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Tomography

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Contents

1 Introduction

Tomography is a branch of science that involves the study of three dimensional structures with the help of two dimensional sections or slices. The word tomography originates from Ancient Greek words $\tau\acute{o}\mu o\varsigma$ - tomos -, which means slice or section and $\gamma\rho\alpha\varphi\omega$ - grapho -, which means to write or, in this context, to describe. Thus the word tomography means something to describe with sections. For example in geology when a piece of stone or mineral is cut into thin pieces to reveal the inner structure, or in pathology a series of thin segments of an organ is produced to retrieve three dimensional information then it's considered a kind of tomography. However there are many situations when we don't want to or unable to cut the object. For example if the piece of stone is an important fossil or the organ is inside a living body. Another example is nondestructive testing used in science and technology industry to evaluate the properties of a material, component or system without causing damage. In such situations computed tomography comes to the rescue, when the cross-section are not physically made, but only computed with the help of measurements on some physical property. The most widely known application is X-ray transmission computed tomography, also know as CT, which is a medical imaging technique used in radiology. It's based on the fact that when x-ray photons enter a piece of material, then some of the photons are absorbed or scattered, but rest of then are transmitted through. The ratio of the amount of photons transmitted over the total number photons heavily depends on the energy of the photons and the type of the material. This is called the attenuation value of that piece of material at a fixed energy. During X-ray transmission computed tomography images of cross sections of the human body are produced from data obtained by measuring the attenuation of x-rays along a large number of lines through the cross section. The measurement of the attenuation of x-ray along a line gives an estimate of the line integral of the attenuation values of all pieces of material along that line. Thus to produce an image of the cross section we need to know how to reconstruct it from the estimates of its line integrals along finite number of lines of known locations. This is a true mathematical problem and the current text is mainly devoted to reveal the mathematical tools required for the reconstruction.

Most people think immediately on a medical imaging technique when computed tomography (CT) is mentioned, however other kind of objects can be put inside a CT scanner. Computed tomography has been found useful in non-destructive testing. A collection of transmission beam neutron radiographs can be used to reconstruct (and hence inspect) objects such as turbine blades and even whole engines, which shouldn't be destroyed. The use of neutron beam is due to the fact that such metallic objects are not penetrated well by x-rays. Another discipline where computed tomography can be used is electron crystallography to determine the arrangement of atoms in solids. Emission tomography is somewhat different, since it's not the attenuation of a transmitted wave from an outer source is measured, but instead the amount of electromagnetic radiation emitted from inside the object. In medical imaging emission tomography is used for the quantitative determination of the moment-to-moment changes in the chemistry and flow physiology of injected or inhaled compounds labeled with radioactive atoms. However emission tomography can be used in astrophysics too. We demonstrate now how image reconstruction from projections can be used in such situation.

Let's assume that we are interested in the distribution of radio sources of a small rectangular area of the sky. Unfortunately existing instruments for the detection of radio waves are of too low resolution, and thus we're able to measure only the total amount of radio waves emitted by radio sources in that rectangular area. That, of course, can't help us to record any details. However, if the moon regularly moves across the portion of the sky that is of interest, then it can help us. We assume that the rectangular area is so small compared to the moon, that the portion of the outline of the moon inside the rectangle can't be distinguished from a straight line. As the moon crosses over the rectangular area, we measure the total intensity of radio waves emitted by the uncovered part in discrete time steps. Subtraction of the measured value of the total intensity at any time instance from the measured value of the total intensity at the next time instance provides us with the total intensity in each of a set of parallel abutting thin strips of known locations. Then we estimate the line integral of the intensity along the central lines of these strips by dividing the total intensity along the strip by the width of the strip. A set of estimated line integrals along lines parallel to each other is called a view. The direction of the paths of the moon across the sky vary, providing us with a number of views. From a large number of views the two-dimensional distribution of radio sources can be reconstructed.

Now let's demonstrate the mathematical tools of reconstruction on a small problem. Let the rectangle $R = [0,4] \times [0,3]$ be given, whose sides are parallel to the coordinate directions and whose projection onto the x-axis is the $[0,4]$ interval and the projection onto the y-axis is the $[0,3]$ interval. Divide the rectangle R into the union of twelve non-overlapping rectangles (or squares actually)

$$\begin{array}{lll} R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\ R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\ R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\ R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1] \end{array}$$

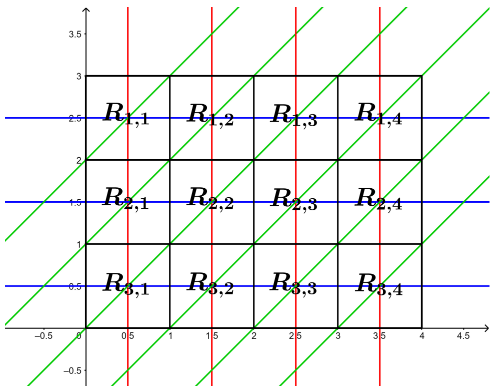


Figure 1.1: The screen R, the pixels $R_{i,j}$, and the lines l_1, l_2, l_3 (blue), lines l_4, l_5, l_6, l_7 (red), lines $l_9, l_9, l_{10}, l_{11}, l_{12}, l_{13}$ (green)

We may call the rectangle R the screen, and the squares $R_{i,j}$ the pixels. Let the point $P_{i,j}$ denote the center of the square $R_{i,j}$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$. These are

$$\begin{array}{lll} P_{1,1} = \left(\frac{1}{2}, \frac{5}{2}\right) & P_{1,2} = \left(\frac{1}{2}, \frac{3}{2}\right) & P_{3,1} = \left(\frac{1}{2}, \frac{1}{2}\right) \\ P_{1,2} = \left(\frac{3}{2}, \frac{5}{2}\right) & P_{2,2} = \left(\frac{3}{2}, \frac{3}{2}\right) & P_{3,2} = \left(\frac{3}{2}, \frac{1}{2}\right) \\ P_{1,3} = \left(\frac{5}{2}, \frac{5}{2}\right) & P_{2,3} = \left(\frac{5}{2}, \frac{3}{2}\right) & P_{3,3} = \left(\frac{5}{2}, \frac{1}{2}\right) \\ P_{1,4} = \left(\frac{7}{2}, \frac{5}{2}\right) & P_{2,4} = \left(\frac{7}{2}, \frac{3}{2}\right) & P_{3,4} = \left(\frac{7}{2}, \frac{1}{2}\right) \end{array}$$

Consider three direction vectors $\underline{v}_1 = (1, 0)$, $\underline{v}_2 = (0, 1)$ and $\underline{v}_3 = (1, 1)$ define the lines l_1, l_2, l_3 that are all parallel to $\underline{v}_1 = (1, 0)$ and passing through the points $P_{1,1}$, $P_{2,1}$ and $P_{3,1}$ respectively (see blue lines in Figure 1.1). Also define the lines l_4, l_5, l_6, l_7 that are all parallel to $\underline{v}_2 = (0, 1)$ and passing through the points $P_{3,1}$, $P_{3,2}$, $P_{3,3}$ and $P_{3,4}$ respectively (see red lines in Figure 1.1). Furthermore take the lines $l_8, l_9, l_{10}, l_{11}, l_{12}, l_{13}$ that are all parallel to $\underline{v}_3 = (1, 1)$ and passing through the points $P_{1,1}$, $P_{2,1}$, $P_{3,1}$, $P_{3,2}$, $P_{3,3}$, and $P_{3,4}$ respectively (see green lines in Figure 1.1).

Assume that the exact values of line integrals of some unknown function f defined on the rectangle R are known along the lines l_k, $k \in \{1, 2, \ldots, 13\}$. The functions f can be the two-dimensional distribution of some physical property, like x-ray attenuation values or intensity of light or other electromagnetic radiation. Let m_k denote the value of the line integral of f along the line l_k. These values are

$$\begin{array}{llllll} m_1 = 3, & m_2 = 2, & m_3 = 2, & & & \\ m_4 = 2, & m_5 = 2, & m_6 = 2, & m_7 = 1, & & \\ m_8 = \sqrt{2}, & m_9 = \sqrt{2}, & m_{10} = 2\sqrt{2}, & m_{11} = 2\sqrt{2}, & m_{12} = \sqrt{2}, & m_{13} = 0. \end{array}$$

We would like to define a function g over the rectangle R which takes the constant value $x_{i,j}$ over the square $R_{i,j}$ for each $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$, and whose line integral along the line l_k equals to m_k for all $k \in \{1, 2, \ldots, 13\}$. The question is how to choose the values of $x_{i,j}$ to ensure that the line integrals equal to the values m_k.

The line integral of a constant function along a straight line segment equals to the length of the segment multiplied with the constant value of the function. Since the squares $R_{i,j}$ don't overlap and their union covers the rectangle R, we can calculate the line integral of the function g along the line l_k by taking the sum of the values $x_{i,j}$ multiplied by the length of the intersection of l_k with the square $R_{i,j}$. It may happen that l_k doesn't intersect some of the squares and the corresponding values $x_{i,j}$ are multiplied by zero.

Now l_1 intersects only the squares $R_{1,1}$, $R_{1,2}$, $R_{1,3}$ and $R_{1,4}$, each in a line segment of length equal to 1. Hence the values $x_{1,1}$, $x_{1,2}$, $x_{1,3}$, $x_{1,4}$ must satisfy the linear equation

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = m_1$$

Similarly l_2 intersects only the squares $R_{2,1}$, $R_{2,2}$, $R_{2,3}$ and $R_{2,4}$, and l_3 intersects only the squares $R_{3,1}$, $R_{3,2}$, $R_{3,3}$ and $R_{3,4}$ in line segments of length equal to 1. Hence the values $x_{2,1}$, $x_{2,2}$, $x_{2,3}$, $x_{2,4}$ and $x_{3,1}$, $x_{3,2}$, $x_{3,3}$, $x_{3,4}$ must satisfy

$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = m_2$$

and

$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = m_3$$

The line l_4 intersects the squares $R_{1,1}$, $R_{2,1}$, $R_{3,1}$, the line l_5 intersects the squares $R_{1,2}$, $R_{2,2}$, $R_{3,2}$, the line l_6 intersects the squares $R_{1,3}$, $R_{2,3}$, $R_{3,3}$, and the line l_7 intersects the squares $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. Each time the length of the intersection equals to 1. Thus these provide the following linear equations:

$$\begin{aligned}
x_{1,1} + x_{2,1} + x_{3,1} &= m_4 \\
x_{1,2} + x_{2,2} + x_{3,2} &= m_5 \\
x_{1,3} + x_{2,3} + x_{3,3} &= m_6 \\
x_{1,4} + x_{2,4} + x_{3,4} &= m_7
\end{aligned}$$

Furthermore the line l_8 intersects only the square $R_{1,1}$. The line l_9 intersects the squares $R_{2,1}$ and $R_{1,2}$, the line l_{10} intersects the squares $R_{3,1}$, $R_{2,2}$, $R_{1,3}$, the line l_{11} intersects the squares $R_{3,2}$, $R_{2,3}$, $R_{1,4}$, the line l_{12} intersects the squares $R_{3,3}$ and $R_{2,4}$, finally the line l_{13} intersects only the square $R_{3,4}$. This time the length of each intersection equals to $\sqrt{2}$. Thus they imply the following equations

$$\begin{aligned}
\sqrt{2}\,x_{1,1} &= m_8 \\
\sqrt{2}\,x_{1,2} + \sqrt{2}\,x_{2,1} &= m_9 \\
\sqrt{2}\,x_{1,3} + \sqrt{2}\,x_{2,2} + \sqrt{2}\,x_{3,1} &= m_{10} \\
\sqrt{2}\,x_{1,4} + \sqrt{2}\,x_{2,3} + \sqrt{2}\,x_{3,2} &= m_{11} \\
\sqrt{2}\,x_{2,4} + \sqrt{2}\,x_{3,3} &= m_{12} \\
\sqrt{2}\,x_{3,4} &= m_{13}
\end{aligned}$$

Collecting all the above equations and substituting the values m_k on the right-hand-sides we have a system of linear equations.

$$\left.\begin{aligned} x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}&=3\\ x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4}&=2\\ x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}&=2\\ x_{1,1}+x_{2,1}+x_{3,1}&=2\\ x_{1,2}+x_{2,2}+x_{3,2}&=2\\ x_{1,3}+x_{2,3}+x_{3,3}&=2\\ x_{1,4}+x_{2,4}+x_{3,4}&=1\\ \sqrt{2}\,x_{1,1}&=\sqrt{2}\\ \sqrt{2}\,x_{1,2}+\sqrt{2}\,x_{2,1}&=\sqrt{2}\\ \sqrt{2}\,x_{1,3}+\sqrt{2}\,x_{2,2}+\sqrt{2}\,x_{3,1}&=2\sqrt{2}\\ \sqrt{2}\,x_{1,4}+\sqrt{2}\,x_{2,3}+\sqrt{2}\,x_{3,2}&=2\sqrt{2}\\ \sqrt{2}\,x_{2,4}+\sqrt{2}\,x_{3,3}&=\sqrt{2}\\ \sqrt{2}\,x_{3,4}&=0 \end{aligned}\right\} \tag{1.1}$$

We need to find the solution of the above system to find appropriate values of $x_{i,j}$. Systems of linear equations can be solved with the help of Gauss elimination, which is discussed in the later sections. This system has more than one solution. More precisely we can choose arbitrary values for $x_{3,2}$ and $x_{3,3}$, and then all other variables can be given in terms of these two in the following way.

$$\begin{aligned} x_{1,1}&=1 & x_{2,1}&=-1+x_{3,2}+x_{3,3} & x_{3,1}&=2-x_{3,2}-x_{3,3}\\ x_{1,2}&=2-x_{3,2}-x_{3,3} & x_{2,2}&=x_{3,3} & x_{3,2}&=x_{3,2}\\ x_{1,3}&=x_{3,2} & x_{2,3}&=2-x_{3,2}-x_{3,3} & x_{3,3}&=x_{3,3}\\ x_{1,4}&=x_{3,3} & x_{2,4}&=1-x_{3,3} & x_{3,4}&=0 \end{aligned}$$

However if the unknown function f is the two-dimensional distribution of the x-ray attenuation values or intensity of some electromagnetic radiation, then we're interested only in the non-negative solutions of the system. This means we can choose only such values of the variables $x_{i,j}$ that satisfy the following system of linear inequalities.

$$\left.\begin{aligned} x_{3,2}&\geq 0\\ x_{3,3}&\geq 0\\ -1+x_{3,2}+x_{3,3}&\geq 0\\ 2-x_{3,2}-x_{3,3}&\geq 0\\ 1-x_{3,3}&\geq 0 \end{aligned}\right\}$$

This can be rearranged as

$$\left.\begin{aligned} x_{3,2} &\geq 0 \\ x_{3,3} &\geq 0 \\ x_{3,3} &\geq 1 - x_{3,2} \\ 2 - x_{3,2} &\geq x_{3,3} \\ 1 &\geq x_{3,3} \end{aligned}\right\}$$

We see that $x_{3,3}$ must be not less than 0 and $1 - x_{3,2}$, on the other hand it must be not more than 1 and $2 - x_{3,2}$. Here $1 - x_{3,2} \leq 2 - x_{3,2}$ is always satisfied. Thus the system may have a solution only if

$$\left.\begin{aligned} 0 &\leq 2 - x_{3,2} \\ 1 - x_{3,2} &\leq 1 \end{aligned}\right\}$$

and besides these $x_{3,2}$ must be non-negative. If we rearrange the above equations, then

$$\left.\begin{aligned} x_{3,2} &\leq 2 \\ 0 &\leq x_{3,2} \end{aligned}\right\}$$

These give a lower and upper bound for $x_{3,2}$ and then we obtain a lower and upper bound for $x_{3,3}$ as well.

$$\begin{aligned} 0 \leq x_{3,2} &\leq 2 \\ \max\{0, 1 - x_{3,2}\} \leq x_{3,3} &\leq \min\{2 - x_{3,2}, 1\} \end{aligned}$$

Finally we can conclude that all the non-negative solutions of the system of linear equations 1.1 are those that satisfy

$$\begin{array}{lll} x_{1,1} = 1 & x_{2,1} = -1 + x_{3,2} + x_{3,3} & x_{3,1} = 2 - x_{3,2} - x_{3,3} \\ x_{1,2} = 2 - x_{3,2} - x_{3,3} & x_{2,2} = x_{3,3} & x_{3,2} = x_{3,2} \\ x_{1,3} = x_{3,2} & x_{2,3} = 2 - x_{3,2} - x_{3,3} & x_{3,3} = x_{3,3} \\ x_{1,4} = x_{3,3} & x_{2,4} = 1 - x_{3,3} & x_{3,4} = 0 \end{array}$$

where

$$\begin{aligned} 0 \leq x_{3,2} &\leq 2 \\ \max\{0, 1 - x_{3,2}\} \leq x_{3,3} &\leq \min\{2 - x_{3,2}, 1\} \end{aligned}$$

If we think that the unknown function f is the two-dimensional distribution of x-ray attenuation values, then the variables $x_{i,j}$ can't be larger than 1, as the x-ray attenuation value is the number of x-ray photons transmitted over the total number of photons. Thus besides non-negativity another restriction on the values can be that all $x_{i,j}$ should be not larger than 1. If we add the this new restriction to the system, then we have

$$\left.\begin{aligned} x_{3,2} &\geq 0 \\ x_{3,3} &\geq 0 \\ -1 + x_{3,2} + x_{3,3} &\geq 0 \\ 2 - x_{3,2} - x_{3,3} &\geq 0 \\ 1 - x_{3,3} &\geq 0 \\ x_{3,2} &\leq 1 \\ x_{3,3} &\leq 1 \\ -1 + x_{3,2} + x_{3,3} &\leq 1 \\ 2 - x_{3,2} - x_{3,3} &\leq 1 \\ 1 - x_{3,3} &\leq 1 \end{aligned}\right\}$$

This can be rearranged as

$$\left.\begin{aligned} x_{3,2} &\geq 0 \\ x_{3,3} &\geq 0 \\ x_{3,3} &\geq 1 - x_{3,2} \\ 2 - x_{3,2} &\geq x_{3,3} \\ 1 &\geq x_{3,3} \\ x_{3,2} &\leq 1 \\ x_{3,3} &\leq 1 \\ x_{3,3} &\leq 2 - x_{3,2} \\ 1 - x_{3,2} &\leq x_{3,3} \\ 0 &\leq x_{3,3} \end{aligned}\right\}$$

Turning each inequality int less-than-or-equal form and deleting the doubles gives

$$\left.\begin{aligned} 0 &\leq x_{3,2} \\ 0 &\leq x_{3,3} \\ 1 - x_{3,2} &\leq x_{3,3} \\ x_{3,3} &\leq 2 - x_{3,2} \\ x_{3,3} &\leq 1 \\ x_{3,2} &\leq 1 \end{aligned}\right\}$$

Thus we have only one inequality which isn't part of the system when we only assumed non-negativity. This new inequality is $x_{3,2} \leq 1$. Hence the upper and lower bounds we obtain for $x_{3,2}$ and $x_{3,3}$ are

$$\begin{aligned} 0 \leq x_{3,2} &\leq 1 \\ \max\{0, 1 - x_{3,2}\} \leq x_{3,3} &\leq \min\{2 - x_{3,2}, 1\} \end{aligned}$$

Here $\max\{0, 1 - x_{3,2}\} = 1 - x_{3,2}$ as $x_{3,2} \leq 1$, and $\min\{2 - x_{3,2}, 1\} = 1$ because of the same reason. Thus the above bounds can be simplified as

$$\begin{aligned} 0 \leq x_{3,2} &\leq 1 \\ 1 - x_{3,2} \leq x_{3,3} &\leq 1 \end{aligned}$$

Finally we can conclude that all the non-negative solutions of the system of linear equations 1.1 are those that satisfy

$$\begin{aligned} &x_{1,1} = 1 && x_{2,1} = -1 + x_{3,2} + x_{3,3} && x_{3,1} = 2 - x_{3,2} - x_{3,3} \\ &x_{1,2} = 2 - x_{3,2} - x_{3,3} && x_{2,2} = x_{3,3} && x_{3,2} = x_{3,2} \\ &x_{1,3} = x_{3,2} && x_{2,3} = 2 - x_{3,2} - x_{3,3} && x_{3,3} = x_{3,3} \\ &x_{1,4} = x_{3,3} && x_{2,4} = 1 - x_{3,3} && x_{3,4} = 0 \end{aligned}$$

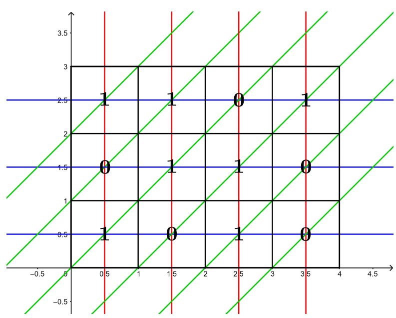
where

$$\begin{aligned} 0 \leq x_{3,2} &\leq 1 \\ 1 - x_{3,2} \leq x_{3,3} &\leq 1 \end{aligned}$$

Among these the integer solutions are the following.

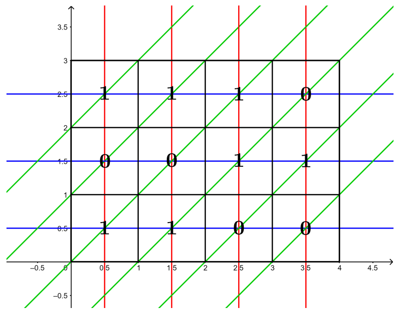
1. If $x_{3,2} = 0$, then $x_{3,3} = 1$ and

$$\begin{aligned} &x_{1,1} = 1 && x_{1,2} = 1 && x_{1,3} = 0 && x_{1,4} = 1 \\ &x_{2,1} = 0 && x_{2,2} = 1 && x_{2,3} = 1 && x_{2,4} = 0 \\ &x_{3,1} = 1 && x_{3,2} = 0 && x_{3,3} = 1 && x_{3,4} = 0 \end{aligned}$$



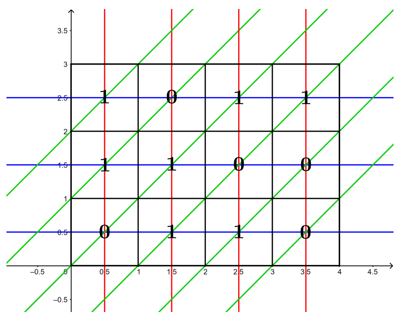
2. If $x_{3,2} = 1$ and $x_{3,3} = 0$, then

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 1 & x_{1,3} = 1 & x_{1,4} = 0 \\ x_{2,1} = 0 & x_{2,2} = 0 & x_{2,3} = 1 & x_{2,4} = 1 \\ x_{3,1} = 1 & x_{3,2} = 1 & x_{3,3} = 0 & x_{3,4} = 0 \end{array}$$



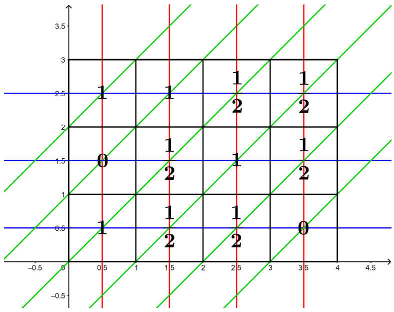
3. If $x_{3,2} = 1$ and $x_{3,3} = 1$, then

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 0 & x_{1,3} = 1 & x_{1,4} = 1 \\ x_{2,1} = 1 & x_{2,2} = 1 & x_{2,3} = 0 & x_{2,4} = 0 \\ x_{3,1} = 0 & x_{3,2} = 1 & x_{3,3} = 1 & x_{3,4} = 0 \end{array}$$



Furthermore if $x_{3,2} = \frac{1}{2}$ and $x_{3,3} = \frac{1}{2}$, then a non-integer solution is

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 1 & x_{1,3} = \frac{1}{2} & x_{1,4} = \frac{1}{2} \\ x_{2,1} = 0 & x_{2,2} = \frac{1}{2} & x_{2,3} = 1 & x_{2,4} = \frac{1}{2} \\ x_{3,1} = 1 & x_{3,2} = \frac{1}{2} & x_{3,3} = \frac{1}{2} & x_{3,4} = 0 \end{array}$$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2 Mathematical preliminaries

2.1 Matrices

A matrix is a rectangular array of numbers arranged in rows and columns. The numbers in a matrix are called entries or elements and we can refer to the entries with a pair of indices, the row index and and column index. The row are indexed by positive integers from top to the bottom starting with 1. Columns are indexed by positive integers from left to right starting with 1. Matrices are often denoted by capital latin letters, while their entries are denoted by the corresponding lower case letter with a pair of subscript indices. First we write the row index and then the column index separated by a comma. An example of a matrix is the following:

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & -2 \\ 0 & 1 & 1 & 5 \end{pmatrix}$$

The matrix A above has 3 rows and 4 columns. We can refer to the entry in the 2nd row and 3rd column as $a_{2,3}$ and it equals to 0, while the entry in the 3rd row and 2nd column is referred as $a_{3,2}$ and it equals to 1. We note here that if the matrix is not too large and it makes no confusion we can omit the comma between the row and column indices, thus we can refer to the above entries as a_{23} and a_{32} too. In a larger matrix we refer to the the entry in the i-th row and j-th column as $a_{i,j}$ or a_{ij}. The notation

$$A = (a_{ij})$$

is used when we would like to refer to or denote the element of the matrix A in the i-th row and j-th column as a_{ij}. This notation is used only when it's clear what are the ranges of the indices i and j (i.e. what is the number of rows and the number of columns). Otherwise we can specify a matrix with

m rows and n columns as

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

If a matrix has m rows and n columns then we say the size (or dimension) of the matrix is $m \times n$. The sample matrix A above has size 3×4. The set of all real matrices (i.e. matrices whose entries are real numbers) with m rows and n columns is denoted by $\mathcal{M}_{m\times n}(\mathbb{R})$. A matrix with the same number of rows and columns is called a square matrix. The set of all real square matrices with n rows and n columns is denoted by $\mathcal{M}_n(\mathbb{R})$.

2.1.1 Matrix operations

Now we define the basic matrix operations such as addition, scalar multiplication and transposition.

Definition 1 *Let a $A, B \in \mathcal{M}_{m\times n}(\mathbb{R})$ be two matrices of size $m \times n$. The* ***sum of** A **and** B* *is the matrix $C \in \mathcal{M}_{m\times n}(\mathbb{R})$ of size $m \times n$ which satisfies*

$$c_{ij} = a_{ij} + b_{ij}$$

if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. The sum of the matrices A and B is denoted by $A + B$.

Note that the sum of two matrices can be defined only if they have the same size.

Definition 2 *Let a $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a matrix of size $m \times n$ and let $\lambda \in \mathbb{R}$ (lambda) be a real number. The* ***product of the matrix** A **with the number** λ* *is the matrix $C \in \mathcal{M}_{m\times n}(\mathbb{R})$ of size $m \times n$ which satisfies*

$$c_{ij} = \lambda \cdot a_{ij}$$

if $A = (a_{ij})$ and $C = (c_{ij})$. The product of the matrix A with the number λ is denoted by $\lambda \cdot A$ or shortly λA if it makes no confusion.

The above operation is called **scalar multiplication**, but make sure to not confuse it with scalar product which is a different operation defined for vectors. The real number λ in the scalar multiplication is often called scalar (thus the name).

Theorem 1 *The addition and scalar multiplication of matrices have the following properties.*

1. $(A+B)+C = A+(B+C)$ *for any matrices* $A, B, C \in \mathcal{M}_{m\times n}(\mathbb{R})$.
2. *There exists a* ***zero matrix*** *of size* $m \times n$ *denoted by* $0_{m\times n}$ *which satisfies* $A + 0_{m\times n} = A$ *for any matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{R})$.
3. *For any matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ *there exists an opposite matrix of size* $m \times n$ *denoted by* $-A$ *which satisfies* $A + (-A) = 0_{m\times n}$.
4. $A + B = B + A$ *for any matrices* $A, B \in \mathcal{M}_{m\times n}(\mathbb{R})$.
5. $\lambda(A+B) = \lambda A + \lambda B$ *for any matrices* $A, B \in \mathcal{M}_{m\times n}(\mathbb{R})$ *and any scalar* $\lambda \in \mathbb{R}$.
6. $(\lambda + \mu)A = \lambda A + \mu B$ *for any matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ *and any scalars* $\lambda, \mu \in \mathbb{R}$ *(lambda and mu).*
7. $(\lambda \cdot \mu)A = \lambda(\mu A) = \mu(\lambda A)$.

Examples.

The zero matrix of size 2×3 is

$$0_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Furthermore let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 1+2 & 2-4 & -1+0 \\ 3-1 & 4+1 & 0+3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -1 \\ 2 & 5 & 3 \end{pmatrix}$$

$$-A = \begin{pmatrix} -1 & -2 & 1 \\ -3 & -4 & 0 \end{pmatrix} \quad -B = \begin{pmatrix} -2 & 4 & 0 \\ 1 & -1 & -3 \end{pmatrix}$$

$$3 \cdot A = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot (-1) \\ 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -3 \\ 9 & 12 & 0 \end{pmatrix}$$

$$(-2) \cdot B = \begin{pmatrix} (-2) \cdot 2 & (-2) \cdot (-4) & (-2) \cdot 0 \\ (-2) \cdot (-1) & (-2) \cdot 1 & (-2) \cdot 3 \end{pmatrix} = \begin{pmatrix} -4 & 8 & 0 \\ 2 & -2 & -6 \end{pmatrix}$$

Definition 3 *Let a $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a matrix of size $m \times n$ and let $\lambda \in \mathbb{R}$ (lambda) be a real number. The* ***transpose of the matrix*** *A the matrix $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ of size $n \times m$ which satisfies*

$$b_{ij} = \lambda \cdot a_{ji}$$

if $A = (a_{ij})$ and $B = (c_{ij})$. The transpose of the matrix A is denoted by $A^\top$.

As an example let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$$

Then

$$A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ -1 & 0 \end{pmatrix}$$

A square matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called symmetrical if $A = A^\top$. For example the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -5 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

is a symmetrical matrix.

Theorem 2 *Let $A, B \in \mathcal{M}_{m\times n}(\mathbb{R})$ be arbitrary matrices of size $m \times n$ and let $\lambda \in \mathbb{R}$ be an arbitrary scalar. Then*

1. $(A + B)^\top = A^\top + B^\top$,
2. $(\lambda A)^\top = \lambda A^\top$,
3. $\left(A^\top\right)^\top = A$.

2.1.2 Matrix multiplication

The addition and scalar multiplication of matrices defined in the previous section are calculated elementwise. This means that if we want to add the matrices A and B, then we only need to add the corresponding elements. Similarly to multiply a matrix by a scalar we need to multiply each element of the matrix with that scalar. Now we introduce the product of two matrices, which is defined in a different manner (not elementwise).

Definition 4 *Let a $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a matrix of size $m \times n$ and $B \in \mathcal{M}_{n\times l}(\mathbb{R})$ be a matrix of size $n \times l$. The* ***product of*** *A* ***and*** *B is the matrix $C \in \mathcal{M}_{m\times l}(\mathbb{R})$ of size $m \times l$ which satisfies*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. The product of the matrices A and B is denoted by $A \cdot B$ or shortly AB.

Note that the product of A and B is defined only if A has the same number of columns as the the number of rows of B. Note also that the element of the product matrix in the i-th row and j-th column is computed as the product of the i-th row of A and the j-th column of B in the same manner as the dot product of two n-dimensional vectors are defined. To see an example let

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & -2 \\ 0 & 1 & 1 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix}$$

Then the product matrix is $A \cdot B = C = (c_{ij})$, where

$$\begin{aligned} c_{11} &= \sum_{k=1}^{4} a_{1k} \cdot b_{k1} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + a_{14} \cdot b_{41} \\ &= 1 \cdot 2 + 2 \cdot (-1) + (-1) \cdot 0 + 0 \cdot 1 = 0 \\ c_{12} &= \sum_{k=1}^{4} a_{1k} \cdot b_{k2} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} + a_{14} \cdot b_{42} \\ &= 1 \cdot (-1) + 2 \cdot 1 + (-1) \cdot 3 + 0 \cdot 2 = -2 \end{aligned}$$

$$c_{21} = \sum_{k=1}^{4} a_{2k} \cdot b_{k1} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41}$$
$$= 3 \cdot 2 + 4 \cdot (-1) + 0 \cdot 0 + (-2) \cdot 1 = 0$$

$$c_{22} = \sum_{k=1}^{4} a_{2k} \cdot b_{k2} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42}$$
$$= 3 \cdot (-1) + 4 \cdot 1 + 0 \cdot 3 + (-2) \cdot 2 = -3$$

$$c_{31} = \sum_{k=1}^{4} a_{3k} \cdot b_{k1} = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} + a_{34} \cdot b_{41}$$
$$= 0 \cdot 2 + 1 \cdot (-1) + 1 \cdot 0 + 5 \cdot 1 = 4$$

$$c_{32} = \sum_{k=1}^{4} a_{3k} \cdot b_{k2} = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} + a_{44} \cdot b_{42}$$
$$= 0 \cdot (-1) + 1 \cdot 1 + 1 \cdot 3 + 5 \cdot 2 = 14$$

Thus

$$A \cdot B = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & -3 \\ 4 & 14 \end{pmatrix}$$

Until you are not experienced in the multiplication of matrices it makes easier to calculate the product if you arrange the factors A and B not next to each other, but A to the bottom left and B to the top right position of a 2-by-2 arrangement. Then the product matrix is written in the bottom right position as the following formula shows.

$$\begin{array}{cc} & \begin{pmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & -2 \\ 0 & 1 & 1 & 5 \end{pmatrix} & \begin{pmatrix} 0 & -2 \\ 0 & -3 \\ 4 & 14 \end{pmatrix} \end{array}$$

Theorem 3 *The multiplication of matrices has the following properties.*

1. *$A \cdot (B \cdot C) = (A \cdot B) \cdot C$ for any matrices A, B, C with appropriate sizes.*

2. *There exists an identity matrix of size $n \times n$ denoted by I_n, which satisfies $I_n \cdot A = A \cdot I_n = A$ for any square matrix A of size $n \times n$.*

3. *$A \cdot (B + C) = A \cdot B + A \cdot C$ for any matrices A, B, C with appropriate sizes.*

4. *$(A + B) \cdot C = A \cdot C + B \cdot C$ for any matrices A, B, C with appropriate sizes.*

5. *$\lambda(A \cdot B) = (\lambda A) \cdot B = A \cdot (\lambda B)$ for any scalar $\lambda \in \mathbb{R}$ and any matrices A, B with appropriate sizes.*

Examples:

The identity matrices of size 2×2 and 3×3 are

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix of size $n \times n$ is

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

We can also define the identity matrix with the help of the Kronecker-delta, which is

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then the identity matrix of size $n \times n$ is $I_n = (\delta_{ij})$.

Consider now the following matrices.

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Then

$$A \cdot B = \begin{pmatrix} -2 & -2 & -1 \\ 4 & 4 & 2 \end{pmatrix}$$

but the product $B \cdot A$ doesn't exist as B has 3 columns and A has 2 rows. Similarly the product $A \cdot C$ doesn't exist as A has 2 columns and C has 3 rows, but the product $C \cdot A$ exists and

$$C \cdot A = \begin{pmatrix} 2 & -2 \\ -5 & 5 \\ 2 & -2 \end{pmatrix}$$

Both of the products $B \cdot C$ and $C \cdot B$ exist

$$B \cdot C = \begin{pmatrix} 4 & 7 \\ 10 & 12 \end{pmatrix} \quad \text{and} \quad C \cdot B = \begin{pmatrix} 2 & 4 & -2 \\ 10 & 14 & -1 \\ -3 & -4 & 0 \end{pmatrix}$$

but $B \cdot C \neq C \cdot B$ as these product don't even have the same size. Similarly both of the products $A \cdot D$ and $D \cdot A$ exist

$$A \cdot D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D \cdot A = \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}$$

but $A \cdot D \neq D \cdot A$ even though these products have the same size. Finally we see that

$$A^2 = A \cdot A = \begin{pmatrix} 3 & -3 \\ -6 & 6 \end{pmatrix} \quad \text{and} \quad D^2 = D \cdot D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

An important note is that if we interchange the factors in the multiplication of matrices, then the product may be undefined (see $A \cdot B$ and $B \cdot A$, or $A \cdot C$ and $C \cdot A$ above), but even if both products are defined it may happen that they don't equal (see $B \cdot C$ and $C \cdot B$, or $A \cdot D$ and $D \cdot A$). In fact if we randomly take two square matrices of size $n \times n$, A and B (this ensures, that both $A \cdot B$ and $B \cdot A$ are defined and have the same size), then it's very likely that $A \cdot B \neq B \cdot A$.

Another interesting property of the multiplication of matrices is that it may happen that none of the matrices A and B is the zero matrix, but their product is the zero matrix (see $A \cdot D$ above). Note that this is a property which is not valid for the multiplication of real numbers, since if $a, b \in \mathbb{R}$ and $a \cdot b = 0$, then $a = 0$ or $b = 0$ (this is called the zero-product property). We can go even further as there exist a nonzero matrix whose square (i.e. the product with itself) is the zero matrix (see D^2 above).

2.1.3 Matrices of special shape

We already introduced square matrices (i.e. matrices with the same number of rows and columns) in the above sections. Now we define further matrices of special shape. First of all lets mention that a single column matrix (which has only one column) with n rows, or a single row matrix (which has only one row) with n columns can be identified as an element of the n-dimensional coordinate space $\mathbb{R}^n$, whose elements are often called vectors. Thus a single column matrix is called a **column vector**, while a single row matrix is called a **row vector**. Note that the transpose of a column vector is a row vector and the transpose of a row vector is a column vector. In the rest of the text we apply the convention that if nothing else said then by a vector we always mean a column vector. Now let's see further matrices of special shape.

Definition 5 *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $A = (a_{ij})$ be a matrix of size n. The* ***diagonal of*** *A is the sequence of its elements with equal row and column indices, i.e. $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$. The matrix A is called* ***diagonal matrix*** *if all elements of A not contained in the diagonal equal to zero, i.e. $a_{ij} = 0$ if $i \neq j$. The matrix A is called* ***upper triangular matrix*** *if all elements of A below the diagonal equal to zero, i.e. $a_{ij} = 0$ if $i > j$. The matrix A is called* ***lower triangular matrix*** *if all elements of A above the diagonal equal to zero, i.e. $a_{ij} = 0$ if $i < j$.*

We note that a diagonal matrix may contain zeros in the diagonal too. Similarly an upper triangular matrix may have zeros in and above the diagonal, just as a lower triangular matrix may have zeros in and below the diagonal. As examples the zero matrix and the identity matrix can be considered as diagonal, upper triangular or lower triangular matrix. Further instances are the following.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

Above the matrix A is diagonal (and also upper and lower triangular), B is upper triangular, C is lower triangular.

Definition 6 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, $A = (a_{ij})$ be a matrix. If the i-th row of A contains at least one nonzero element, then the leftmost nonzero element is called the **pivot element** of the i-th row. We say that the matrix is A in a **row echelon form** if all rows consisting of only zeros are at the bottom and the pivot element of any row (except the first) is strictly to the right from the pivot element of the previous row. In other words A in a row echelon form if considering the pivot elements from the top to the bottom, the column indices of the pivot elements form a strictly increasing sequence. Furthermore we say A is in **reduced row echelon form** if it's in row echelon form, all pivot elements equal to 1, and each column containing a pivot element has zeros in all other entries.*

Consider the following matrices.

$$A = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 2 & -4 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here A in not in row echelon form since the pivot element of the second row has column index 3, but the pivot element of the third row has column index 2. B in not in row echelon form also, since the column indices of the pivot elements in the second and third rows equal. C is in row echelon form, but not reduced echelon form, since none of the pivot elements equal to 1, and furthermore the second column and fourth column contain nonzero entries besides the pivot element. The matrix D is in reduced row echelon form.

2.1.4 Elementary row operations

We will see in the later chapters that matrices of reduced row echelon form have an important role in the solution of systems of linear equations. Thus now we introduce elementary row operations that can be applied to transform a matrix into reduced row echelon form.

Definition 7 *The elementary row operations are the following.*

1. *Interchange two rows of a matrix.*
2. *Multiply each element of a row in a matrix with a nonzero scalar $\lambda \in \mathbb{R}$.*
3. *Add each element of a row of a matrix to the corresponding elements of another row.*
4. *As a combination of the above two we can add each element of a row multiplied by $\lambda \in \mathbb{R}$ to the corresponding elements of another row.*

Note that during the last operation if we add λ-times the i-th row the j-th row, then only the entries of the j-th row are modified, while the i-t row remain unchanged. Note also that if we multiply a row with the multiplicative inverse nonzero scalar $\lambda \in \mathbb{R}$, then it has the same result as if we divide with $\lambda \in \mathbb{R}$. Thus the second elementary row operation means that we can also divide each element of a row in a matrix with a nonzero scalar $\lambda \in \mathbb{R}$.

Theorem 4 *We can transform any matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ into a matrix of reduced row echelon form with the help of finitely many elementary row operations.*

Now we give a sketch of the proof of the above theorem by telling what steps are required to achieve the reduced row echelon form. The procedure, that transforms an arbitrary matrix to a matrix in reduced row echelon from with the help of elementary row operations, is called **Gaussian elimination**.

First let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be an arbitrary matrix. Let k be the counter of how many pivot elements we found. Set the initial value of k to zero. Let B_k denote the part of A below the k-th row. At the beginning, when $k = 0$ let $B_0 = A$. Now follow the steps below to transform A into a matrix of reduced row echelon form.

1. If B_k is the zero matrix, then A is in reduced row echelon form and the procedure terminates.

2. Otherwise let j be the column index of the leftmost nonzero column of B_k.

3. If the first element of the j-th column of B_k is zero, then look for a row in B_k which contains a nonzero element in the j-th column and interchange it with the first row of B_k (i.e. with the $k+1$-th row of A).

4. Divide each element of the $k+1$-th row of A with $a_{k+1,j}$.

5. Add $-a_{ij}$-times the $k+1$-th row to the i-th row of A for all $i \in \{1, 2, \dots, m\}$, $i \neq k+1$.

6. Increase the value of the counter k by 1.

7. Repeat steps (1)-(6) until the procedure terminates in step (1) or k is increased to n.

The above procedure transforms any matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ into a matrix of reduced row echelon form. Let's illustrate the procedure on the following matrix.

$$A = \begin{pmatrix} 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & 3 & 2 & 3 & 0 & 4 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix}$$

First $k = 0$ and $B_0 = A$ is not the zero matrix. The leftmost nonzero column is the second column. The first element of the second column is zero, thus we need to choose a row whose entry in the second column is nonzero. Let's choose the second row. Now we interchange the first and second row of A. This step is denoted as follows.

$$A = \begin{pmatrix} 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & 3 & 2 & 3 & 0 & 4 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 0 & 3 & 2 & 3 & 0 & 4 \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix}$$

Note that we don't write equality between the above matrices, because they are not the same, but they are similar in some sense. In the above procedure we refer to this new matrix as A, since we assume that the new matrix overwrites A. Now we apply the same notation, i.e. when we refer to an element of A, then we always mean the latest version of the matrix. The next step is to divide each element of the first row of A with $a_{12} = 3$. This step is denoted as follows.

$$\begin{pmatrix} 0 & 3 & 2 & 3 & 0 & 4 \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{3}\cdot R1} \begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix}$$

Now we add $-a_{22} = 0$-times the first row to the second row, $-a_{32} = 2$-times the first row to the third row, $-a_{42} = -1$-times the first row to the fourth row, and $-a_{52} = 0$-times the first row to the fifth row. However note that if we add 0-times a row to another row, then nothing changes, thus it's enough to work with those rows, which contain a nonzero element in the second column. Considering this we only add 2-times the first row to the third row, and (-1)-times the first row to the fourth row. This step is denoted as follows.

$$\begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & -2 & 1 & 5 & 1 & -3 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix} \xrightarrow[R4-1\cdot R1]{R3+2\cdot R1} \begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & 0 & \frac{7}{3} & 7 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} & -2 & 0 & \frac{2}{3} \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix}$$

Then we set $k = 1$. Here B_1 is not the zero matrix and its leftmost nonzero column is the third column. The first element of the third column in $B1$ is nonzero, thus we divide the second row by $a_{23} = 3$.

$$\begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 3 & 9 & 1 & -1 \\ 0 & 0 & \frac{7}{3} & 7 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} & -2 & 0 & \frac{2}{3} \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{3}\cdot R2} \begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{7}{3} & 7 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} & -2 & 0 & \frac{2}{3} \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix}$$

Then we add $-a_{13} = -\frac{2}{3}$-times the second row to the first row, $-a_{33} = -\frac{7}{3}$-times the second row to the third row, $-a_{43} = \frac{2}{3}$-times the second row to the fourth row, and $-a_{53} = 1$-times the second row to the fifth row (the later means only that we add the second row to the fifth row).

$$\begin{pmatrix} 0 & 1 & \frac{2}{3} & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{7}{3} & 7 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{2}{3} & -2 & 0 & \frac{2}{3} \\ 0 & 0 & -1 & -3 & 1 & 3 \end{pmatrix} \xrightarrow[\substack{R4+\frac{2}{3}\cdot R2 \\ R5+R2}]{\substack{R1-\frac{2}{3}\cdot R2 \\ R3-\frac{7}{3}\cdot R2}} \begin{pmatrix} 0 & 1 & 0 & -1 & -\frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{3} & \frac{8}{3} \end{pmatrix}$$

Now set $k = 2$. Here B_2 is not the zero matrix and its leftmost nonzero column is the fifth column. The first element of the fifth column in B_2 is nonzero, thus we divide the third row by $a_{35} = \frac{2}{9}$.

$$\begin{pmatrix} 0 & 1 & 0 & -1 & -\frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{3} & \frac{8}{3} \end{pmatrix} \xrightarrow{\frac{9}{2}\cdot R3} \begin{pmatrix} 0 & 1 & 0 & -1 & -\frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{3} & \frac{8}{3} \end{pmatrix}$$

Then we add $-a_{15} = \frac{2}{9}$-times the third row to the first row, $-a_{25} = -\frac{1}{3}$-times the third row to the second row, $-a_{45} = -\frac{2}{9}$-times the third row to the fourth row, and $-a_{55} = -\frac{4}{3}$-times the third row to the fifth row.

$$\begin{pmatrix} 0 & 1 & 0 & -1 & -\frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & 3 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{3} & \frac{8}{3} \end{pmatrix} \xrightarrow[\substack{R4-\frac{2}{9}\cdot R3 \\ R5-\frac{4}{3}\cdot R3}]{\substack{R1+\frac{2}{9}\cdot R3 \\ R2-\frac{1}{3}\cdot R3}} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally set $k = 3$. Now B_3 is the zero matrix, thus the procedure terminates and we claim that the resulting matrix is in reduced row echelon form, and it clearly is.

2.2 Systems of linear equations

A linear equation for the unknowns (or variables) $x_1, x_2, x_3, \ldots x_n$ is an equation of the form where a linear combination of $x_1, x_2, \ldots x_n$ equals to a constant. If the coefficients in the linear combination are $a_1, a_2, \ldots, a_n$, and the constant is b, then the linear equation is

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$$

Sometimes the same unknowns must satisfy not just one, but several linear equations. Then we talk about a system of linear equations. As the combining coefficients in different equations vary, it's better to use double indexing form these coefficients. For example the coefficient of the unknown x_j is the i-th equation can be denoted by a_{ij}. The constants may be different in different equations, hence these should be also indexed. Let b_i denote the constant on the right in the i-th equation. Then a system of m equations for the unknowns $x_1, x_2, x_3, \ldots x_n$ is of the following form.

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\}$$

The coefficients in the above system naturally define a matrix of size $m \times n$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

which is called the **coefficient matrix** of the system. The coefficient matrix together with the multiplication of matrices gives a good opportunity to make a short notation of systems of linear equalities. If $\underline{x}$ denotes the column vector (i.e. single column matrix) containing the unknowns and $\underline{b}$ denotes the column vector (i.e. single column matrix) containing the the constants on the right, then the system can be written as

$$A \cdot \underline{x} = \underline{b}$$

which is called the matrix form of the system. Clearly the product on the left has m rows and only one column (as $\underline{x}$ has only one column). The column vector $\underline{b}$ has the same size as $A \cdot \underline{x}$, and the two column vectors equal to each other if all corresponding components equal to each other, which means exactly that the unknowns must satisfy the system of linear equalities above. We note here that the elements of the n-dimensional coordinate space $\mathbb{R}^n$ as vectors are usually denoted by underlined lowercase letters in order to emphasize that they are vectors and not to confuse them with real numbers, which are that scalars. Thus the notation $\underline{x}$ and $\underline{b}$ of column vectors.

Definition 8 *If $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $\underline{b}$ is column vector of m elements, then the* ***extended coefficient matrix*** *of the system of linear equalities $A \cdot \underline{x} = \underline{b}$ is the matrix of size $m \times (n+1)$ whose first n columns are exactly the same as the columns of A and the last column equals to the column vector $\underline{b}$. The extended coefficient matrix is denoted by $(A|\underline{b})$*

The last column of the extended coefficient matrix is often separated by a vertical line because of it's special role. Now let's see the following example of a system of linear equations.

$$\left.\begin{aligned} 3x_1 - 2x_2 + 4x_3 - x_4 &= 8 \\ x_1 + x_2 - 5x_4 &= -1 \\ 2x_1 + 3x_3 + x_4 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 - x_3 - 2x_4 &= 6 \end{aligned}\right\}$$

The coefficient matrix and the column vector of right-hand-side constants are

$$A = \begin{pmatrix} 3 & -2 & 4 & -1 \\ 1 & 1 & 0 & -5 \\ 2 & 0 & 3 & 1 \\ -1 & -1 & 2 & -3 \\ 0 & 1 & -1 & -2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 6 \end{pmatrix}$$

and the extended coefficient matrix is

$$\left(\begin{array}{cccc|c} 3 & -2 & 4 & -1 & 8 \\ 1 & 1 & 0 & -5 & -1 \\ 2 & 0 & 3 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 1 & -1 & -2 & 6 \end{array}\right)$$

Note that whenever the coefficient of an unknown seems to be missing in the system (see for example x_1 and x_2 in the second equation) then it only means that the coefficient is 1. It's also possible that some of the unknowns are missing is some equations (see for example x_3 in the second equation or x_2 is the third equation). Then the corresponding coefficient is zero.

Definition 9 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and let $\underline{b}$ be a column vector of m elements. The system of linear equations $A \cdot \underline{x} = \underline{b}$ is called **underdetermined** if the number of equations is less than the number of unknowns (i.e. $m < n$), and called **overdetermined** if the number of equations is larger than the number of unknowns (i.e. $n < m$). We say that the system is **solvable** if it has at least one solution, and otherwise we say it's **unsolvable**. Furthermore a solvable system $A \cdot \underline{x} = \underline{b}$ is called **determined** if it has exactly one solution, and called **undetermined** if it has more than one solution.*

Please note the difference between the properties underdetermined and undetermined. An underdeterminded system can be unsolvable, while a system is undetermined if it's solvable and has several solutions. However a solvable underdetermined system is undetermined. An overdetermined system can be solvable or unsolvable, and can be determined or undetermined when solvable.

An interesting property of systems of linear equations is that if a system has at least two solutions, then it has infinitely many solutions. To see this let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be an aritrary matrix and let $\underline{b}$ be an arbitrary column vector of m elements. Assume that two different column vectors $\underline{u}$ and $\underline{v}$ of m elements are solutions of the system $A \cdot \underline{x} = \underline{b}$. This means that $A \cdot \underline{u} = \underline{b}$ and $A \cdot \underline{v} = \underline{b}$. Now choose and arbitrary scalar $t \in \mathbb{R}$ and construct the column vector $(1-t)\underline{u} + t\underline{v}$. Then

$$A \cdot \big((1-t)\underline{u} + t\underline{v}\big) = (1-t)\, A \cdot \underline{u} + t\, A \cdot \underline{v} =$$

$$(1-t)\, \underline{b} + t\, \underline{b} = (1-t+t)\, \underline{b} = 1 \cdot \underline{b} = \underline{b}$$

This means $A \cdot \big((1-t)\underline{u} + t\underline{v}\big) = \underline{b}$ and it implies that $(1-t)\underline{u} + t\underline{v}$ is also a solution of the system $A \cdot \underline{x} = \underline{b}$. This show the system has infinitely many solutions because $\underline{u}$ and $\underline{v}$ are different and we can choose infinitely many different scalars $t \in \mathbb{R}$.

Theorem 5 *Let $A \cdot \underline{x} = \underline{b}$ be a system of linear equations. If the we apply any of the elementary row operations to the extended coefficient matrix $(A|\underline{b})$, then the set of solutions of system defined by the new extended coefficient matrix is exactly the same as the set of solutions of $A \cdot \underline{x} = \underline{b}$.*

The consequence of the above theorem is that if the extended coefficient matrix of the system $A \cdot \underline{x} = \underline{b}$ is transformed into reduced row echelon form with the help of elementary row operations, then the set of solutions remains the same as for the original system. Since any matrix can be transformed into reduced row echelon form with the help of elementary row operations, it's enough to investigate the set of solutions of systems with extended coefficient matrix in reduced row echelon form.

Theorem 6 *Let $A \cdot \underline{x} = \underline{b}$ be a system of linear equations and assume that the extended coefficient matrix $(A|\underline{b})$ is in reduced row echelon form.*

1. *If $(A|\underline{b})$ has a row which contains at least one nonzero element, and the pivot element in that row is in the last column, then the system is unsolvable. Otherwise the system is solvable.*
2. *If $A \cdot \underline{x} = \underline{b}$ is solvable and the number of nonzero rows in $(A|\underline{b})$ equals to the number of unknowns, then the system is determined. The only solution is $\widehat{\underline{b}}$, which consists of those elements of $\underline{b}$, which are not included in any zero row of $(A|\underline{b})$.*
3. *If $A \cdot \underline{x} = \underline{b}$ is solvable and the number of nonzero rows in $(A|\underline{b})$ is less than the number of unknowns, then the system is undetermined.*

Note that if a matrix is in reduced row echelon form, then the number of nonzero rows can't be larger than the number of columns, since each nonzero row contains a pivot element, but in reduced row echelon form each column may contain at most one pivot element. By the above theorem we know what is the set of solutions if the system is unsolvable, or solvable and the number of nonzero rows in $(A|\underline{b})$ equals to the number of unknowns (in the former case it's the empty set). We also know that if a system is undetermined, then it has infinitely many solutions. Yet, we would like to give a method to characterize all the solutions in such a situation.

Let $A \cdot \underline{x} = \underline{b}$ be a system of linear equations whose extended coefficient matrix $(A|\underline{b})$ is in reduced row echelon form. Assume that the system is solvable and the number of nonzero rows in $(A|\underline{b})$ is less than the number of unknowns. Then at least one column of the coefficient matrix A contains no pivot element. Let $\underline{\widehat{x}}$ denote the column vector of those unknowns, whose corresponding columns don't contain a pivot element. The ordering of the elements in $\underline{\widehat{x}}$ is the same as in $\underline{x}$. The unknowns in the column vector $\underline{\widehat{x}}$ are called the **free variables** (since soon we will see that their values can be chosen freely). Let B be a matrix which consists of the nonzero rows of the coefficient matrix A and the opposites of those columns of A which contain no pivot element. The opposite of a column means that each element of that column is multiplied by -1. The ordering of the columns in B is the same as in A. The matrix B is called the **coefficient matrix of the free variables**.

Theorem 7 *Let $A \cdot \underline{x} = \underline{b}$ be a system of linear equations whose extended coefficient matrix $(A|\underline{b})$ is in reduced row echelon form. Assume that the system is solvable and the number of nonzero rows in $(A|\underline{b})$ is less than the number of unknowns. Then the system is undetermined and*

$$\underline{v} = B \cdot \underline{u} + \widehat{\underline{b}}$$

is a solution of the system, where B is the coefficient matrix of the free variables, $\widehat{\underline{b}}$ is the same as in the previous theorem, and $\underline{u}$ is a column vector of arbitrary real numbers with appropriate size. Moreover for all solution $\underline{v}$ of the system $A \cdot \underline{x} = \underline{b}$ there exists a column vector $\underline{u}$ such that $\underline{v} = B \cdot \underline{u} + \widehat{\underline{b}}$.

In the above theorem the column vector $\underline{u}$ determines the values of the free variables. The appropriate size means that the number of elements in $\underline{u}$ is the same as the number of columns in the matrix B. Then the product $B \cdot \underline{u}$ is well defined and its dimension is the same as the dimension of $\widehat{\underline{b}}$, thus the sum $B \cdot \underline{u} + \widehat{\underline{b}}$ is also well defined.

Now we are able to determine the set of solutions of any system of linear equations by transforming its extended coefficient matrix into reduced row echelon form with the help of elementary row operations, and then applying one of the above theorems.

Example 1.

$$\left.\begin{aligned} x_1 - 3x_2 + 2x_3 + x_4 &= 4 \\ 5x_3 - 2x_4 &= 12 \\ -x_1 + 2x_2 + x_3 - 4x_4 &= 5 \\ -2x_2 + 4x_3 + 3x_4 &= 5 \\ x_1 - x_2 - 11x_4 &= 12 \end{aligned}\right\}$$

This is an overdetermined system. The extended coefficient matrix of the above system is

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 4 \\ 0 & 0 & 5 & -2 & 12 \\ -1 & 2 & 1 & -4 & 5 \\ 0 & -2 & 4 & 3 & 5 \\ 1 & -1 & 0 & -11 & 12 \end{array}\right)$$

Transform this into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 4 \\ 0 & 0 & 5 & -2 & 12 \\ -1 & 2 & 1 & -4 & 5 \\ 0 & -2 & 4 & 3 & 5 \\ 1 & -1 & 0 & -11 & 12 \end{array}\right) \xrightarrow[R5-1\cdot R1]{R3+1\cdot R1} \left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 4 \\ 0 & 0 & 5 & -2 & 12 \\ 0 & -1 & 3 & -3 & 9 \\ 0 & -2 & 4 & 3 & 5 \\ 0 & 2 & -2 & -12 & 8 \end{array}\right) \xrightarrow{R2\leftrightarrow R3}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 4 \\ 0 & -1 & 3 & -3 & 9 \\ 0 & 0 & 5 & -2 & 12 \\ 0 & -2 & 4 & 3 & 5 \\ 0 & 2 & -2 & -12 & 8 \end{array}\right) \xrightarrow{(-1)\cdot R2} \left(\begin{array}{cccc|c} 1 & -3 & 2 & 1 & 4 \\ 0 & 1 & -3 & 3 & -9 \\ 0 & 0 & 5 & -2 & 12 \\ 0 & -2 & 4 & 3 & 5 \\ 0 & 2 & -2 & -12 & 8 \end{array}\right) \xrightarrow[R5-2\cdot R2]{\substack{R1+3\cdot R2 \\ R4+2\cdot R2}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -7 & 10 & -23 \\ 0 & 1 & -3 & 3 & -9 \\ 0 & 0 & 5 & -2 & 12 \\ 0 & 0 & -2 & 9 & -13 \\ 0 & 0 & 4 & -18 & 26 \end{array}\right) \xrightarrow{\frac{1}{5}\cdot R3} \left(\begin{array}{cccc|c} 1 & 0 & -7 & 10 & -23 \\ 0 & 1 & -3 & 3 & -9 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{12}{5} \\ 0 & 0 & -2 & 9 & -13 \\ 0 & 0 & 4 & -18 & 26 \end{array}\right) \xrightarrow[\substack{R4+2\cdot R3 \\ R5-4\cdot R3}]{\substack{R1+7\cdot R3 \\ R2+3\cdot R3}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{36}{5} & -\frac{31}{5} \\ 0 & 1 & 0 & \frac{9}{5} & -\frac{9}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{12}{5} \\ 0 & 0 & 0 & \frac{41}{5} & -\frac{41}{5} \\ 0 & 0 & 0 & -\frac{82}{5} & \frac{82}{5} \end{array}\right) \xrightarrow{\frac{5}{41}\cdot R4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{36}{5} & -\frac{31}{5} \\ 0 & 1 & 0 & \frac{9}{5} & -\frac{9}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{12}{5} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -\frac{82}{5} & \frac{82}{5} \end{array}\right) \xrightarrow[\substack{R3+\frac{2}{5}\cdot R4 \\ R5+\frac{82}{5}\cdot R4}]{\substack{R1-\frac{36}{5}\cdot R4 \\ R2-\frac{9}{5}\cdot R4}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now this is in reduced row echelon form. There's no row whose pivot element is in the last column, thus the system is solvable. The number of nonzero rows is 4 just as the number of unknowns, thus the system is determined and the only solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

Clearly the first row of the above matrix in reduced row echelon form gives the equation $1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 1$, that is $x_1 = 1$. The second row gives the equation $0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0$, that is $x_2 = 0$. The third and fourth rows are similar.

Example 2.

$$\left.\begin{aligned} x_1 - 2x_2 + 4x_3 &= 10 \\ x_1 - x_2 + 3x_3 + 3x_4 &= 10 \\ 2x_2 - x_3 - 3x_4 &= -7 \\ 3x_1 - 2x_2 + 9x_3 + 3x_4 &= 14 \end{aligned}\right\}$$

This system is neither underdetermined nor overdetermined. The extended coefficient matrix of the above system is

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} 1 & -2 & 4 & 0 & 10 \\ 1 & -1 & 3 & 3 & 10 \\ 0 & 2 & -1 & -3 & -7 \\ 3 & -2 & 9 & 3 & 14 \end{array}\right)$$

Transform this into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & -2 & 4 & 0 & 10 \\ 1 & -1 & 3 & 3 & 10 \\ 0 & 2 & -1 & -3 & -7 \\ 3 & -2 & 9 & 3 & 14 \end{array}\right) \xrightarrow[R2-1\cdot R1]{R4-3\cdot R1} \left(\begin{array}{cccc|c} 1 & -2 & 4 & 0 & 10 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 2 & -1 & -3 & -7 \\ 0 & 4 & -3 & 3 & -16 \end{array}\right) \xrightarrow[R4-4\cdot R2]{\substack{R1+2\cdot R2 \\ R3-2\cdot R2}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 6 & 10 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & -9 & -7 \\ 0 & 0 & 1 & -9 & -16 \end{array}\right) \xrightarrow[R4-1\cdot R3]{\substack{R1-2\cdot R3 \\ R2+1\cdot R3}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 24 & 24 \\ 0 & 1 & 0 & -6 & -7 \\ 0 & 0 & 1 & -9 & -7 \\ 0 & 0 & 0 & 0 & -9 \end{array}\right) \xrightarrow{-\frac{1}{9}\cdot R4}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 24 & 24 \\ 0 & 1 & 0 & -6 & -7 \\ 0 & 0 & 1 & -9 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3-7\cdot R4]{\substack{R1-24\cdot R4 \\ R2+7\cdot R4}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 24 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now this is in reduced row echelon form. The pivot element in the last row is in the last column, thus the system is unsolvable. Clearly the last row of the above matrix in reduced row echelon form gives the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 1$, that is $0 = 1$, which is impossible.

Example 3.

$$\left.\begin{aligned} x_1 + 2x_3 + 3x_4 &= 0 \\ -x_1 + 2x_2 + x_3 - x_4 &= 3 \\ 2x_2 + 3x_3 + 2x_4 &= 3 \\ 3x_1 - 2x_2 + 3x_3 + 7x_4 &= -3 \\ -2x_1 + 6x_2 + 5x_3 &= 9 \end{aligned}\right\}$$

This is an overdetermined system. The extended coefficient matrix of the above system is

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ -1 & 2 & 1 & -1 & 3 \\ 0 & 2 & 3 & 2 & 3 \\ 3 & -2 & 3 & 7 & -3 \\ -2 & 6 & 5 & 0 & 9 \end{array}\right)$$

Transform this into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ -1 & 2 & 1 & -1 & 3 \\ 0 & 2 & 3 & 2 & 3 \\ 3 & -2 & 3 & 7 & -3 \\ -2 & 6 & 5 & 0 & 9 \end{array}\right) \xrightarrow[R5+2\cdot R1]{\substack{R2+1\cdot R1 \\ R4-3\cdot R1}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 2 & 3 \\ 0 & 2 & 3 & 2 & 3 \\ 0 & -2 & -3 & -2 & -3 \\ 0 & 6 & 9 & 6 & 9 \end{array}\right) \xrightarrow{\frac{1}{2}\cdot R2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 2 & 3 & 2 & 3 \\ 0 & -2 & -3 & -2 & -3 \\ 0 & 6 & 9 & 6 & 9 \end{array}\right) \xrightarrow[R5-6\cdot R2]{\substack{R3-2\cdot R1 \\ R4+2\cdot R1}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now this is in reduced row echelon form. There's no row whose pivot element is in the last column, thus the system is solvable. The number of nonzero rows is 2, but the number of unknowns is 4, thus the system is undetermined. The free variables are x_3 and x_4 as the third and fourth columns of the coefficient matrix don't contain any pivot element. This means all unknowns can be given as a linear combination of the free variables plus and additive constant. Clearly the first row gives the equation $1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 + 3 \cdot x_4 = 0$ and the second row gives the equation $0 \cdot x_1 + 1 \cdot x_2 + \frac{3}{2} \cdot x_3 + 1 \cdot x_4 = \frac{3}{2}$. The third, fourth and fifth rows give only the identity $0 = 0$. Thus we have

$$\left.\begin{aligned} x_1 + 2x_3 + 3x_4 &= 0 \\ x_2 + \tfrac{3}{2}x_3 + x_4 &= \tfrac{3}{2} \end{aligned}\right\}$$

which can be rearranged as

$$\left.\begin{aligned} x_1 &= -2x_3 - 3x_4 \\ x_2 &= -\tfrac{3}{2}x_3 - x_4 + \tfrac{3}{2} \end{aligned}\right\}$$

In matrix form this is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -\frac{3}{2} & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

Here

$$B = \begin{pmatrix} -2 & -3 \\ -\frac{3}{2} & -1 \end{pmatrix}$$

is the coefficient matrix of the free variables and

$$\underline{\widehat{b}} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

is just as it's defined in Theorem 6. Now we can choose arbitrary real values for x_3 and x_4 and then computing $x_1 = -2x_3 - 3x_4$ and $x_2 = -\frac{3}{2}x_3 - x_4 + \frac{3}{2}$ we get a solution of the system. For example $x_3 = 0$ and $x_4 = 0$ gives $x_1 = 0$ and $x_2 = \frac{3}{2}$, thus it's a solution of the system. Another solution is $x_3 = 1$, $x_4 = 1$ which gives $x_1 = -5$, $x_2 = -1$.

2.3 Systems of linear inequalities

A linear inequality for the unknowns (or variables) $x_1, x_2, x_3, \ldots x_n$ is inequaltilty, where a linear combination of $x_1, x_2, \ldots x_n$ is less than or larger than a constant. If the coefficients in the linear combination are $a_1, a_2, \ldots, a_n$, and the constant is b, then the linear inequality is

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n \leq b$$

or

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n \geq b$$

For the sake of simplicity we deal only with inequalities where equality is allowed (i.e. less than or equal, larger or equal). Note that a greater-or-equal-type inequality can be transformed to a less-than-or-equal-type inequality by multiplying it with (-1). Thus it's enough to deal with linear inequalities where a linear combination of the unknowns is less than or equal to a constant. A system of m linear inequalities for the unknowns $x_1, x_2, x_3, \ldots x_n$ is of the following form.

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{array}\right\}$$

The coefficients in the above system naturally define a matrix of size $m \times n$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

which is called the **coefficient matrix** of the system. If $\underline{x}$ denotes the column vector containing the unknowns and $\underline{b}$ denotes the column vector containing the the right-hand-side constants, then the matrix form of the system of inequalities is

$$A \cdot \underline{x} \leq \underline{b}$$

where the column vector $A \cdot \underline{x}$ is less than or equal to the column vector $\underline{b}$ if each entry of $A \cdot \underline{x}$ is less than or equal to the corresponding entry of

$\underline{b}$. Solving systems of linear inequalities can be quite challenging. Even the decision problem whether a given system has a solution or not is just complicated as a linear programming problem. The linear programming problem is to minimize (or maximize) a given linear combination of the unknowns under linear equality and inequality constraints. There are computationally efficient algorithms to solve linear programming problems, such as simplex method or interior point method, but the presentation of these algorithms is way beyond the scope of this text. Instead, we would like to introduce two methods which are much more simple to describe and can be used to solve small systems linear inequalities. These are the graphical solution method and Fourier-Motzkin elimination. Although the simplex method and interior point method are efficient for large system of inequalities, they are designed to find only one optimal solution and they're unable to find all the solutions of the system. An advantage of graphical solution method and Fourier-Motzkin elimination is that it's possible to give all solutions of the system with the help of them.

2.3.1 Graphical solution method

The graphical solution method can be applied to a system of linear inequalities for two variables. The unknowns in this situation are typically denoted by x and y instead of x_1 and x_2. The idea behind this method is that the set of points in the plane whose Cartesian coordinates (x, y) satisfy the linear equation $a_1x + a_2y = b$ is a line which is perpendicular to the vector (a_1, a_2). The set of points in the plane whose Cartesian coordinates (x, y) satisfy the linear inequality $a_1x + a_2y \leq b$ is a closed half-plane bounded by the line $a_1x + a_2y = b$. More precisely the linear inequality $a_1x + a_2y \leq b$ determines the opposite of the half-plane where the vector (a_1, a_2) points to. Thus the closed half-plane determined by $a_1x + a_2y \leq b$ can be presented in a figure by drawing the line $a_1x + a_2y = b$ and a vector (more precisely a directed line segment) with the same direction as $(-a_1, -a_2)$ and initial point on the line $a_1x + a_2y = b$.

The pair of real numbers (x, y) is a solution of the system of linear inequalities

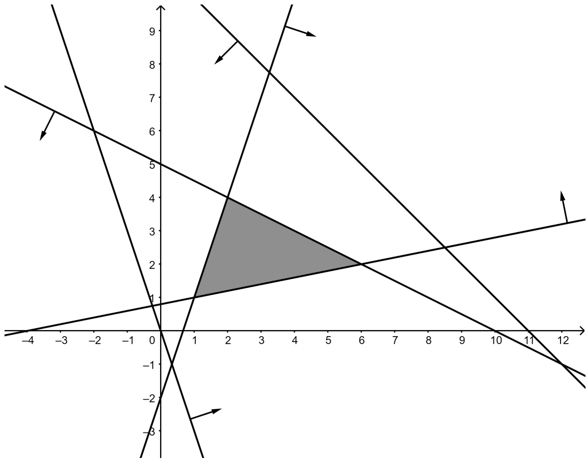
$$\left.\begin{aligned} a_{11}x + a_{12}y &\leq b_1 \\ a_{21}x + a_{22}y &\leq b_2 \\ \vdots \qquad\quad &\ \ \vdots \\ a_{m1}x + a_{m2}y &\leq b_m \end{aligned}\right\}$$

if and only if the point in the plane with Cartesian coordinates is in the intersection of all half-planes determined by the inequalities in the system. Thus it's enough to present all the half-planes determined by the inequalities in the system and look for their intersection.

Example 1.

$$\left.\begin{aligned} x + 2y &\leq 10 \\ x - 5y &\leq -4 \\ -3x + y &\leq -2 \\ x + y &\leq 11 \\ -3x - y &\leq 0 \end{aligned}\right\}$$

The line $x + 2y = 10$ can be drawn easily if we find two points of it. A point of the line can be found by substituting an arbitrary value to x or y and then solve the equation for the other variable. Let's choose now $x = 0$. Then y must satisfy $0 + 2y = 10$, which gives $y = 5$. This means $(0, 5)$ is a point of the line $x + 2y = 10$. If we choose $y = 0$, then x must satisfy $x + 2 \cdot 0 = 10$, which gives $x = 10$. This means $(10, 0)$ is a point of the line $x+2y = 10$. Thus $x+2y = 10$ is the line which passes through the points $(0, 5)$ and $(10, 0)$. The closed half-plane determined by $x + 2y \leq 10$ can be presented by drawing (besides the corresponding line) a vector with the same direction as $(-1, -2)$ and initial point on the line $x + 2y = 10$. The rest of the half-planes are presented in a similar way.



We can see that the system is solvable and the set of solutions is a triangle (shaded area in the figure). We also found that the inequalities $x + y \leq 11$ and $-3x - y \leq 0$ are redundant, which means that the set of solutions doesn't change if we omit these inequalities from the system. The triangle is bounded from below by the line $x - 5y = -4$ and bounded from above by the lines $-3x + y = -2$ and $x + 2y = 10$. We can rearrange the inequality corresponding to the lower bounding line as $\frac{1}{5}x + \frac{4}{5} \leq y$ and we can rearrange the inequalities corresponding to the upper bounding lines as $y \leq 3x - 2$ and $y \leq -\frac{1}{2}x + 5$. Furthermore we can see that the first component of any point in the triangle is between 1 and 6. Thus the set of solutions can be algebraically characterized as the set of points in the plane whose Cartesian coordinates (x, y) satisfy

$$\left.\begin{array}{c} 1 \leq x \leq 6 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq \min\left\{3x - 2, -\frac{1}{2}x + 5\right\} \end{array}\right\}$$

We can go even further and say $3x - 2 \leq -\frac{1}{2}x + 5$ is satisfied if and only if $x \leq 2$, hence the set of solutions is the set of points in the plane whose Cartesian coordinates (x, y) satisfy

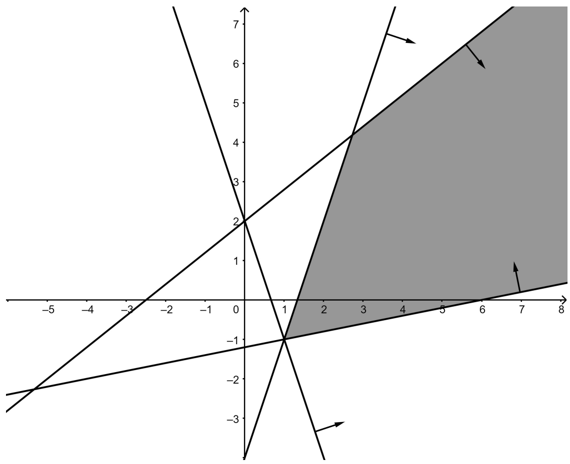
$$\left.\begin{array}{c} 1 \leq x \leq 2 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq 3x - 2 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{c} 2 \leq x \leq 6 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq -\frac{1}{2}x + 5 \end{array}\right\}$$

Geometrically this mean we cut the triangle into two pieces with the help of the vertical line $x = 2$ and say a point presents a solution if it's an element of the left-hand-part or an element of the right-hand-part.

Example 2.

$$\left.\begin{aligned} x - 5y &\le 6 \\ -3x + y &\le -4 \\ -3x - y &\le -2 \\ -4x + 5y &\le 10 \end{aligned}\right\}$$

The half-planes determined by the inequalities in the system are presented in a similar way as in Example 1. above.



We can see that the system is solvable, but this time the set of solutions is an unbounded area in the plane (shaded in the figure). We find that the inequality $-3x - y \le -2$ is redundant. The set of solutions is bounded from below by the line $x - 5y = 6$ and bounded from above by the lines $-3x + y = -4$ and $-4x + 5y = 10$. We can rearrange the inequality corresponding to the lower bounding line as $\frac{1}{5}x - \frac{6}{5} \le y$ and we can rearrange the inequalities corresponding to the upper bounding lines as $y \le 3x - 4$ and $y \le \frac{4}{5}x + 2$. The first component of any point in the set of solutions is larger than 1. Thus the set of solutions can be algebraically described as the set of points in the plane whose Cartesian coordinates (x, y) satisfy

$$\left.\begin{aligned} 1 \le x &< +\infty \\ \tfrac{1}{5}x - \tfrac{6}{5} \le y &\le \min\left\{3x - 4, \tfrac{4}{5}x + 2\right\} \end{aligned}\right\}$$

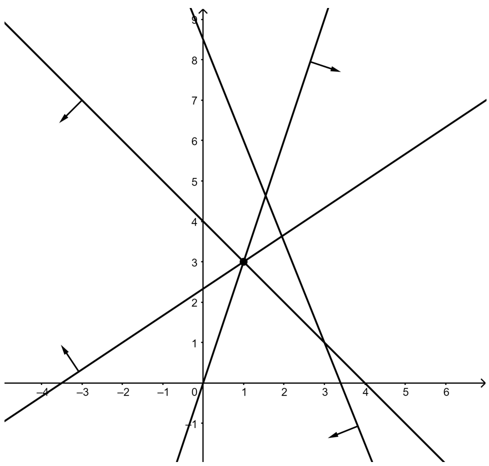
Here $3x - 4 \leq \frac{4}{5}x + 2$ is satisfied if and only if $x \leq \frac{30}{11}$, hence the set of solutions is the set of points in the plane whose Cartesian coordinates (x, y) satisfy

$$\left.\begin{array}{r} 1 \leq x \leq \frac{30}{11} \\ \frac{1}{5}x - \frac{6}{5} \leq y \leq 3x - 4 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{r} \frac{30}{11} \leq x < +\infty \\ \frac{1}{5}x - \frac{6}{5} \leq y \leq \frac{4}{5}x + 2 \end{array}\right\}$$

Example 3.

$$\left.\begin{array}{r} x + y \leq 4 \\ 2x - 3y \leq -7 \\ 5x + 2y \leq 17 \\ -3x + y \leq 0 \end{array}\right\}$$

The half-planes determined by the inequalities in the system are presented in a similar way as in Example 1.



We can see that the system is solvable, but this time the set of solutions contains only the single point $(1, 3)$. We find that the inequality $5x + 2y \leq 17$ is redundant. To prove algebraically that there's no solution besides the point $(1, 3)$, let's express y from all the relevant (i.e. non-redundant) inequalities. Then we have

$$\left.\begin{array}{r} y \leq -x + 4 \\ \frac{2}{3}x + \frac{7}{3} \leq y \\ y \leq 3x \end{array}\right\}$$

This shows that

$$\frac{2}{3}x + \frac{7}{3} \le y \le \min\{-x+4, 3x\}$$

but this can satisfied only if the following two inequalities hold.

$$\left.\begin{array}{l} \frac{2}{3}x + \frac{7}{3} \le -x + 4 \\ \frac{2}{3}x + \frac{7}{3} \le 3x \end{array}\right\}$$

These can be rearranged as

$$\left.\begin{array}{r} \frac{5}{3}x \le \frac{5}{3} \\ -\frac{7}{3}x \le -\frac{7}{3} \end{array}\right\}$$

and then

$$\left.\begin{array}{r} x \le 1 \\ x \ge 1 \end{array}\right\}$$

The only solution of the above system is $x = 1$. Substituting this back into the last system which contained y gives

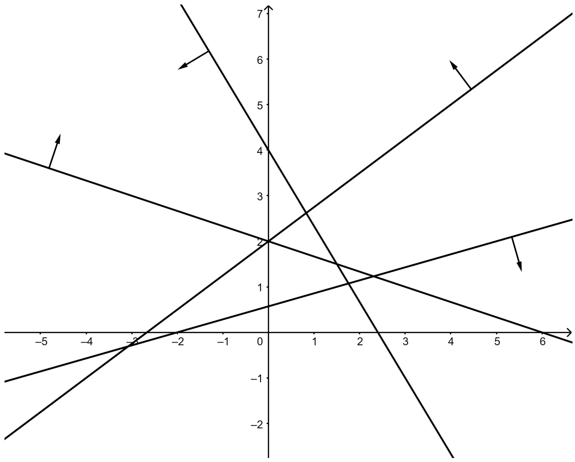
$$\left.\begin{array}{r} y \le 3 \\ 3 \le y \\ y \le 3 \end{array}\right\}$$

which shows that $y = 3$. Thus the only solution is clearly $(x, y) = (1, 3)$.

Example 4.

$$\left.\begin{array}{r} 3x - 4y \le -8 \\ -x - 3y \le -6 \\ 5x + 3y \le 12 \\ -2x + 7y \le 4 \end{array}\right\}$$

The half-planes determined by the inequalities in the system are presented in the following figure.



We can see that the system is has no solution, as no point of the intersection of the half-planes determined by the first three inequalities is an element of the half-plane determined by the last inequality. To see this algebraically, let's express y from all inequalities. Then we have

$$\left.\begin{aligned} \tfrac{3}{4}x + 2 &\leq y \\ -\tfrac{1}{3}x + 2 &\leq y \\ y &\leq -\tfrac{5}{3}x + 4 \\ y &\leq \tfrac{2}{7}x + \tfrac{4}{7} \end{aligned}\right\}$$

This shows that

$$\max\left\{\frac{3}{4}x + 2, -\frac{1}{3}x + 2\right\} \leq y \leq \min\left\{-\frac{5}{3}x + 4, \frac{2}{7}x + \frac{4}{7}\right\}$$

but this can satisfied only if the following four inequalities hold.

$$\left.\begin{aligned} \tfrac{3}{4}x + 2 &\leq -\tfrac{5}{3}x + 4 \\ \tfrac{3}{4}x + 2 &\leq \tfrac{2}{7}x + \tfrac{4}{7} \\ -\tfrac{1}{3}x + 2 &\leq -\tfrac{5}{3}x + 4 \\ -\tfrac{1}{3}x + 2 &\leq \tfrac{2}{7}x + \tfrac{4}{7} \end{aligned}\right\}$$

These can be rearranged as

$$\left.\begin{aligned} \tfrac{29}{12}x &\le 2 \\ \tfrac{13}{28}x &\le -\tfrac{10}{7} \\ \tfrac{4}{3}x &\le 2 \\ -\tfrac{13}{21}x &\le -\tfrac{10}{7} \end{aligned}\right\}$$

and then

$$\left.\begin{aligned} x &\le \tfrac{24}{29} \\ x &\le \tfrac{40}{13} \\ x &\le \tfrac{3}{2} \\ x &\ge \tfrac{30}{13} \end{aligned}\right\}$$

Here $\frac{24}{29} < 1$ and $1 < \frac{30}{13}$, hence there's no $x \in \mathbb{R}$ which could satisfy the above inequalities.

2.3.2 Fourier-Motzkin elimination

The graphical solution method is a good tool to visualize the set of solutions of a system of linear inequalities. We can also easily find the redundant inequalities in the system with the help of it. However sometimes we need to apply an algebraic approach for example when the lines corresponding to the inequalities intersect each other in points which are very close to each other. In some cases it's hard to make a decision upon the graphical image whether an intersection point of two lines is an element of a third line or not. Furthermore the graphical solution method can be applied to systems with only two unknowns. We could develop an extended version for the case of three unknowns since inequalities can be visualized then as half-spaces of the three dimensional coordinate space, but the shape of the intersection of such half-spaces can be very complicated and hard to find a good projection onto the plane. Besides there's no hope to visualize inequalities when the number unknowns is larger than 3, since these could be presented only in higher dimensional spaces.

Now we introduce an algebraical approach to find all the solutions of a system of linear inequalities. This method was first developed by French mathematician J.B.J. Fourier in 1826 and later T. S. Motzkin rediscovered it in 1936 to solve linear programming problems, and hence the method is named after them. The Fourier-Motzkin elimination method eliminates the unknowns from the system one-by-one until only one unknown is left. Then it can be easily told whether the system is solvable or not, and what's the possible range of that single unknown when the system is solvable. After that the possible values of the variables can be substituted back step-by-step to find all the solutions of the system. Now we first show how to eliminate the last unknown from any system and then this can be repeated until it's necessary and only one unknown is left.

Consider the following system of linear inequalities:

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{array}\right\}$$

We can collect here first those inequalities where the coefficient of x_n is positive, then collect the those inequalities where the coefficient of x_n is negative, and finally collect those where the coefficient of x_n is zero (i.e. x_n is missing). Assume that the above system is already in this form, that is there are non-negative integers $0 \leq k \leq l \leq m$ such that

$$\begin{array}{lll} a_{in} > 0 & \text{for all} & i \in \{1, 2, \ldots, k\} \\ a_{in} < 0 & \text{for all} & i \in \{k+1, k+2, \ldots, l\} \\ a_{in} = 0 & \text{for all} & i \in \{l+1, l+2, \ldots, m\} \end{array}$$

In other words the coefficient of x_n is positive in the first k inequalities, negative in the next $l-k$ inequalities, and zero in the rest of the inequalities.

$$\left.\begin{array}{r} a_{1\,1}x_1 + a_{1\,2}x_2 + \ldots + a_{1\,n}x_n \leq b_1 \\ a_{2\,1}x_1 + a_{2\,2}x_2 + \ldots + a_{2\,n}x_n \leq b_2 \\ \vdots \qquad\qquad \vdots \\ a_{k\,1}x_1 + a_{k\,2}x_2 + \ldots + a_{k\,n}x_n \leq b_k \\ a_{k+1\,1}x_1 + a_{k+1\,2}x_2 + \ldots + a_{k+1\,n}x_n \leq b_{k+1} \\ \vdots \qquad\qquad \vdots \\ a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n}x_n \leq b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n}x_n \leq b_{l+1} \\ \vdots \qquad\qquad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n}x_n \leq b_m \end{array}\right\}$$

This can be rearranged then as

$$\left.\begin{array}{r}
x_n \le \frac{b_1}{a_{1\,n}} - \frac{a_{1\,1}}{a_{1\,n}} x_1 - \frac{a_{1\,2}}{a_{1\,n}} x_2 - \ldots - \frac{a_{1\,n-1}}{a_{1\,n}} x_{n-1} \\
x_n \le \frac{b_2}{a_{2\,n}} - \frac{a_{2\,1}}{a_{2\,n}} x_1 - \frac{a_{2\,2}}{a_{2\,n}} x_2 - \ldots - \frac{a_{2\,n-1}}{a_{2\,n}} x_{n-1} \\
\vdots \qquad\qquad \vdots \qquad\qquad \\
x_n \le \frac{b_k}{a_{k\,n}} - \frac{a_{k\,1}}{a_{k\,n}} x_1 - \frac{a_{k\,2}}{a_{k\,n}} x_2 - \ldots - \frac{a_{k\,n-1}}{a_{k\,n}} x_{n-1} \\
\frac{b_{k+1}}{a_{k+1\,n}} - \frac{a_{k+1\,1}}{a_{k+1\,n}} x_1 - \frac{a_{k+1\,2}}{a_{k+1\,n}} x_2 - \ldots - \frac{a_{k+1\,n-1}}{a_{k+1\,n}} x_{n-1} \le x_n \\
\vdots \qquad\qquad \vdots \\
\frac{b_l}{a_{l\,n}} - \frac{a_{l\,1}}{a_{l\,n}} x_1 - \frac{a_{l\,2}}{a_{l\,n}} x_2 - \ldots - \frac{a_{l\,n-1}}{a_{l\,n}} x_{n-1} \le x_n \\
a_{l+1\,1} x_1 + a_{l+1\,2} x_2 + \ldots + a_{l+1\,n-1} x_{n-1} \le b_{l+1} \\
\vdots \qquad\qquad \vdots \\
a_{m\,1} x_1 + a_{m\,2} x_2 + \ldots + a_{m\,n-1} x_{n-1} \le b_m
\end{array}\right\}$$

Let's see the following cases.

1. If all coefficients of x_n are zero (i.e. $k = l = 0$ and x_n is missing from each inequality), then x_n may take any value independently of the values of the other unknowns, however the system can be still unsolvable.

2. If the coefficients of x_n are positive in all inequalities of the system (i.e. $k = l = m$), then we have only upper bounds for x_n. In this situation we can choose arbitrary values for $x_1, x_2, \ldots, x_{n-1}$ and then x_n must be not larger than the minimum of the upper bounds, which can be computed after substituting the values of $x_1, x_2, \ldots, x_{n-1}$.

3. If the coefficients of x_n are negative in all inequalities of the system (i.e. $k = 0$ and $l = m$), then we have only lower bounds for x_n. In this situation we can choose arbitrary values for $x_1, x_2, \ldots, x_{n-1}$ and then x_n must be not less than the maximum of the lower bounds, which can be computed after substituting the values of $x_1, x_2, \ldots, x_{n-1}$.

4. If the coefficients of x_n are positive or zero (but not negative) in all inequalities of the system (i.e. $k = l < m$), then we have to find the set

of solutions of the system

$$\left.\begin{array}{r} a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n-1}x_{n-1} \leq b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n-1}x_{n-1} \leq b_{l+1} \\ \vdots \qquad\qquad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n-1}x_{n-1} \leq b_m \end{array}\right\}$$

If there's a solution, then x_n must be not larger than the minimum of the upper bounds, which can computed with the help of the solutions of the above system.

5. If the coefficients of x_n are negative or zero (but not positive) in all inequalities of the system (i.e. $k = 0$ and $0 < l < m$), then we have to find the set of solutions of the system

$$\left.\begin{array}{r} a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n-1}x_{n-1} \leq b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n-1}x_{n-1} \leq b_{l+1} \\ \vdots \qquad\qquad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n-1}x_{n-1} \leq b_m \end{array}\right\}$$

   If there's a solution, then x_n must be not less than the maximum of the lower bounds, which can computed with the help of the solutions of the above system.

6. If there are both positive and negative coefficients of x_n in the system (i.e. $0 < k < l \leq m$), then we have both upper and lower bounds for x_n. Thus x_n must be not larger than the minimum of the upper bounds and not less than the maximum of the lower bounds. We can choose a value for x_n if and only if the maximum of the lower bounds is less then or equal to the minimum of the upper bounds. This property holds exactly when every lower bound for x_n is not larger than any of the upper bounds. This yields to a system of inequalities together with those where the coefficient of x_n is zero if there's any. This new system contains only the unknowns $x_1, x_2, \ldots, x_{n-1}$. A solution of that system can be substituted back to get the lower and upper bound for x_n.

Now we discuss what to do when only one unknown is left. Assume we have system

$$\left.\begin{aligned} a_{1\,1}x_1 &\le b_1 \\ a_{2\,1}x_1 &\le b_2 \\ \vdots \quad & \quad \vdots \\ a_{k\,1}x_1 &\le b_k \\ a_{k+1\,1}x_1 &\le b_{k+1} \\ \vdots \quad & \quad \vdots \\ a_{l\,1}x_1 &\le b_k \\ a_{l+1\,1}x_1 &\le b_{l+1} \\ \vdots \quad & \quad \vdots \\ a_{m\,1}x_1 &\le b_m \end{aligned}\right\}$$

where

$$\begin{array}{ll} a_{i1} > 0 & \text{if } \ i \in \{1, 2, \ldots, k\} \\ a_{i1} < 0 & \text{if } \ i \in \{k+1, k+2, \ldots, l\} \\ a_{i1} = 0 & \text{if } \ i \in \{l+1, l+2, \ldots, m\} \end{array}$$

The later means we possibly have inequalities of the form $0 \le b_i$. Such inequalities can be produced during the preceding steps. If there's at least one $i \in \{l+1, l+2, \ldots, m\}$ such that $b_i < 0$, then the system has no solution. Otherwise those inequalities are irrelevant and it's enough to treat the first l inequalities.

1. If the coefficients of x_1 are positive in all inequalities of the system, then we have only upper bounds for x_n. In this situation the system is solvable and the only restriction for x_1 is that it must be not larger than the minimum of the upper bounds.

2. If the coefficients of x_1 are negative in all inequalities of the system, then we have only lower bounds for x_1. In this situation the system is solvable and the only restriction for x_1 is that it must be not less than the maximum of the lower bounds.

3. If there are both positive and negative coefficients of x_1 in the system, then we have both upper and lower bounds for x_1. Thus x_1 must be not larger than the minimum of the upper bounds and not less than the maximum of the lower bounds. We can choose a value for x_1 if and only if the maximum of the lower bounds is less then or equal to the minimum of the upper bounds. Otherwise the system is unsolvable.

Now can solve any system of linear inequalities with the help of the above procedure. The drawback of the Fourier-Motzkin elimination method is that the number of inequalities can exponentially grow during the procedure, thus we can use it for systems with relatively few inequalities.

Example 1.

$$\left.\begin{array}{r} 3x_1 - x_2 + 2x_3 \leq -2 \\ x_1 - 3x_2 + x_3 \leq 6 \\ 4x_1 + x_2 - x_3 \leq -1 \\ -x_1 + 2x_2 - 3x_3 \leq 0 \end{array}\right\}$$

Here the coefficients of x_3 are positive in the first and second inequalities, and negative in the third and fourth inequalities. Now we rearrange them to get lower and upper bounds for x_3.

$$\left.\begin{array}{rl} & x_3 \leq -\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1 \\ & x_3 \leq -x_1 + 3x_2 + 6 \\ 4x_1 + x_2 + 1 \leq x_3 & \\ -\frac{1}{3}x_1 + \frac{2}{3}x_2 \leq x_3 & \end{array}\right\}$$

Thus

$$\max\left\{4x_1 + x_2 + 1, -\frac{1}{3}x_1 + \frac{2}{3}x_2\right\} \leq x_3 \leq \min\left\{-\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1, -x_1 + 3x_2 + 6\right\}$$

The system may have a solution only if each lower bound for x_3 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{r} 4x_1 + x_2 + 1 \leq -\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1 \\ 4x_1 + x_2 + 1 \leq -x_1 + 3x_2 + 6 \\ -\frac{1}{3}x_1 + \frac{2}{3}x_2 \leq -\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1 \\ -\frac{1}{3}x_1 + \frac{2}{3}x_2 \leq -x_1 + 3x_2 + 6 \end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r} \frac{11}{2}x_1 + \frac{1}{2}x_2 \leq -2 \\ 5x_1 - 2x_2 \leq 5 \\ \frac{7}{6}x_1 + \frac{1}{6}x_2 \leq -1 \\ \frac{2}{3}x_1 - \frac{7}{3}x_2 \leq 6 \end{array}\right\}$$

Collect the those inequalities first where the coefficient of x_2 is positive.

$$\left.\begin{array}{r} \frac{11}{2}x_1 + \frac{1}{2}x_2 \leq -2 \\ \frac{7}{6}x_1 + \frac{1}{6}x_2 \leq -1 \\ 5x_1 - 2x_2 \leq 5 \\ \frac{2}{3}x_1 - \frac{7}{3}x_2 \leq 6 \end{array}\right\}$$

Here the coefficients of x_2 are positive in the first and second inequalities, and negative in the third and fourth inequalities. Rearrange them to get lower and upper bounds for x_2.

$$\left.\begin{array}{rl} & x_2 \leq -11x_1 - 4 \\ & x_2 \leq -7x_1 - 6 \\ \frac{5}{2}x_1 - \frac{5}{2} \leq x_2 & \\ \frac{2}{7}x_1 - \frac{18}{7} \leq x_2 & \end{array}\right\}$$

Thus

$$\max\left\{\frac{5}{2}x_1 - \frac{5}{2}, \frac{2}{7}x_1 - \frac{18}{7}\right\} \leq x_2 \leq \min\{-11x_1 - 4, -7x_1 - 6\}$$

The system may have a solution only if each lower bound for x_2 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{r} \frac{5}{2}x_1 - \frac{5}{2} \leq -11x_1 - 4 \\ \frac{5}{2}x_1 - \frac{5}{2} \leq -7x_1 - 6 \\ \frac{2}{7}x_1 - \frac{18}{7} \leq -11x_1 - 4 \\ \frac{2}{7}x_1 - \frac{18}{7} \leq -7x_1 - 6 \end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r} \frac{27}{2}x_1 \leq -\frac{3}{2} \\ \frac{19}{2}x_1 \leq -\frac{7}{2} \\ \frac{79}{7}x_1 \leq -\frac{10}{7} \\ \frac{51}{7}x_1 \leq -\frac{24}{7} \end{array}\right\}$$

Here all coefficients of x_1 are positive, hence we have only upper bound for x_1, but no lower bound. The upper bounds are

$$\left.\begin{array}{l} x_1 \leq -\frac{1}{9} \\ x_1 \leq -\frac{7}{19} \\ x_1 \leq -\frac{10}{79} \\ x_1 \leq -\frac{8}{17} \end{array}\right\}$$

Thus

$$x_1 \leq \min\left\{-\frac{1}{9}, -\frac{7}{19}, -\frac{10}{79}, -\frac{8}{17}\right\} = -\frac{8}{17}$$

Finally we see that the solutions of the system are those triples (x_1, x_2, x_3) which satisfy

$$\left.\begin{array}{c} -\infty < x_1 \leq -\frac{8}{17} \\ \max\left\{\frac{5}{2}x_1 - \frac{5}{2}, \frac{2}{7}x_1 - \frac{18}{7}\right\} \leq x_2 \leq \min\left\{-11x_1 - 4, -7x_1 - 6\right\} \\ \max\left\{4x_1 + x_2 + 1, -\frac{1}{3}x_1 + \frac{2}{3}x_2\right\} \leq x_3 \leq \min\left\{-\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1, -x_1 + 3x_2 + 6\right\} \end{array}\right\}$$

To give one solution let's choose $x_1 = -1$ first. Then

$$\max\left\{-5, -\frac{20}{7}\right\} \leq x_2 \leq \min\left\{7, 1\right\},$$

that is

$$-\frac{20}{7} \leq x_2 \leq 1$$

Let's choose $x_2 = -1$. Then

$$\max\left\{-4, -\frac{1}{3}\right\} \leq x_3 \leq \min\left\{0, 4\right\},$$

that is

$$-\frac{1}{3} \leq x_1 \leq 0$$

Finally choose $x_3 = 0$. This shows that $(-1, -1, 0)$ is a solution of the system.

Example 2.

$$\left.\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &\le -7 \\ x_1 + x_2 + 2x_3 - x_4 &\le 8 \\ x_1 + x_2 + x_3 - 3x_4 &\le 4 \\ -x_1 - 2x_2 - x_4 &\le 1 \\ 3x_1 + 2x_2 + x_3 &\le 4 \\ -x_1 + 2x_2 &\le 5 \end{aligned}\right\}$$

Here the coefficient of x_4 is positive in the first inequality, they are negative in the second, third and fourth inequalities, and zero in the fifth and sixth inequalities. Rearrange the first four inequalities to get lower and upper bounds for x_4.

$$\left.\begin{aligned} x_4 \le -\tfrac{1}{4}x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{4}x_3 - \tfrac{7}{4}& \\ x_1 + x_2 + 2x_3 - 8 &\le x_4 \\ \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 - \tfrac{4}{3} &\le x_4 \\ -x_1 - 2x_2 &\le x_4 \\ 3x_1 + 2x_2 + x_3 &\le 4 \\ -x_1 + 2x_2 &\le 5 \end{aligned}\right\}$$

Thus

$$\max\left\{x_1 + x_2 + 2x_3 - 8, \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}, -x_1 - 2x_2 - 1\right\} \le x_4$$

and

$$x_4 \le -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4}$$

The system may have a solution only if each lower bound for x_4 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{aligned} x_1 + x_2 + 2x_3 - 8 &\le -\tfrac{1}{4}x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{4}x_3 - \tfrac{7}{4} \\ \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 - \tfrac{4}{3} &\le -\tfrac{1}{4}x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{4}x_3 - \tfrac{7}{4} \\ -x_1 - 2x_2 - 1 &\le -\tfrac{1}{4}x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{4}x_3 - \tfrac{7}{4} \\ 3x_1 + 2x_2 + x_3 &\le 4 \\ -x_1 + 2x_2 &\le 5 \end{aligned}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r} \frac{5}{4}x_1 + \frac{3}{2}x_2 + \frac{5}{4}x_3 \le \frac{25}{4} \\ \frac{7}{12}x_1 + \frac{5}{6}x_2 - \frac{5}{12}x_3 \le -\frac{5}{12} \\ -\frac{3}{4}x_1 - \frac{3}{2}x_2 - \frac{3}{4}x_3 \le -\frac{3}{4} \\ 3x_1 + 2x_2 + x_3 \le 4 \\ -x_1 + 2x_2 \le 5 \end{array}\right\}$$

Collect the those equations first where the coefficient of x_3 is positive.

$$\left.\begin{array}{r} \frac{5}{4}x_1 + \frac{3}{2}x_2 + \frac{5}{4}x_3 \le \frac{25}{4} \\ 3x_1 + 2x_2 + x_3 \le 4 \\ \frac{7}{12}x_1 + \frac{5}{6}x_2 - \frac{5}{12}x_3 \le -\frac{5}{12} \\ -\frac{3}{4}x_1 - \frac{3}{2}x_2 - \frac{3}{4}x_3 \le -\frac{3}{4} \\ -x_1 + 2x_2 \le 5 \end{array}\right\}$$

Here the coefficients of x_3 are positive in the first and second inequalities, negative in the third and fourth inequalities, and zero in the fifth inequality. Rearrange them to get lower and upper bounds for x_3.

$$\left.\begin{array}{rl} & x_3 \le -x_1 - \frac{6}{5}x_2 + 5 \\ & x_3 \le -3x_1 - 2x_2 + 4 \\ \frac{7}{5}x_1 + 2x_2 + 1 \le x_3 & \\ -x_1 - 2x_2 + 1 \le x_3 & \\ -x_1 + 2x_2 \le 5 & \end{array}\right\}$$

Thus

$$\max\left\{\frac{7}{5}x_1 + 2x_2 + 1, -x_1 - 2x_2 + 1\right\} \le x_3 \le \min\left\{-x_1 - \frac{6}{5}x_2 + 5, -3x_1 - 2x_2 + 4\right\}$$

The system may have a solution only if each lower bound for x_3 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{l} \frac{7}{5}x_1 + 2x_2 + 1 \le -x_1 - \frac{6}{5}x_2 + 5 \\ \frac{7}{5}x_1 + 2x_2 + 1 \le -3x_1 - 2x_2 + 4 \\ -x_1 - 2x_2 + 1 \le -x_1 - \frac{6}{5}x_2 + 5 \\ -x_1 - 2x_2 + 1 \le -3x_1 - 2x_2 + 4 \\ -x_1 + 2x_2 \le 5 \end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r} \frac{12}{5}x_1 + \frac{16}{5}x_2 \le 4 \\ \frac{22}{5}x_1 + 4x_2 \le 3 \\ -\frac{4}{5}x_2 \le 4 \\ 2x_1 \le 3 \\ -x_1 + 2x_2 \le 5 \end{array}\right\}$$

Collect the equations upon the coefficients of x_2.

$$\left.\begin{array}{r} \frac{12}{5}x_1 + \frac{16}{5}x_2 \le 4 \\ \frac{22}{5}x_1 + 4x_2 \le 3 \\ -x_1 + 2x_2 \le 5 \\ -\frac{4}{5}x_2 \le 4 \\ 2x_1 \le 3 \end{array}\right\}$$

Here the coefficients of x_2 are positive in the first three inequalities, negative in the fourth inequality, and zero in the fifth. Rearrange them to get lower and upper bounds for x_2.

$$\left.\begin{array}{l} x_2 \le -\frac{3}{4}x_1 + \frac{5}{4} \\ x_2 \le -\frac{11}{10}x_1 + \frac{3}{4} \\ x_2 \le \frac{1}{2}x_1 + \frac{5}{2} \\ -5 \le x_2 \\ 2x_1 \le 3 \end{array}\right\}$$

Thus

$$-5 \le x_2 \le \min\left\{-\frac{3}{4}x_1 + \frac{5}{4}, -\frac{11}{10}x_1 + \frac{3}{4}, \frac{1}{2}x_1 + \frac{5}{2}\right\}$$

The system may have a solution only if each lower bound for x_2 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{l} -5 \le -\frac{3}{4}x_1 + \frac{5}{4} \\ -5 \le -\frac{11}{10}x_1 + \frac{3}{4} \\ -5 \le \frac{1}{2}x_1 + \frac{5}{2} \\ 2x_1 \le 3 \end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r}\frac{3}{4}x_1 \le \frac{25}{4}\\ \frac{11}{10}x_1 \le \frac{23}{4}\\ -\frac{1}{2}x_1 \le \frac{15}{2}\\ 2x_1 \le 3\end{array}\right\}$$

Collect the equations upon the coefficients of x_1.

$$\left.\begin{array}{r}\frac{3}{4}x_1 \le \frac{25}{4}\\ \frac{11}{10}x_1 \le \frac{23}{4}\\ 2x_1 \le 3\\ -\frac{1}{2}x_1 \le \frac{15}{2}\end{array}\right\}$$

Rearrange the inequalities to get lower and upper bounds for x_1.

$$\left.\begin{array}{r}x_1 \le \frac{25}{3}\\ x_1 \le \frac{115}{22}\\ x_1 \le \frac{3}{2}\\ -15 \le x_1\end{array}\right\}$$

Thus

$$-15 \le x_2 \le \min\left\{\frac{25}{3}, \frac{115}{22}, \frac{3}{2}\right\} = \frac{3}{2}$$

Finally we see that the solutions of the system are those (x_1, x_2, x_3, x_4) which satisfy

$$\left.\begin{array}{r}-15 \le x_1 \le \frac{3}{2}\\ -5 \le x_2 \le \min\left\{-\frac{3}{4}x_1 + \frac{5}{4}, -\frac{11}{10}x_1 + \frac{3}{4}, \frac{1}{2}x_1 + \frac{5}{2}\right\}\\ \max\left\{\frac{7}{5}x_1 + 2x_2 + 1, -x_1 - 2x_2 + 1\right\} \le x_3 \le \min\left\{-x_1 - \frac{6}{5}x_2 + 5, -3x_1 - 2x_2 + 4\right\}\\ \max\left\{x_1 + x_2 + 2x_3 - 8, \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}\ x_3 - \frac{4}{3}, -x_1 - 2x_2 - 1\right\} \le x_4\\ x_4 \le -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4}\end{array}\right\}$$

To give one solution let's choose $x_1 = 1$ first. Then

$$-5 \le x_2 \le \min\left\{\frac{1}{2}, -\frac{7}{20}, 3\right\},$$

that is

$$-5 \leq x_2 \leq -\frac{7}{20}$$

Let's choose $x_2 = -1$. Then

$$\max\left\{\frac{2}{5}, 2\right\} \leq x_3 \leq \min\left\{\frac{26}{5}, 3\right\},$$

that is

$$2 \leq x_3 \leq 3$$

Let's choose $x_3 = 2$. Then

$$\max\left\{-4, -\frac{2}{3}, 0\right\} \leq x_4 \leq 0,$$

that is

$$0 \leq x_4 \leq 0$$

Now we don't have a choice for x_4, since it must equal to zero. This shows that $(1, -1, 2, 0)$ is a solution of the system.

3 Computed tomography

3.1 Pictures, digitization and phantoms

A picture consists of two components: the picture region and the picture function. The **picture region** is a rectangle R whose sides are parallel to the axises of a Cartesian coordinate system in the plane. Such a rectangle can be easily given as the direct product of two intervals $[a, b]$ and $[c, d]$.

$$R = \{(x, y) \mid x \in [a, b],\ y \in [c, d]\}$$

Here $[a, b]$ and $[c, d]$ are just the orthogonal projection of R onto the coordinate axises. The **picture function** is a function f of two variables, which is constant zero outside of the picture region. We often refer to the value of the picture at a point (x, y) as the **density** at (x, y).

A partition of the interval $[a, b]$ is a sequence of points $a_0, a_1, \ldots a_n \in [a, b]$, where $a = a_0 < a_1 < \ldots a_{n-1} < a_n = b$. A partition naturally divides the the interval $[a, b]$ into the union of the intervals $[a_{i-1}, a_i]$, $i \in \{1, 2, \ldots n\}$. The uniform partition of $[a, b]$ is when all the intervals $[a_{i-1}, a_i]$ have the same length, that is when

$$a_i = a + i \cdot \frac{b - a}{n}, \quad i \in \{0, 1, \ldots n\}$$

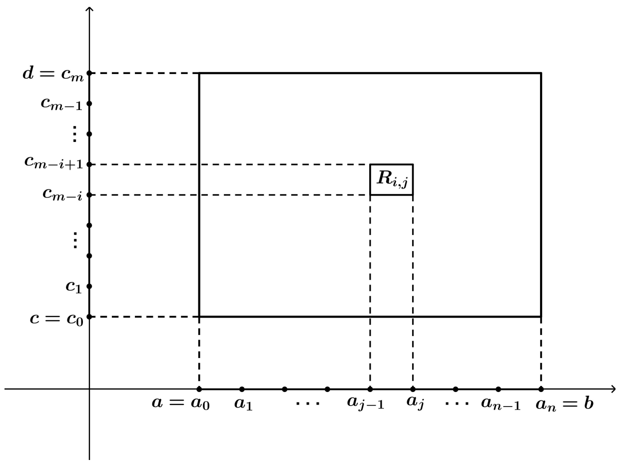
Similarly a partition of the interval $[c, d]$ is a sequence of points $c_0, c_1, \ldots c_m \in [a, b]$, where $c = c_0 < c_1 < \ldots c_{n-1} < c_m = d$. A partition naturally divides the the interval $[c, d]$ into the union of the intervals $[c_{i-1}, c_i]$, $i \in \{1, 2, \ldots n\}$. The uniform partition of $[c, d]$ is when

$$c_i = c + i \cdot \frac{d - c}{m}, \quad i \in \{0, 1, \ldots m\}$$

These two partitions together give a partition of the picture region R, which is the set of rectangles obtained by taking the direct product of each pair of the intervals $[a_{i-1}, a_i]$ and $[c_{j-1}, c_j]$, $i \in \{0, 1, \ldots n\}$ and $j \in \{0, 1, \ldots m\}$. We call the elements of the partition of the picture region R pixels if they're obtained by uniform partitions of $[a, b]$ and $[c, d]$. We can refer to the pixels by double indexing in the same manner as in the case of matrices if

$$R_{i,j} = [a_{j-1}, a_j] \times [c_{m-i}, c_{m-i+1}], \quad i \in \{1, 2, \ldots m\}$$

Then we say $R_{i,j}$ is the pixel of the picture region in the i-th row and j-th column.



An $m \times n$ digitized picture is one whose value in the interior of any pixel of a uniform partition of the picture region is constant. The $m \times n$ digitization of a picture is an $m \times n$ digitized picture such that the double integral of the original picture over any pixel equals to the double integral of the digitized picture over the same pixel. In X-ray transmission computed tomography the picture region is the reconstruction region and the the density of the picture at a point (x, y) is the relative linear attenuation number at an effective energy of the tissue at the point (x, y).

While the aim of X-ray transmission computed tomography is the reconstruction of real objects from their actual x-ray projections, the theoretical development of CT was based on experiments on mathematically described objects from computer simulated projection data. The basic reason for this that computer simulation enables us to investigate individually various effects that can't be separated physically. Such mathematically described objects are called **phantoms**. A test phantom is nothing but a picture on which we wish to test reconstruction algorithms or data collection methods. A simple phantom can be defined for example by specifying the constant values of the picture function over each pixel of a digitized picture. Such phantom can be specified with the help of a matrix $A = (a_{ij})$ where a_{ij} determines the constant value over the pixel $R_{i,j}$. Another type of simple phantom is a polygon inside the picture region and specifying the picture function equal to 1 inside the polygon and equal to zero outside the polygon.

3.2 Data collection

Let a picture with picture region $R = [a, b] \times [c, d]$ and picture function f be given, which acts as a phantom. A line l can be specified by giving a point and a direction vector of the line. The parametric equation of the line l passing through the point P and parallel to the vector $\underline{v}$ is

$$l = \left\{ P + t \cdot \underline{v} \,\middle|\, t \in \mathbb{R} \right\}$$

The parametrization of the line is not unique, however the value of the line integral is independent of the choice of the parametrization. If the parametric equation of the l is given, then the line integral of the picture function f can be computed as

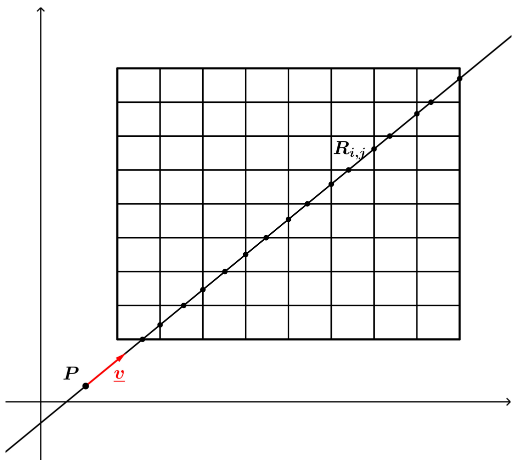
$$\int_l f = \int_{-\infty}^{\infty} f(P + t \cdot \underline{v}) \cdot |\underline{v}| \; dt$$

where $|\underline{v}|$ denotes the Euclidean length of the vector $\underline{v}$. If $\underline{v} = (v_1, v_2)$, then $|\underline{v}| = \sqrt{(v_1)^2 + (v_2)^2}$. Despite the integral is taken over an unbounded interval, the picture function is zero outside of a finite domain (the picture region), thus it's enough to take the above integral over a finite interval which depends on position of the line to the picture region and depends on the parametrization.

$$\int_l f = \int_{l \cap R} f$$

If a uniform partition of the picture region is also given with pixels $R_{i,j}$, then the line integral of f along the line l can be computed as the sum of the line integrals along the intersections of l with each pixel.

$$\int_{l\cap R} f = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{l\cap R_{i,j}} f$$



The line integral of a constant function over a bounded domain equals to the value of the function multiplied by the length of intersection of the line and the domain. Thus if f is simple phantom which takes the constant value $a_{i,j}$ over the pixel $R_{i,j}$ of an $m \times n$ uniform partition of the picture region R then

$$\int_{l} f = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{l\cap R_{i,j}} f = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} \cdot \lambda_1 \left(l \cap R_{i,j}\right)$$

where λ_1 denotes the one-dimensional Lebesgue measure (i.e. length).

Example

Let $R = [0,4] \times [0,3]$, $m = 3$, $n = 4$. Then the uniform partition of $[0,4]$ is $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and the uniform partition of $[0,3]$ is $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. These imply the pixels

$$\begin{array}{lll}
R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\
R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\
R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\
R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1]
\end{array}$$

Let f be the simple phantom defined by the matrix

$$A = \begin{pmatrix} \frac{1}{4} & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

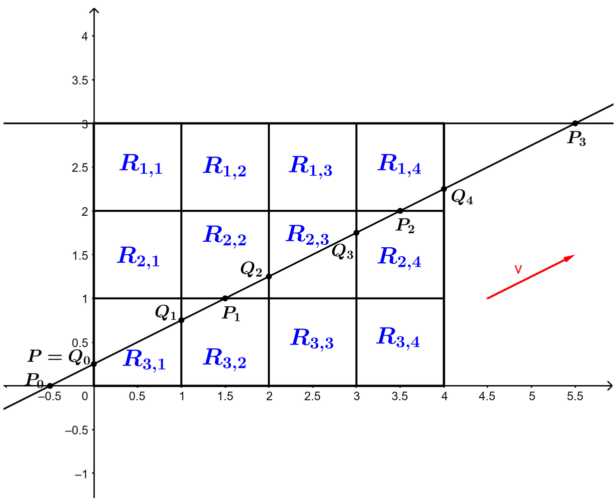
and let l be the line passing through the point $P = \left(0, \frac{1}{4}\right)$ and parallel to the vector $\underline{v} = (2, 1)$. The parametrization of l is

$$l = \left\{ P + t \cdot \underline{v} \,\middle|\, t \in \mathbb{R} \right\}$$

This means every point of the line l can be written in the form

$$P + t \cdot \underline{v} = \left(0, \frac{1}{4}\right) + t \cdot (2, 1) = \left(0 + 2t, \frac{1}{4} + t\right) = \left(2t, \frac{1}{4} + t\right)$$

where $t \in \mathbb{R}$.



Let P_0, P_1, P_2, P_3 denote the intersection points of l with the horizontal lines $y = 0$, $y = 1$, $y = 2$, $y = 3$ respectively, and let Q_0, Q_1, Q_2, Q_3, Q_4 denote the intersection points of l with the vertical lines $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$ respectively. Then the corresponding parameters of P_0, P_1, P_2, P_3 can be computed by making $\frac{1}{4}+t$ equal to 0, 1, 2, 3, and then the first components of P_0, P_1, P_2, P_3 are obtained by substituting the parameter to $2t$. That is

$$\begin{array}{lllll}
\frac{1}{4}+t=0 & \Rightarrow\ t=-\frac{1}{4} & \Rightarrow\ 2t=-\frac{1}{2} & \Rightarrow\ P_0=\left(-\frac{1}{2},0\right) \\
\frac{1}{4}+t=1 & \Rightarrow\ t=\frac{3}{4} & \Rightarrow\ 2t=\frac{3}{2} & \Rightarrow\ P_1=\left(\frac{3}{2},1\right) \\
\frac{1}{4}+t=2 & \Rightarrow\ t=\frac{7}{4} & \Rightarrow\ 2t=\frac{7}{2} & \Rightarrow\ P_2=\left(\frac{7}{2},2\right) \\
\frac{1}{4}+t=3 & \Rightarrow\ t=\frac{11}{4} & \Rightarrow\ 2t=\frac{11}{2} & \Rightarrow\ P_3=\left(\frac{11}{2},3\right)
\end{array}$$

Similarly the corresponding parameters of Q_0, Q_1, Q_2, Q_3, Q_4 can be computed by making $2t$ equal to 0, 1, 2, 3, 4, and then the second components of Q_0, Q_1, Q_2, Q_3, Q_4 are obtained by substituting the parameter to $t+\frac{1}{4}$. That is

$$\begin{array}{lllll}
2t=0 & \Rightarrow\ t=0 & \Rightarrow\ t+\frac{1}{4}=\frac{1}{4} & \Rightarrow\ Q_0=\left(0,\frac{1}{4}\right) \\
2t=1 & \Rightarrow\ t=\frac{1}{2} & \Rightarrow\ t+\frac{1}{4}=\frac{3}{4} & \Rightarrow\ Q_1=\left(1,\frac{3}{4}\right) \\
2t=2 & \Rightarrow\ t=1 & \Rightarrow\ t+\frac{1}{4}=\frac{5}{4} & \Rightarrow\ Q_2=\left(2,\frac{5}{4}\right) \\
2t=3 & \Rightarrow\ t=\frac{3}{2} & \Rightarrow\ t+\frac{1}{4}=\frac{7}{4} & \Rightarrow\ Q_3=\left(3,\frac{7}{4}\right) \\
2t=4 & \Rightarrow\ t=2 & \Rightarrow\ t+\frac{1}{4}=\frac{9}{4} & \Rightarrow\ Q_4=\left(4,\frac{9}{4}\right)
\end{array}$$

Thus the line l intersects only the pixels $R_{3,1}$, $R_{3,2}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$, $R_{1,4}$. The intersection with the pixel $R_{3,1}$ is the line segment $\overline{Q_0Q_1}$, which can be shortly written as $l \cap R_{3,1} = \overline{Q_0Q_1}$. Furthermore

$$\begin{array}{lll}
l \cap R_{3,2} = \overline{Q_1P_1} & l \cap R_{2,2} = \overline{P_1Q_2} & l \cap R_{2,3} = \overline{Q_2Q_3} \\
l \cap R_{2,4} = \overline{Q_3P_2} & l \cap R_{1,4} = \overline{P_2Q_4} &
\end{array}$$

The length of any line segment $\overline{PQ}$ connecting the points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ is computed by the formula $\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}$. Hence

$$\begin{aligned}
\lambda_1\left(l \cap R_{3,1}\right) &= \lambda_1\left(\overline{Q_0Q_1}\right) = \sqrt{(0-1)^2+\left(\tfrac{1}{4}-\tfrac{3}{4}\right)^2} = \tfrac{\sqrt{5}}{2} \\
\lambda_1\left(l \cap R_{3,2}\right) &= \lambda_1\left(\overline{Q_1P_1}\right) = \sqrt{\left(1-\tfrac{3}{2}\right)^2+\left(\tfrac{3}{4}-1\right)^2} = \tfrac{\sqrt{5}}{4} \\
\lambda_1\left(l \cap R_{2,2}\right) &= \lambda_1\left(\overline{P_1Q_2}\right) = \sqrt{\left(\tfrac{3}{2}-2\right)^2+\left(1-\tfrac{5}{4}\right)^2} = \tfrac{\sqrt{5}}{4} \\
\lambda_1\left(l \cap R_{2,3}\right) &= \lambda_1\left(\overline{Q_2Q_3}\right) = \sqrt{(2-3)^2+\left(\tfrac{5}{4}-\tfrac{7}{4}\right)^2} = \tfrac{\sqrt{5}}{2} \\
\lambda_1\left(l \cap R_{2,4}\right) &= \lambda_1\left(\overline{Q_3P_2}\right) = \sqrt{\left(3-\tfrac{7}{2}\right)^2+\left(\tfrac{7}{4}-2\right)^2} = \tfrac{\sqrt{5}}{4} \\
\lambda_1\left(l \cap R_{1,4}\right) &= \lambda_1\left(\overline{P_2Q_4}\right) = \sqrt{\left(\tfrac{7}{2}-4\right)^2+\left(2-\tfrac{9}{4}\right)^2} = \tfrac{\sqrt{5}}{4}
\end{aligned}$$

Now we can calculate the line integral of the phantom f along the line l.

$$\int_l f = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \cdot \lambda_1 \left(l \cap R_{i,j} \right) =$$

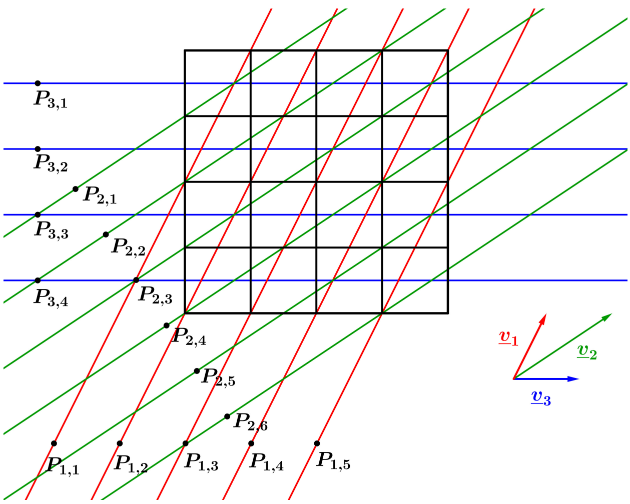
$$= a_{3,1} \cdot \lambda_1 \left(l \cap R_{3,1} \right) + a_{3,2} \cdot \lambda_1 \left(l \cap R_{3,2} \right) + a_{2,2} \cdot \lambda_1 \left(l \cap R_{2,2} \right) +$$

$$+ a_{2,3} \cdot \lambda_1 \left(l \cap R_{2,3} \right) + a_{2,4} \cdot \lambda_1 \left(l \cap R_{2,4} \right) + a_{1,4} \cdot \lambda_1 \left(l \cap R_{1,4} \right) =$$

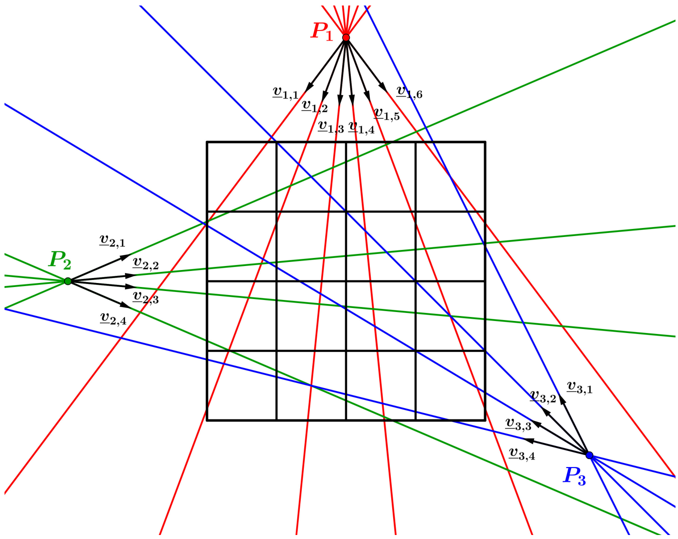
$$= \frac{1}{3} \cdot \frac{\sqrt{5}}{2} + \frac{1}{2} \cdot \frac{\sqrt{5}}{4} + \frac{1}{3} \cdot \frac{\sqrt{5}}{4} + \frac{1}{2} \cdot \frac{\sqrt{5}}{2} + \frac{1}{3} \cdot \frac{\sqrt{5}}{4} + \frac{1}{2} \cdot \frac{\sqrt{5}}{4} = \frac{5}{6}\sqrt{5}$$

3.3 Typical line sets for data collection

Basically there are two types of line sets which are commonly used for data collection in CT: one related to parallel beam x-rays and the other one related to divergent beam x-rays. We talk about parallel beam x-ray, when there's a finite set of directions $\underline{v}_1$, $\underline{v}_2$, …, $\underline{v}_k$, and for each direction $\underline{v}_i$ there are points $P_{i,1}$, $P_{i,2}$, …, P_{i,s_i} along a line which is not parallel to $\underline{v}_i$, and adjacent points have the same distance from each other. Then the set of lines used for measurements is $\{l_{i,j} \,|\, i \in \{1, 2, \ldots, k\}, \, j \in \{1, 2, \ldots, s_i\}\}$, where $l_{i,j}$ is the line passing through $P_{i,j}$ and parallel to $\underline{v}_i$.



We talk about divergent beam x-rays when there's a finite set of points P_1, P_2, …, P_k, and for each point P_i there are directions $\underline{v}_{i,1}$, $\underline{v}_{i,2}$, …, $\underline{v}_{i,s_i}$. Then the set of lines used for measurements is $\{l_{i,j} \mid i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, s_i\}\}$, where $l_{i,j}$ is the line passing through P_i and parallel to $\underline{v}_{i,j}$.



3.4 Series expansion method

The idea behind the series expansion method is that, given the picture region, we choose a set of **basis functions** $b_1, b_2, \ldots, b_J$, each of which is a picture function for the given picture region. These must be chosen such that for any picture function f that we want to reconstruct, there exists a linear combination of the basis functions that we consider an adequate approximation of f. If the line integrals of an unknown function f are given along a set of lines, then we choose a linear combination of the basis functions whose line integrals along the same lines are as close to the measurements of f as possible.

There are many possible choices of the basis functions, such as the generalized Kaiser-Bessel window functions, also known as blobs, which are widely used in X-ray transmission tomography. However, given an $m \times n$ uniform partition of the picture region, another typical choice is the set of characteristic functions of the pixels. The characteristic function of the pixel $R_{i,j}$ is the function $r_{i,j}$ of two variables defined by

$$r_{i,j}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R_{i,j} \\ 0, & \text{if } (x,y) \notin R_{i,j} \end{cases}$$

Then a linear combination of these characteristic functions is the picture function

$$g = x_{i,j} \cdot r_{i,j}$$

where $x_{i,j} \in \mathbb{R}$. Note that g is nothing but the picture function which takes the constant value $x_{i,j}$ over the pixel $R_{i,j}$ for each $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$. Thus the line integral of such function g along any line l is a linear combination of the variables $x_{i,j}$, where the coefficient of $x_{i,j}$ is exactly the length of the intersection of l with the pixel $R_{i,j}$. Making the measurements of an unknown function f equal to the linear combinations of the variables $x_{i,j}$, defined by the line integrals along a given set of lines, we obtain a system of linear equations. Unfortunately not all picture function f can be given as a linear combination of the characteristic functions $r_{i,j}$. Besides there are different sources of error during the data collection, hence the system may be unsolvable. In such situation we need to introduce an extra variable e_k for each equation of the system, which presents difference between measurement corresponding to that equation and the line integral of the picture function provided by the variables $x_{i,j}$. Then our task is to find a solution of the system which minimizes the square sum of the extra variables, i.e. $(e_1)^2 + (e_2)^2 + \ldots + (e_K)^2$.

Example 1

Let $R = [0,4] \times [0,3]$, $m = 3$, $n = 4$. Then the uniform partition of $[0,4]$ is $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and the uniform partition of $[0,3]$ is $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. These imply the pixels

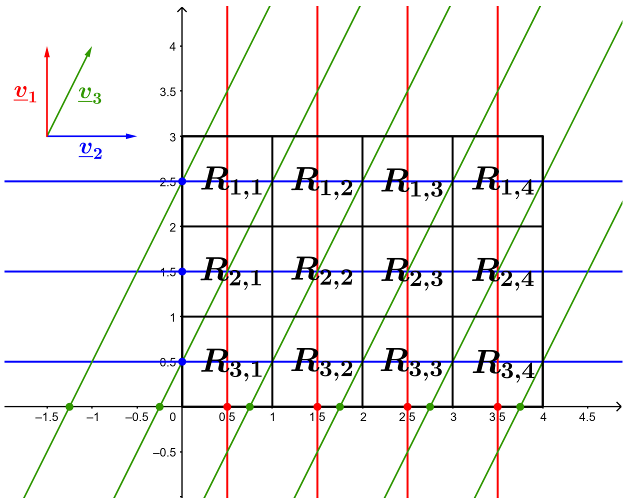
$$\begin{array}{lll} R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\ R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\ R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\ R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1] \end{array}$$

Consider the following set of lines:

- l_1, l_2, l_3, l_4 are lines parallel to $\underline{v}_1 = (0,1)$ and passing through the points $P_{1,1} = \left(\frac{1}{2}, 0\right)$, $P_{1,2} = \left(\frac{3}{2}, 0\right)$, $P_{1,3} = \left(\frac{5}{2}, 0\right)$, $P_{1,4} = \left(\frac{7}{2}, 0\right)$ respectively.

- l_5, l_6, l_7 are lines parallel to $\underline{v}_2 = (1, 0)$ and passing through the points $P_{2,1} = \left(0, \frac{5}{2}\right)$, $P_{2,2} = \left(0, \frac{3}{2}\right)$, $P_{2,3} = \left(0, \frac{1}{2}\right)$ respectively.

- $l_8, l_9, l_{10}, l_{11}, l_{12}, l_{13}$ are lines parallel to $\underline{v}_3 = (1, 2)$ and passing through the points $P_{3,1} = \left(-\frac{5}{4}, 0\right)$, $P_{3,2} = \left(-\frac{1}{4}, 0\right)$, $P_{3,3} = \left(\frac{3}{4}, 0\right)$, $P_{3,4} = \left(\frac{7}{4}, 0\right)$, $P_{3,5} = \left(\frac{11}{4}, 0\right)$, $P_{3,6} = \left(\frac{15}{4}, 0\right)$ respectively.



Let m_k be the line integral of an unknown function f along line l_k for $k = 1, 2, \ldots, 13$, where

$$\begin{aligned}
&m_1 = 1, \quad m_2 = 3, \quad m_3 = 0, \quad m_4 = 2\\
&m_5 = 2, \quad m_6 = 2, \quad m_7 = 2,\\
&m_8 = \frac{\sqrt{5}}{4}, \; m_9 = \frac{\sqrt{5}}{2}, \; m_{10} = \sqrt{5}, \; m_{11} = \frac{\sqrt{5}}{4}, \; m_{12} = \frac{3\sqrt{5}}{4}, \; m_{13} = \frac{\sqrt{5}}{4}.
\end{aligned}$$

We try to find the values $x_{i,j}$, such that the function g, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines l_k equal to m_k for all $k = 1, 2, \ldots 13$.

The line l_1 intersects the pixels $R_{1,1}$, $R_{2,1}$, $R_{3,1}$, the line l_2 intersects the pixels $R_{1,2}$, $R_{2,2}$, $R_{3,2}$, the line l_3 intersects the pixels $R_{1,3}$, $R_{2,3}$, $R_{3,3}$, and the line l_4 intersects the pixels $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. Each time the length of the intersection is 1. Thus the line integrals of g along the lines l_1, l_2, l_3, l_4 are

$$\begin{array}{c} x_{1,1} + x_{2,1} + x_{3,1} \\ x_{1,2} + x_{2,2} + x_{3,2} \\ x_{1,3} + x_{2,3} + x_{3,3} \\ x_{1,4} + x_{2,4} + x_{3,4} \end{array}$$

respectively. The line l_5 intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{1,4}$, the line l_6 intersects the pixels $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$, and the line l_7 intersects the pixels $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, $R_{3,4}$. Each time the length of the intersection equals to 1. Thus the line integrals of g along the lines l_5, l_6, l_7 are

$$\begin{array}{c} x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} \\ x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} \\ x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} \end{array}$$

respectively. Furthermore the line l_8 intersects only the pixel $R_{1,1}$ in a line segment of length $\frac{\sqrt{5}}{4}$. The line l_9 intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{3,1}$ in line segments of length $\frac{\sqrt{5}}{4}$, and intersects $R_{2,1}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line l_{10} intersects the pixels $R_{1,2}$, $R_{1,3}$, $R_{3,1}$, $R_{3,2}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,2}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line l_{11} intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{3,2}$, $R_{3,3}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,3}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line l_{12} intersects the pixels $R_{1,4}$, $R_{3,3}$, $R_{3,4}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,4}$ in a line segment of length $\frac{\sqrt{5}}{2}$. Finally l_{13} intersects only the pixel $R_{3,4}$ in a line segment of length $\frac{\sqrt{5}}{4}$. Thus the line integrals of g along the lines l_8, l_9, l_{10}, l_{11}, l_{12}, l_{13} are

$$\begin{array}{l} \frac{\sqrt{5}}{4} x_{1,1} \\ \frac{\sqrt{5}}{4} x_{1,1} + \frac{\sqrt{5}}{4} x_{1,2} + \frac{\sqrt{5}}{2} x_{2,1} + \frac{\sqrt{5}}{4} x_{3,1} \\ \frac{\sqrt{5}}{4} x_{1,2} + \frac{\sqrt{5}}{4} x_{1,3} + \frac{\sqrt{5}}{2} x_{2,2} + \frac{\sqrt{5}}{4} x_{3,1} + \frac{\sqrt{5}}{4} x_{3,2} \\ \frac{\sqrt{5}}{4} x_{1,3} + \frac{\sqrt{5}}{4} x_{1,4} + \frac{\sqrt{5}}{2} x_{2,3} + \frac{\sqrt{5}}{4} x_{3,2} + \frac{\sqrt{5}}{4} x_{3,3} \\ \frac{\sqrt{5}}{4} x_{1,4} + \frac{\sqrt{5}}{2} x_{2,4} + \frac{\sqrt{5}}{4} x_{3,3} + \frac{\sqrt{5}}{4} x_{3,4} \\ \frac{\sqrt{5}}{4} x_{3,4} \end{array}$$

respectively. Hence making all these line integrals equal to the corresponding line integrals of the unknown function f yields the system of equations

$$\left.\begin{aligned}
x_{1,1}+x_{2,1}+x_{3,1}&=1\\
x_{1,2}+x_{2,2}+x_{3,2}&=3\\
x_{1,3}+x_{2,3}+x_{3,3}&=0\\
x_{1,4}+x_{2,4}+x_{3,4}&=2\\
x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}&=2\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4}&=2\\
x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}&=2\\
\tfrac{\sqrt{5}}{4}\,x_{1,1}&=\tfrac{\sqrt{5}}{4}\\
\tfrac{\sqrt{5}}{4}\,x_{1,1}+\tfrac{\sqrt{5}}{4}x_{1,2}+\tfrac{\sqrt{5}}{2}x_{2,1}+\tfrac{\sqrt{5}}{4}x_{3,1}&=\tfrac{\sqrt{5}}{2}\\
\tfrac{\sqrt{5}}{4}\,x_{1,2}+\tfrac{\sqrt{5}}{4}x_{1,3}+\tfrac{\sqrt{5}}{2}x_{2,2}+\tfrac{\sqrt{5}}{4}x_{3,1}+\tfrac{\sqrt{5}}{4}x_{3,2}&=\sqrt{5}\\
\tfrac{\sqrt{5}}{4}\,x_{1,3}+\tfrac{\sqrt{5}}{4}x_{1,4}+\tfrac{\sqrt{5}}{2}x_{2,3}+\tfrac{\sqrt{5}}{4}x_{3,2}+\tfrac{\sqrt{5}}{4}x_{3,3}&=\tfrac{\sqrt{5}}{4}\\
\tfrac{\sqrt{5}}{4}\,x_{1,4}+\tfrac{\sqrt{5}}{2}x_{2,4}+\tfrac{\sqrt{5}}{4}x_{3,3}+\tfrac{\sqrt{5}}{4}x_{3,4}&=\tfrac{3\sqrt{5}}{4}\\
\tfrac{\sqrt{5}}{4}\,x_{3,4}&=\tfrac{\sqrt{5}}{4}
\end{aligned}\right\}$$

Multiplying each of the last 6 equations by $\frac{4}{\sqrt{5}}$ gives

$$\left.\begin{aligned}
x_{1,1}+x_{2,1}+x_{3,1}&=1\\
x_{1,2}+x_{2,2}+x_{3,2}&=3\\
x_{1,3}+x_{2,3}+x_{3,3}&=0\\
x_{1,4}+x_{2,4}+x_{3,4}&=2\\
x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}&=2\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4}&=2\\
x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}&=2\\
x_{1,1}&=1\\
x_{1,1}+x_{1,2}+2x_{2,1}+x_{3,1}&=2\\
x_{1,2}+x_{1,3}+2x_{2,2}+x_{3,1}+x_{3,2}&=4\\
x_{1,3}+x_{1,4}+2x_{2,3}+x_{3,2}+x_{3,3}&=1\\
x_{1,4}+2x_{2,4}+x_{3,3}+x_{3,4}&=3\\
x_{3,4}&=1
\end{aligned}\right\}$$

Here the unknowns have double index, so before we write the matrix form of the above system we need to fix an ordering of the above unknowns. This can be for example the lexicographic order, where $x_{i,j} \le x_{k,l}$ holds if $i < k$, or $i = k$ and $j \leq l$. Hence the ordering is

$$(x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4})$$

Then the above system of equations in matrix form is $A \cdot \underline{x} = \underline{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\underline{x} = (x_{1,1}\ x_{1,2}\ x_{1,3}\ x_{1,4}\ x_{2,1}\ x_{2,2}\ x_{2,3}\ x_{2,4}\ x_{3,1}\ x_{3,2}\ x_{3,3}\ x_{3,4})^\top$$

$$\underline{b} = (1\ 3\ 0\ 2\ 2\ 2\ 2\ 1\ 2\ 4\ 1\ 3\ 1)^\top$$

The extended coefficient matrix is

$$\left(\begin{array}{cccccccccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

This can transformed to reduced row echelon form with Gauss elimination. The reduced row echelon form is

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This shows that we have two free variables $x_{3,2}$ and $x_{3,3}$. The rest of the variables can be given as

$$\begin{pmatrix} x_{1,1} \\ x_{1,2} \\ x_{1,3} \\ x_{1,4} \\ x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ x_{2,4} \\ x_{3,1} \\ x_{3,4} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{3,2} \\ x_{3,3} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This can be arranged in the same manner as in the case of a matrix:

$$\begin{array}{llll}
x_{1,1} = 1 & x_{1,2} = 2 - x_{3,2} - x_{3,3} & x_{1,3} = x_{3,2} - 1 & x_{1,4} = x_{3,3} \\
x_{2,1} = x_{3,2} + x_{3,3} - 1 & x_{2,2} = x_{3,3} + 1 & x_{2,3} = 1 - x_{3,2} - x_{3,3} & x_{2,4} = 1 - x_{3,3} \\
x_{3,1} = 1 - x_{2,3} - x_{3,3} & x_{3,2} = x_{3,2} & x_{3,3} = x_{3,3} & x_{3,4} = 1
\end{array}$$

Now let's find the non-negative solutions. Then we have the following system of linear inequalities:

$$\left.\begin{aligned} 2 - x_{3,2} - x_{3,3} &\geq 0 \\ x_{3,2} - 1 &\geq 0 \\ x_{3,3} &\geq 0 \\ x_{3,2} + x_{3,3} - 1 &\geq 0 \\ x_{3,3} + 1 &\geq 0 \\ 1 - x_{3,2} - x_{3,3} &\geq 0 \\ 1 - x_{3,3} &\geq 0 \\ x_{3,2} &\geq 0 \end{aligned}\right\}$$

Here the fourth and sixth inequalities imply that $x_{3,2} + x_{3,3} - 1 = 0$, and hence $x_{3,3} = 1 - x_{3,2}$. This can be substituted into the rest of the inequalities.

$$\left.\begin{aligned} 1 &\geq 0 \\ x_{3,2} - 1 &\geq 0 \\ 1 - x_{3,2} &\geq 0 \\ 2 - x_{3,2} &\geq 0 \\ x_{3,2} &\geq 0 \\ x_{3,2} &\geq 0 \end{aligned}\right\}$$

Here the second and third inequalities together imply that $x_{3,2} - 1 = 0$, that is $x_{3,2} = 1$ and then $x_{3,3} = 1 - x_{3,2} = 0$. It's easy to check that $x_{3,2} = 1$ and $x_{3,3} = 0$ is a solution of the above system of inequalities. Thus substituting $x_{3,2} = 1$ and $x_{3,3} = 0$ into the solutions of the system of equations gives that the only non-negative solution is

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 1 & x_{1,3} = 0 & x_{1,4} = 0 \\ x_{2,1} = 0 & x_{2,2} = 1 & x_{2,3} = 0 & x_{2,4} = 1 \\ x_{3,1} = 0 & x_{3,2} = 1 & x_{3,3} = 0 & x_{3,4} = 1 \end{array}$$

Example 2

Let $R = [0,4] \times [0,3]$, $m = 3$, $n = 4$. Then the uniform partition of $[0,4]$ is $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and the uniform partition of $[0,3]$ is $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. These imply the pixels

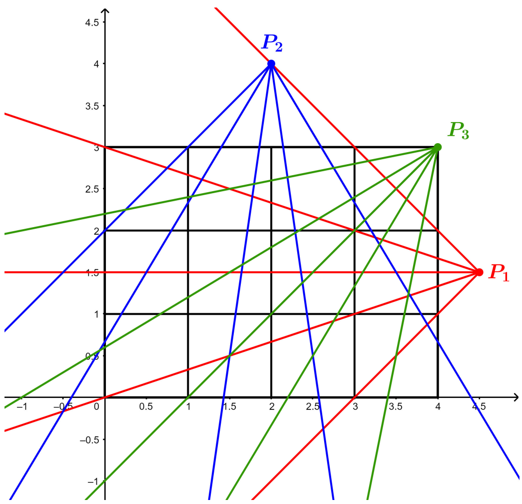
$$\begin{array}{lll} R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\ R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\ R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\ R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1] \end{array}$$

Consider the following set of lines:

- l_1, l_2, l_3, l_4, l_5 are lines passing through the point $P_1 = \left(\frac{9}{2}, \frac{3}{2}\right)$ and parallel to the vectors $\underline{v}_{1,1} = (1, -1)$, $\underline{v}_{1,2} = (3, -1)$, $\underline{v}_{1,3} = (1, 0)$, $\underline{v}_{1,4} = (3, 1)$, $\underline{v}_{1,5} = (1, 1)$ respectively.

- $l_6, l_7, l_8, l_9, l_{10}$ are lines passing through the point $P_2 = (2, 4)$ and parallel to the vectors $\underline{v}_{2,1} = (1, 1)$, $\underline{v}_{2,2} = (3, 5)$, $\underline{v}_{2,3} = (1, 7)$, $\underline{v}_{2,4} = (1, -7)$, $\underline{v}_{2,5} = (3, -5)$ respectively.

- $l_{11}, l_{12}, l_{13}, l_{14}, l_{15}$ are lines passing through the point $P_3 = (4, 3)$ and parallel to the vectors $\underline{v}_{3,1} = (5, 1)$, $\underline{v}_{3,2} = (5, 3)$, $\underline{v}_{3,3} = (1, 1)$, $\underline{v}_{3,4} = (3, 5)$, $\underline{v}_{3,5} = (1, 5)$ respectively.



Let m_k be the line integral of an unknown function f along line l_k for $k = 1, 2, \ldots, 15$, where

$$\begin{array}{lllll} m_1 = 0, & m_2 = \frac{2\sqrt{10}}{3}, & m_3 = 3, & m_4 = \frac{2\sqrt{10}}{3} & m_5 = \sqrt{2} \\ m_6 = 0, & m_7 = \frac{\sqrt{34}}{5}, & m_8 = \frac{10\sqrt{2}}{7}, & m_9 = \frac{5\sqrt{2}}{7} & m_{10} = \frac{4\sqrt{34}}{15} \\ m_{11} = \frac{\sqrt{26}}{5}, & m_{12} = \frac{2\sqrt{34}}{5}, & m_{13} = \sqrt{2}, & m_{14} = \frac{\sqrt{34}}{5}, & m_{15} = \frac{2\sqrt{26}}{5} \end{array}$$

We try to find the values $x_{i,j}$, such that the function g, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines l_k equal to m_k for all $k = 1, 2, \ldots 15$.

- The line l_1 intersects only the pixel $R_{1,4}$ and the length of the intersection is $\sqrt{2}$.
- The line l_2 intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{10}}{3}$ in each case.
- The line l_3 intersects the pixels $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$. The length of the intersection equals to 1 in each case.
- The line l_4 intersects the pixels $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{10}}{3}$ in each case.
- The line l_5 intersects only the pixel $R_{3,4}$ and the length of the intersection is $\sqrt{2}$.
- The line l_6 intersects only the pixel $R_{1,1}$ and the length of the intersection is $\sqrt{2}$.
- The line l_7 intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{2,1}$, $R_{3,1}$. The lengths of the intersections are $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, respectively.
- The line l_8 intersects the pixels $R_{1,2}$, $R_{2,2}$, $R_{3,2}$. The length of the intersection equals to $\frac{5\sqrt{2}}{7}$ in each case.
- The line l_9 intersects the pixels $R_{1,3}$, $R_{2,3}$, $R_{3,3}$. The length of the intersection equals to $\frac{5\sqrt{2}}{7}$ in each case.
- The line l_{10} intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. The lengths of the intersections are $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, respectively.
- The line l_{11} intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{26}}{5}$ in each case.
- The line l_{12} intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{3,1}$. The lengths of the intersections are $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, respectively.
- The line l_{13} intersects the pixels $R_{1,4}$, $R_{2,3}$, $R_{3,2}$. The length of the intersection equals to $\sqrt{2}$ in each case.

- The line l_{14} intersects the pixels $R_{1,4}$, $R_{2,3}$, $R_{2,4}$, $R_{3,3}$. The lengths of the intersections are $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$ respectively.

- The line l_{15} intersects the pixels $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. The length of the intersection equals to $\frac{\sqrt{26}}{5}$ in each case.

Hence making all the line integrals of g equal to the corresponding line integrals of the unknown function f yields the system of equations

$$\left.\begin{aligned}
\sqrt{2}x_{1,4} &= 0 \\
\tfrac{\sqrt{10}}{3}x_{1,1} + \tfrac{\sqrt{10}}{3}x_{1,2} + \tfrac{\sqrt{10}}{3}x_{1,3} + \tfrac{\sqrt{10}}{3}x_{2,4} &= \tfrac{2\sqrt{10}}{3} \\
x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} &= 3 \\
\tfrac{\sqrt{10}}{3}x_{3,1} + \tfrac{\sqrt{10}}{3}x_{3,2} + \tfrac{\sqrt{10}}{3}x_{3,3} + \tfrac{\sqrt{10}}{3}x_{2,4} &= \tfrac{2\sqrt{10}}{3} \\
\sqrt{2}x_{3,4} &= \sqrt{2} \\
\sqrt{2}x_{1,1} &= 0 \\
\tfrac{\sqrt{34}}{15}x_{1,1} + \tfrac{2\sqrt{34}}{15}x_{1,2} + \tfrac{\sqrt{34}}{5}x_{2,1} + \tfrac{\sqrt{34}}{15}x_{3,1} &= \tfrac{\sqrt{34}}{5} \\
\tfrac{5\sqrt{2}}{7}x_{1,2} + \tfrac{5\sqrt{2}}{7}x_{2,2} + \tfrac{5\sqrt{2}}{7}x_{3,2} &= \tfrac{10\sqrt{2}}{7} \\
\tfrac{5\sqrt{2}}{7}x_{1,3} + \tfrac{5\sqrt{2}}{7}x_{2,3} + \tfrac{5\sqrt{2}}{7}x_{3,3} &= \tfrac{5\sqrt{2}}{7} \\
\tfrac{\sqrt{34}}{15}x_{1,3} + \tfrac{2\sqrt{34}}{15}x_{1,4} + \tfrac{\sqrt{34}}{5}x_{2,4} + \tfrac{\sqrt{34}}{15}x_{3,4} &= \tfrac{4\sqrt{34}}{15} \\
\tfrac{\sqrt{26}}{5}x_{1,1} + \tfrac{\sqrt{26}}{5}x_{1,2} + \tfrac{\sqrt{26}}{5}x_{1,3} + \tfrac{\sqrt{26}}{5}x_{2,4} &= \tfrac{\sqrt{26}}{5} \\
\tfrac{2\sqrt{34}}{15}x_{1,3} + \tfrac{\sqrt{34}}{5}x_{1,4} + \tfrac{\sqrt{34}}{15}x_{2,1} + \tfrac{\sqrt{34}}{5}x_{2,2} + \tfrac{\sqrt{34}}{15}x_{2,3} + \tfrac{2\sqrt{34}}{15}x_{3,1} &= \tfrac{2\sqrt{34}}{5} \\
\sqrt{2}x_{1,4} + \sqrt{2}x_{2,3} + \sqrt{2}x_{3,2} &= \sqrt{2} \\
\tfrac{\sqrt{34}}{5}x_{1,4} + \tfrac{\sqrt{34}}{15}x_{2,3} + \tfrac{2\sqrt{34}}{15}x_{2,4} + \tfrac{\sqrt{34}}{5}x_{3,3} &= \tfrac{\sqrt{34}}{5} \\
\tfrac{\sqrt{26}}{5}x_{1,4} + \tfrac{\sqrt{26}}{5}x_{2,4} + \tfrac{\sqrt{26}}{5}x_{3,4} &= \tfrac{2\sqrt{26}}{5}
\end{aligned}\right\}$$

Here we multiply the the first equation by $\frac{1}{\sqrt{2}}$, the second equation by $\frac{3}{\sqrt{10}}$, the fourth equation by $\frac{3}{\sqrt{10}}$, the fifth and sixth equations by $\frac{1}{\sqrt{2}}$, the seventh equation by $\frac{15}{\sqrt{34}}$, the eighth and ninth equations by $\frac{7}{5\sqrt{2}}$, the tenth equation by $\frac{15}{\sqrt{34}}$, the eleventh equation by $\frac{5}{\sqrt{26}}$, the twelfth equation by $\frac{15}{\sqrt{34}}$, the thirteenth equation by $\frac{1}{\sqrt{2}}$, the fourteenth equation by $\frac{15}{\sqrt{34}}$, and the fifteenth equation by $\frac{5}{\sqrt{26}}$. Then all coefficients of the system are integers.

$$
\left.\begin{aligned}
x_{1,4} &= 0\\
x_{1,1} + x_{1,2} + x_{1,3} + x_{2,4} &= 2\\
x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} &= 3\\
x_{3,1} + x_{3,2} + x_{3,3} + x_{2,4} &= 2\\
x_{3,4} &= 1\\
x_{1,1} &= 0\\
x_{1,1} + 2x_{1,2} + 3x_{2,1} + x_{3,1} &= 3\\
x_{1,2} + x_{2,2} + x_{3,2} &= 2\\
x_{1,3} + x_{2,3} + x_{3,3} &= 1\\
x_{1,3} + 2x_{1,4} + 3x_{2,4} + x_{3,4} &= 4\\
x_{1,1} + x_{1,2} + x_{1,3} + x_{2,4} &= 1\\
2x_{1,3} + 3x_{1,4} + x_{2,1} + 3x_{2,2} + x_{2,3} + 2x_{3,1} &= 6\\
x_{1,4} + x_{2,3} + x_{3,2} &= 1\\
3x_{1,4} + x_{2,3} + 2x_{2,4} + 3x_{3,3} &= 3\\
x_{1,4} + x_{2,4} + x_{3,4} &= 2
\end{aligned}\right\}
$$

Assuming lexicographic order of the variables the extended coefficient matrix is

$$
(A|\underline{b})
\left(\begin{array}{cccccccccccc|c}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 4\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 2 & 3 & 1 & 3 & 1 & 0 & 2 & 0 & 0 & 0 & 6\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 0 & 3 & 0 & 0 & 1 & 2 & 0 & 0 & 3 & 0 & 3\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2
\end{array}\right)
$$

If we transform it into reduced row echelon form with the help of Gaussian elimination, then we get

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Each column contains a pivot element except the last one, which means the system is solvable, and has a unique solution. This solution is

$$\begin{array}{llll}
x_{1,1} = 0 & x_{1,2} = 1 & x_{1,3} = 0 & x_{1,4} = 0 \\
x_{2,1} = 0 & x_{2,2} = 1 & x_{2,3} = 1 & x_{2,4} = 1 \\
x_{3,1} = 1 & x_{3,2} = 0 & x_{3,3} = 0 & x_{3,4} = 1
\end{array}$$

4 Discrete tomography

4.1 Reconstruction of binary matrices with prescribed row and column sums

A matrix A is called **binary matrix**, if all elements of A are either one or zero. The row sum vector of the binary matrix A of size $m \times n$ is the vector $R = (r_1, r_2, \dots, r_m)$, where

$$\sum_{j=1}^{n} a_{i,j} = r_i, \qquad i \in \{1, 2, \dots, m\}$$

In other words r_i equals to number of ones in the i-th row of A for each $i \in \{1, 2, \dots, m\}$. The column sum vector of the binary matrix A of size $m \times n$ is the vector $S = (s_1, s_2, \dots, s_n)$, where

$$\sum_{i=1}^{m} a_{i,j} = s_j, \qquad j \in \{1, 2, \dots, n\}$$

In other words s_j equals to number of ones in the j-th column of A for each $j \in \{1, 2, \dots, n\}$.

It's easy to see, that if A is any binary matrix of size $m \times n$, and its row and column sum vectors are R and S, then

1. all elements of R and S are non-negative integers,
2. R has no element which is larger than n, and S has no element which is larger than m,
3. the sum of the elements in R equals to the sum of the elements in S, that is
$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.$$

The first two statements is quite trivial, while the third is just the consequence of the fact, that both $\sum_{i=1}^{m} r_i$ and $\sum_{j=1}^{n} s_j$ equals to the total number of ones in the matrix A.

Definition 10 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two vectors, whose elements are non-negative integers. The vectors R and S are called compatible if*

1. *$r_i \leq n$ for all $i \in \{1, 2, \ldots, m\}$,*
2. *$s_j \leq m$ for all $j \in \{1, 2, \ldots, n\}$,*
3. $$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.$$

Note that if the vectors $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are not compatible, then there's no binary matrix of size $m \times n$ whose row sum vector is R and column sum vector is S.

Given the row sum vector $R = (r_1, r_2, \ldots, r_m)$ and the number of columns n the maximal matrix corresponding to R is the binary matrix A of size $m \times n$ which satisfies

$$a_{ij} = \begin{cases} 1 & \text{if } j \leq r_i, \\ 0 & \text{if } r_i < j. \end{cases} \qquad \text{for all } i \in \{1, 2, \ldots, m\}$$

In other words the first r_i elements of the i-th row of the maximal matrix equal to one, while the rest of the elements in the i-th row equal to zero. The maximal matrix with n columns corresponding to the row sum vector R is denoted by $\overline{A}$, and the column sum vector of the maximal matrix is denoted by $\overline{S} = (\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_n)$. Furthermore let's denote the nonincreasing permutation of any column sum vector S by $S' = (s'_1, s'_2, \ldots, s'_n)$.

Theorem 8 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two compatible integer vectors. There exists a binary matrix with row sum vector R and column sum vector S if and only if*

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \textit{for all } k \in \{1, 2, \ldots, n\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities.

The above theorem can be used to decide whether there's a binary matrix with given row sum and column sum vector or not, and also to decide whether the solution is unique or not. However it tell nothing about how to find such matrix. If we know that there's a solution of the problem then the following procedure can be used.

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two compatible integer vectors that satisfy the condition

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1, 2, \ldots, n\}$$

Let the matrix A be equal to the maximal matrix $\overline{A}$ at the beginning of the procedure. Then

1. If there's no column index k such that s'_k is larger than the sum of the elements of A in the k-th column, then go to step 6.

2. Choose the largest column index k such that s'_k is larger than the sum of the elements of A in the k-th column.

3. Choose the largest column index l, such that it's less than k and the l-th column of A contains at least one nonzero element.

4. Let i denote the largest row index such that $a_{i,l} = 1$. Then change the value of $a_{i,l}$ to zero and change the value of $a_{i,k}$ to one (which must be zero before the change if the above conditions are satisfied).

5. Repeat steps (2)-(4) until there's no column index k such that s'_k is larger than the sum of the elements of A in the k-th column.

6. Now the matrix A must have row sum vector R and column sum vector S'. Find a permutation that transforms S' into S and apply the same permutation for the columns of A

By the end of the above procedure A has row sum vector R and column sum vector S.

Example 1

Let $R = (2, 6, 4, 3, 3)$ and $S = (3, 4, 6, 2, 2, 1)$. Then there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because S has an element larger than 5, and hence the vectors R and S are not compatible.

Example 2

Let $R = (5, 4, 2, 3, 3)$ and $S = (2, 3, 1, 4, 4, 2)$. Then there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because

$$\sum_{i=1}^{5} r_i = 5 + 4 + 2 + 3 + 3 = 17 \quad \text{and} \quad \sum_{j=1}^{6} s_j = 2 + 3 + 1 + 4 + 4 + 2 = 16$$

and hence the vectors R and S are not compatible.

Example 3

Let $R = (3, 5, 2, 3, 1)$ and $S = (0, 2, 4, 4, 4, 0)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $S' = (4, 4, 4, 2, 0, 0)$ and $\overline{S} = (5, 4, 3, 1, 1, 0)$. Here

$$\begin{aligned} 4 \leq 5 &\implies s'_1 \leq \overline{s}_1 \\ 4 + 4 \leq 5 + 4 &\implies s'_1 + s'_2 \leq \overline{s}_1 + \overline{s}_2 \\ 4 + 4 + 4 \leq 5 + 4 + 3 &\implies s'_1 + s'_2 + s'_3 \leq \overline{s}_1 + \overline{s}_2 + \overline{s}_3 \\ 4 + 4 + 4 + 2 > 5 + 4 + 3 + 1 &\implies s'_1 + s'_2 + s'_3 + s'_4 > \overline{s}_1 + \overline{s}_2 + \overline{s}_3 + \overline{s}_4 \end{aligned}$$

Hence there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j$$

is not satisfied for all $k \in \{1, 2, \ldots, 6\}$.

Example 4

Let $R = (2, 3, 5, 5, 3)$ and $S = (1, 3, 4, 5, 4, 1)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus $S' = (5, 4, 4, 3, 1, 1)$ and $\overline{S} = (5, 5, 4, 2, 2, 0)$. Here

$$\begin{aligned} 5 &\leq 5 \\ 5+4 &\leq 5+5 \\ 5+4+4 &\leq 5+5+4 \\ 5+4+4+3 &\leq 5+5+4+2 \\ 5+4+4+3+1 &\leq 5+5+4+2+2 \\ 5+4+4+3+1+1 &\leq 5+5+4+2+2+0 \end{aligned}$$

Hence

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j$$

is satisfied for all $k \in \{1, 2, \ldots, 6\}$, and there exists a binary matrix of size 5×6 which has row sum vector R and column sum vector S. This matrix is not unique, because some of the above inequalities are strict. Now let's find a binary matrix with row sum vector R and column sum vector S.

$$\begin{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 5 & 4 & 4 & 3 & 1 & 1 \end{matrix} \end{matrix} \longrightarrow \begin{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 5 & 4 & 4 & 3 & 1 & 1 \end{matrix} \end{matrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$
$$\quad\ \ 5 \quad 4 \quad 4 \quad 3 \quad 1 \quad 1 \qquad\qquad 5 \quad 4 \quad 4 \quad 3 \quad 1 \quad 1$$

Now the last matrix has row sum vector R and column sum vector S'. Let's permute the columns to get column sum vector S.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$
$$5 \quad 4 \quad 4 \quad 3 \quad 1 \quad 1 \qquad\qquad 1 \quad 3 \quad 4 \quad 5 \quad 4 \quad 1$$

The matrix on the right has row sum vector R and column sum vector S.

Example 5

Let $R = (6, 3, 4, 2, 2)$ and $S = (1, 2, 5, 5, 3, 1)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $S' = (5, 5, 3, 2, 1, 1)$ and $\overline{S} = (5, 5, 3, 2, 1, 1)$. Here $S' = \overline{S}$, and hence

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j$$

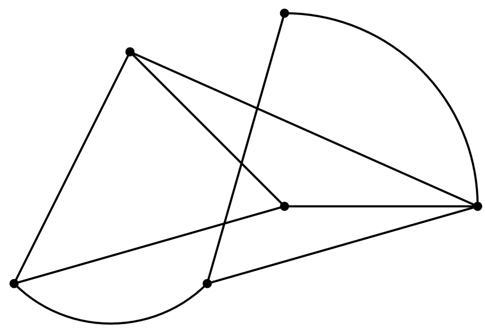
is satisfied for all $k \in \{1, 2, \ldots, 6\}$, and there exists a binary matrix of size 5×6 which has row sum vector R and column sum vector S. This matrix is unique, because the above inequalities are all satisfied with equalities. To find a binary matrix with row sum vector R and column sum vector S it's enough to permute the columns of the maximal matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$
$$\begin{matrix} 5 & 5 & 3 & 2 & 1 & 1 \end{matrix} \qquad\qquad \begin{matrix} 1 & 2 & 5 & 5 & 3 & 1 \end{matrix}$$

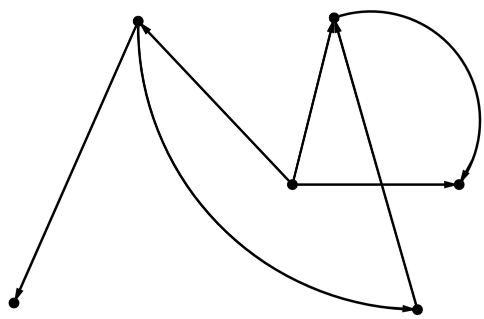
The matrix on the right has row sum vector R and column sum vector S.

4.2 Graphs

A **(simple) graph** consists of a (non-empty) finite set V of elements called **vertices** (or **nodes**), a finite set E of elements called **edges**, and a function f which assigns a subset of two elements of V to every element of E, such that if $e_1 \neq e_2$, then $f(e_1) \neq f(e_2)$. If $f(e) = \{u, v\}$ for an $e \in E$ and some $u, v \in V$ then we say the edge **connects** the vertices u and v. In such case the vertices u and v are called **adjacent**, and we say u is a **neighbor** of v, and v is a neighbor of u. We may also write $uv \in E$ and say uv is an edge of the graph, if there exists $e \in E$ such that $f(e) = uv$. We note that if uv is an edge, then vu is the same edge of the graph. A graph can be visualized in the plane by presenting the vertices as points of the plane and presenting the edges as curves or line segments connecting the vertices. It doesn't matter how we arrange the points, or how the shape of the curves presenting the edges looks like, the only important thing is to define which vertices are connected by which edges. The edges may cross each other in points which are not vertices of the graph.



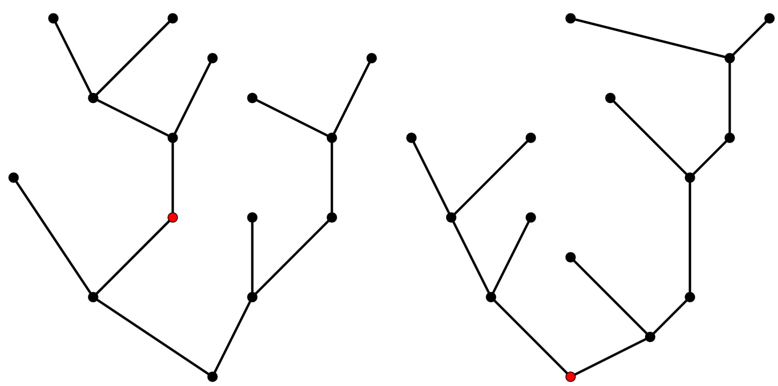
A **(simple) directed graph** consists of a (non-empty) finite set V of elements called **vertices** (or **nodes**), a finite set E of elements called **edges**, and a function f which assigns an ordered pair of elements of V to every element of E, such that if $e_1 \neq e_2$, then $f(e_1) \neq f(e_2)$. We say the vertex u is **connected to** v if there exists $e \in E$, such that $f(e) = (u, v)$. In this case we may write $uv \in E$ and say uv is and edge of the directed graph, and v is called an **out-neighbor** of u, while u is called an **in-neighbor** of v. If $f(e) = (u, v)$, then u is called the **initial vertex** and v is celled the **terminal vertex** of the directed edge e. Note that if uv is an edge of a directed graph it my happen that vu is not an edge. A directed graph can be visualized in the plane by presenting the vertices as points of the plane and presenting the edges as directed curves or directed line segments connecting the vertices.



Given a graph G, a **walk** in the graph is a finite sequence of edges $e_1, e_2, \ldots, e_m$ such that e_{i-1} and e_i have a common vertex for all $i \in \{2, 3, \ldots, m\}$. A **(di-**

rected) walk in a directed graph is a finite sequence of edges $e_1, e_2, \dots, e_m$ such that the terminal vertex of e_{i-1} is the initial vertex of e_i for all $i \in \{2, 3, \dots, m\}$. Any walk determines the sequence of vertices $v_0, v_1, \dots, v_m$, where e_i connects the vertices v_{i-1} and v_i (or v_{i-1} is connected to v_i by the edge e_i in the case of a directed graph). A walk in which all the edges are distinct is called a **trail**. If, in addition, all the vertices $v_0, v_1, \dots, v_k$ determined by the path/trail are distinct (except possibly $v_0 = v_m$), then it's called a **path**. A path or trail is **closed** if $v_0 = v_m$, and a closed path is called a **cycle**. The length of a path of r

We say that a graph is **connected** if there exists a path between any pair of vertices. A connected graph which contains no cycle is called a **tree**.



The **length** of a path or a cycle is the number of edges in it. The **distance of the vertices** s and t in a connected graph is the length the shortest path connecting s and t. Given a vertex of $s \in V$ of any (directed) graph there's an easy method to find the distance of any further vertex form s which can be connected to s with a path. It's based on assigning labels to the veritces that give the distances from s.

1. Assign the label zero to the vertex s.

2. Assuming that the highest label of the vertices of the graph is $k \in \mathbb{Z}$, look for the unlabeled neighbors (or out-neighbors) of the vertices with label k. If there's at least one such neighbor (or out-neighbor) then assign the label $k + 1$ to them.

3. Repeat step 2 as long as possible.

4. If the vertices with the highest label have no unlabeled neighbors (or out-neighbors) then we found all the distances of the vertices of the graph which can be connected to s with a path. If there are still unlabeled vetrices, then those cannot be accessed from s via any path.

The labels assigned by the above procedure give the distances form s. Then it's also possible to find a shortest path connecting s to a vetrex t with the help of these distances. If t has no labeled then it cannot be accessed from s. Otherwise let the label of t be $k \in \mathbb{Z}$. Then t must have a neighbor (or in-neighbor) denoted by v_{k-1} which has label $k-1$. The vertex v_{k-1} must have a neighbor (or in-neighbor) denoted by v_{k-2} which has label $k-2$. Tis can be continued until a vertex v_1 with label 1 is found. Then s must be a neighbor (or in-neighbor) of v_1, thus the vertices $s, v_1, v_2, \ldots, v_{k-1}, t$ determine a shortest path form s to t.

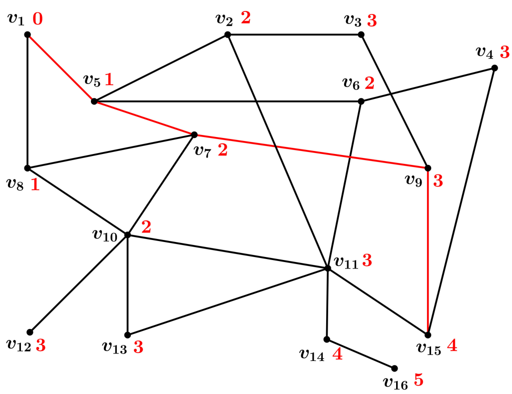


Figure 4.1: The distances from the vertex v_1 and the shortest path connecting v_1 and v_{15}

4.3 Networks and flows

A **network** is a directed graph, where non-negative real numbers are assigned to the directed edges, which are called **capacities**. The capacity of the edge e

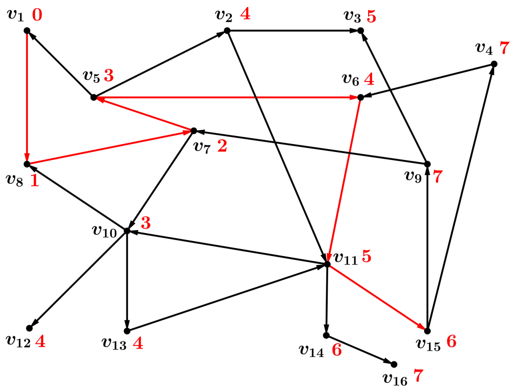


Figure 4.2: The distances in a directed graph from the vertex v_1 and the shortest directed path connecting v_1 and v_{15}

is denoted by $U(e)$, and U is called the capacity function. The out-degree of a vertex v is the sum of the capacities of all edges vu where u is an out-neighbor of v. The in-degree of a vertex v is the sum of the capacities of all edges uv where u is an in-neighbor of v. Given two specified vetrices s and t, which are called source and sink, a flow is a function $Y\colon E \to \mathbb{R}$ with non-negative values such that

1. $Y(e) \leq U(e)$ for all $e \in E$,
2. the out-degree and the in-degree equal to each other for any vertex, except for s and t.

Given a network and the flow, an edge e is called **saturated** if $Y(e) = U(e)$, otherwise it's called **unsaturated**. The **size of a flow** is the sum of the values of the flow on the edges whose initial vertex is s. The size of the flow must also be equal to the sum of the values of the flow on the edges whose terminal vertex is t. A flow is called **maximal** if there's no other flow on the same network, whose size would be larger. A flow is called **integral flow**, if all its values are integers. A **flow-augmenting path** is an undirected path from the source s to the sink t, which satisfies the following: moving form s to t along an edge e of the undirected path,

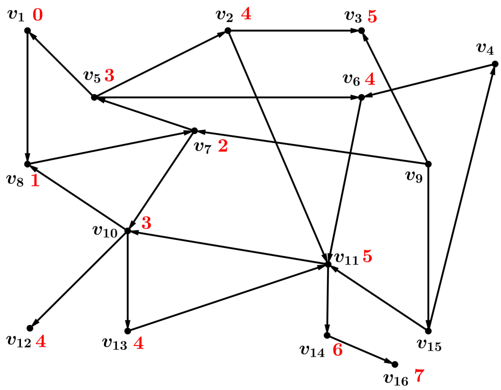


Figure 4.3: The distances in a directed graph from the vertex v_1 when there's no directed path connecting v_1 and v_{15}

1. if the movement has the same direction as the direction of e in the directed graph, then $Y(e) < U(e)$,

2. if the movement has the opposite direction as the direction of e in the directed graph, then $0 < Y(e)$.

If there's an flow-augmenting path then we can compute the minimum of the values $U(e) - Y(e)$ for edges of the path when the movement has the same direction as the direction of e together with the values $Y(e)$ for edges of the path when the movement has the opposite direction as the direction of e. Let this minimum be denoted by α. Then we can increase the values of the flow by α along edges of the path, when the movement has the same direction as the direction of e, and decrease the values of the flow by α for edges of the path, when the movement has the opposite direction as the direction of e. This change in the values of the flow results a valid flow on the same network with a larger size.

Theorem 9 *A flow on a network is maximal if and only if there exists no flow-augmenting path for the flow.*