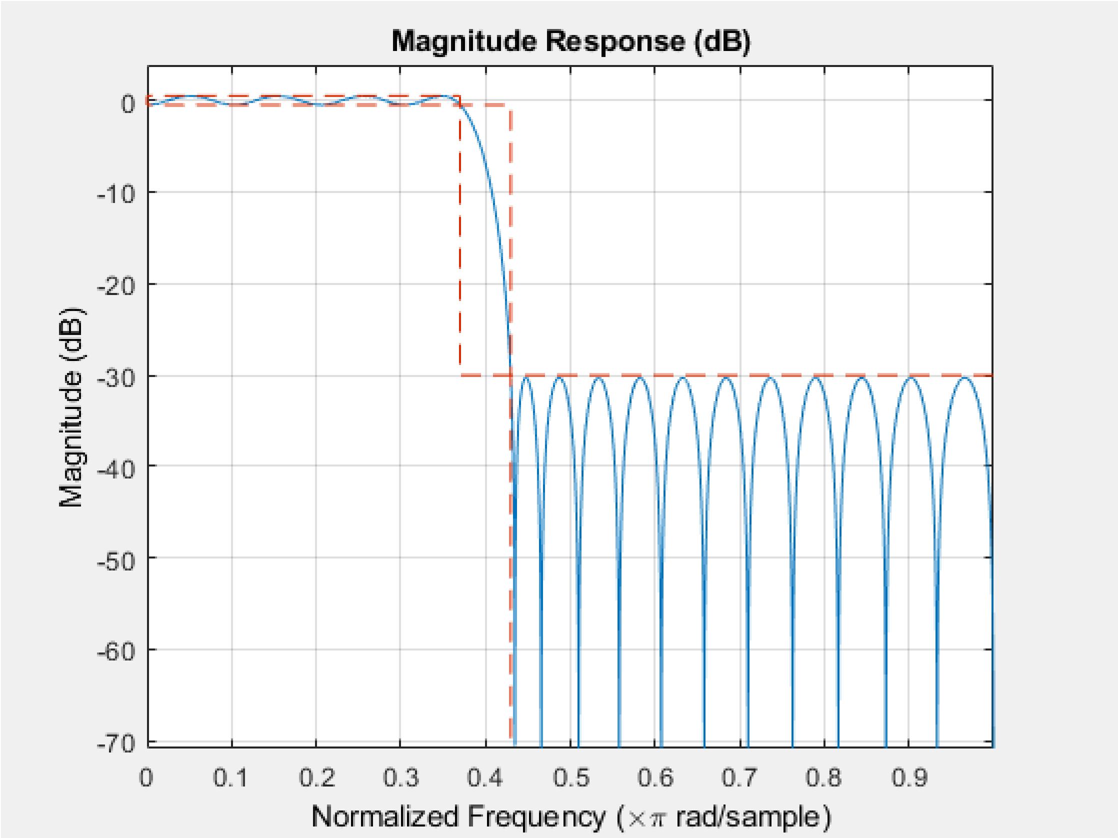
Design a minimum-order lowpass FIR filter with a passband frequency of 0.37*pi rad/sample, a stopband frequency of 0.43*pi rad/sample (hence the transition width equals 0.06*pi rad/sample), a passband ripple of 1 dB and a stopband attenuation of 30 dB.

Minimum-order designs are obtained by specifying passband and stopband frequencies as well as a passband ripple and a stopband attenuation. The design algorithm then chooses the minimum filter length that complies with the specifications.

**Solution**

```
Fpass = 0.37;
Fstop = 0.43;
Ap = 1;
Ast = 30;

d = designfilt('lowpassfir','PassbandFrequency',Fpass,...
```

```
 'StopbandFrequency',Fstop,'PassbandRipple',Ap,'StopbandAttenuation',Ast);
hfvt = fvtool(d);
```



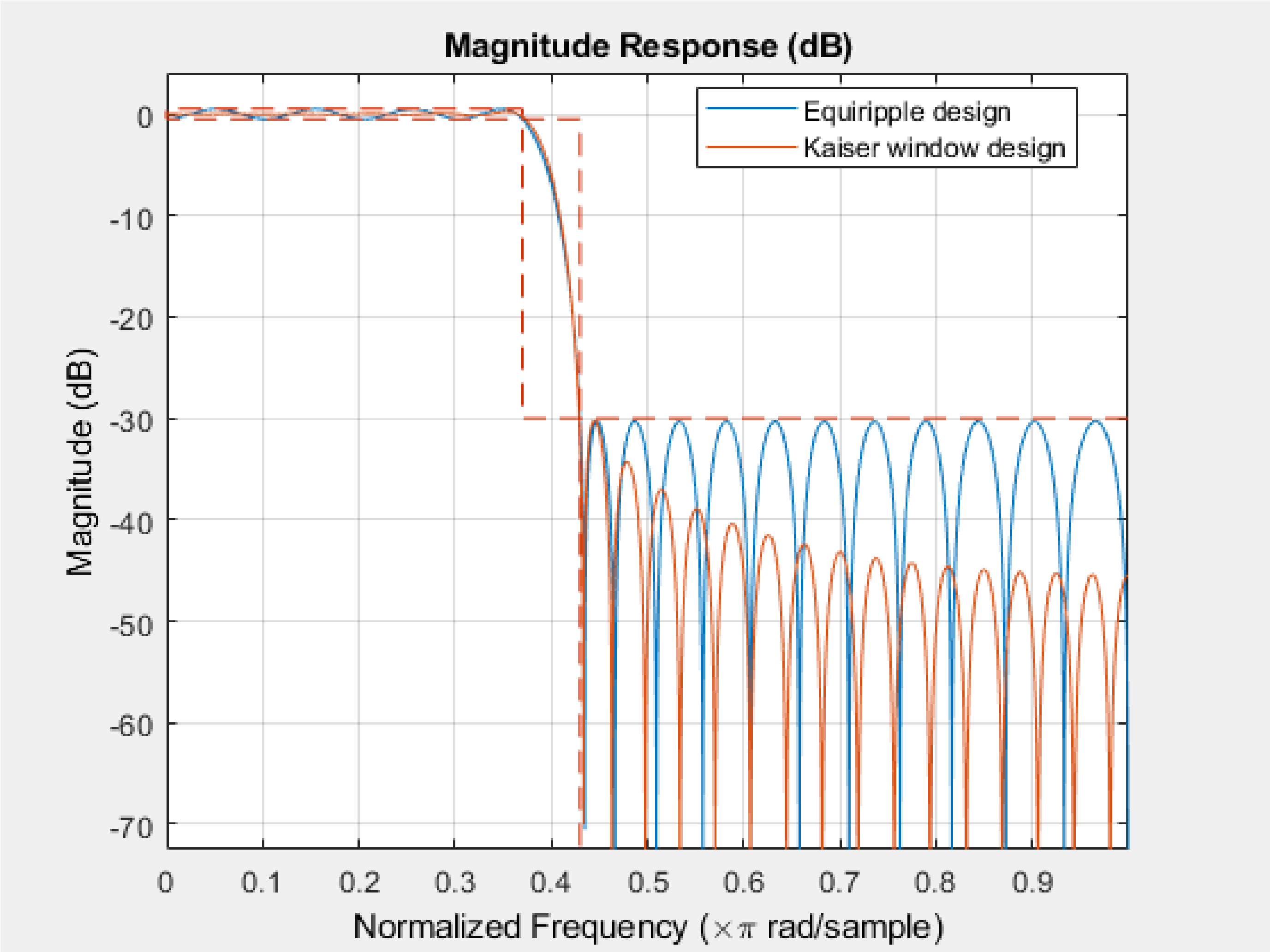
## Exercise

Design a filter with the same specifications as above using the Kaiser window method and compare its response to the equiripple filter.

The Kaiser window method yields a larger filter order for the same specifications, the algorithm is less computationally expensive and less likely to have convergence issues when the design specifications are very stringent. This may occur if the application requires a very narrow transition width or a very large stopband attenuation.

## Solution

```
dk = designfilt('lowpassfir','PassbandFrequency',Fpass,...
  'StopbandFrequency',Fstop,'PassbandRipple',Ap,...
  'StopbandAttenuation',Ast, 'DesignMethod', 'kaiserwin');
addfilter(hfvt,dk);
legend(hfvt,'Equiripple design', 'Kaiser window design')
```


Magnitude Response (dB)
Equiripple design
Kaiser window design
Magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Normalized Frequency (×π rad/sample)

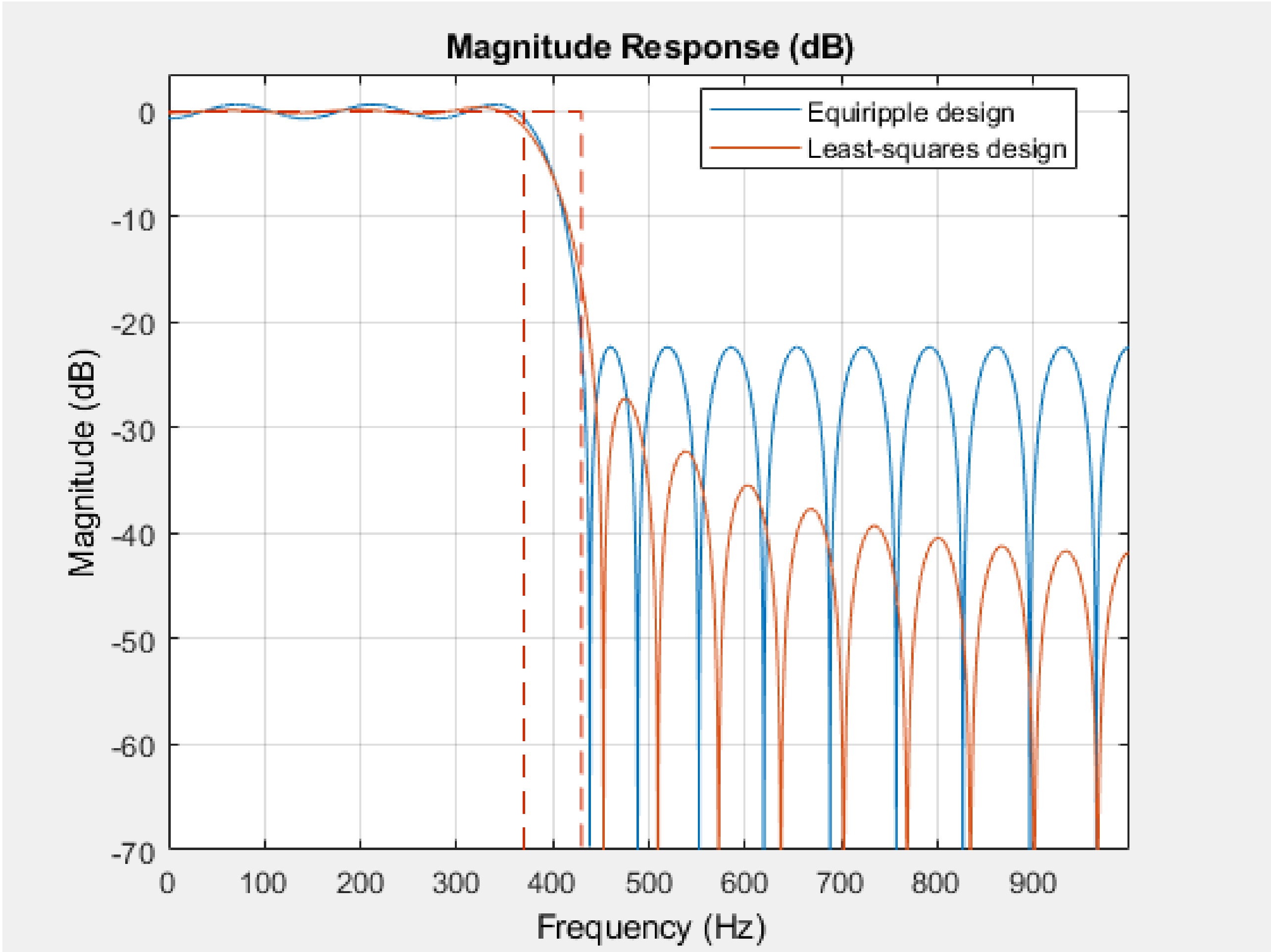
## Exercise

Consider a 30-th order lowpass FIR filter with a passband frequency of 370 Hz, a stopband frequency of 430 Hz, and sample rate of 2 kHz. There are two design methods available for this particular set of specifications: equiripple and least squares. Let us design one filter for each method and compare the results.

## Solution

```
N = 30;
Fpass = 370;
Fstop = 430;
Fs = 2000;
% Design method defaults to 'equiripple' when omitted
deq = designfilt('lowpassfir','FilterOrder',N,'PassbandFrequency',Fpass,...
  'StopbandFrequency',Fstop,'SampleRate',Fs);
dls = designfilt('lowpassfir','FilterOrder',N,'PassbandFrequency',Fpass,...
  'StopbandFrequency',Fstop,'SampleRate',Fs,'DesignMethod','ls');

hfvt = fvtool(deq,dls);
```

```
legend(hfvt,'Equiripple design', 'Least-squares design')
```



# 13 References

[1] Greenberg, M. D., Advanced Engineering Mathematics, Prentice Hall, 1998

[2] Dyke, P., An Introduction to Laplace Transforms and Fourier Series, Springer, 2014

[3] Debnath, L., Bhatta, D., Integral transforms and their applications, CRC Press, 2015

[4] Kurtosis in Random Vibration Control (Brüel & Kjær, downloaded: 21.3.2022), https://www.bksv.com/media/doc/bo0510.pdf

[5] Leonova Infinity User Guide (SPM 71792B manual)

[6] R.B. Randall: Cepstrum Analysis and Gearbox Fault Diagnosis, Brüel&Kjaer Application Notes 233-80, Edition 2. 2015

[7] Matlab applications, Mathworks https://www.mathworks.com/help/examples.html (downloaded: 19.3.2022),

[8] Micheil-Yves-Georges-Jean, Wavelets and their applications, ISTA Ltd., 2007

[9] Allan-Thomas: Harris's Shock and Vibration Handbook. McGrawHill, 2010

[10] Tom M. Mitchell: Machine learning. McGraw-Hill Science, 1997