Exercises

1 Systems of linear equations and inequalities

Exercise 1.1 *Find the set of solutions of the following systems of linear equations.*

(a)

$$\left.\begin{aligned} x_1 + 2x_2 - 4x_3 &= -2 \\ -2x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 5x_2 - 3x_3 + x_4 &= 1 \\ x_1 + x_2 + 3x_4 &= -2 \\ 11x_2 - 6x_3 + 5x_4 &= 0 \end{aligned}\right\}$$

(b)

$$\left.\begin{aligned} x_1 + 3x_3 - 2x_4 &= -1 \\ -x_1 + 2x_2 + 5x_3 + x_4 &= -5 \\ 2x_1 + 3x_2 + 4x_4 &= 7 \\ x_2 + x_3 - 3x_4 &= 0 \\ 2x_1 + 6x_2 + 9x_3 &= 1 \end{aligned}\right\}$$

(c)

$$\left.\begin{aligned} -x_1 + 3x_2 + 4x_3 &= 2 \\ 2x_1 + 2x_2 + x_3 - 1x_4 &= 2 \\ -x_1 + 3x_3 + 2x_4 &= -1 \\ 5x_2 + 8x_3 + x_4 &= 5 \\ 2x_2 + 7x_3 + 3x_4 &= 0 \end{aligned}\right\}$$

(d)

$$\left.\begin{aligned} -2x_2 + 3x_3 + 3x_4 &= 4 \\ x_1 + 4x_2 - x_3 &= 3 \\ 2x_1 - 2x_2 + 4x_3 + x_4 &= 3 \\ -x_1 + 2x_2 + x_3 + 5x_4 &= 8 \\ -12x_2 + 9x_3 + 4x_4 &= 1 \end{aligned}\right\}$$

(e)

$$\left.\begin{aligned} -x_1 - x_2 + 3x_3 + x_5 &= 1 \\ x_2 + x_3 + 2x_4 - 3x_5 &= -2 \\ 4x_1 - 2x_3 + 2x_4 + x_5 &= -2 \\ x_1 - 2x_2 + 8x_3 + 4x_4 + x_5 &= -1 \\ 3x_1 - 2x_2 + 5x_5 &= 1 \end{aligned}\right\}$$

(f)

$$\left.\begin{aligned} x_1 + 2x_2 - 3x_4 + x_5 &= 1 \\ 2x_1 + 5x_2 - x_3 + x_4 &= -8 \\ -3x_1 - 8x_2 + 2x_3 - 5x_4 + x_5 &= 17 \\ 5x_1 + 11x_2 - x_3 - 8x_4 + 3x_5 &= -5 \\ -x_2 + x_3 - 7x_4 + 2x_5 &= 10 \end{aligned}\right\}$$

Exercise 1.2 *Find the set of solutions of the following systems of linear inequalities with help of graphical solution method.*

(a)

$$\left.\begin{aligned} 2x + y &\leq 4 \\ -x + y &\leq 1 \\ -3x + y &\leq 5 \\ -x - 2y &\leq 4 \\ x - 2y &\leq 4 \\ x &\leq 2 \end{aligned}\right\}$$

(b)

$$\left.\begin{aligned} 2x - 5y &\leq 10 \\ x + y &\leq 5 \\ -x + 4y &\leq 8 \\ x + 3y &\leq 9 \\ y &\leq 4 \end{aligned}\right\}$$

(c)

$$\left.\begin{aligned} -x + y &\leq 0 \\ 5x + 3y &\leq 6 \\ x + 5y &\leq 13 \\ -2x + 5y &\leq 4 \\ -x + 6y &\leq -6 \end{aligned}\right\}$$

(d)

$$\left.\begin{aligned} -x + y &\leq 1 \\ -x + 2y &\leq 6 \\ -2x + y &\leq 5 \\ -x - y &\leq 0 \\ -5x - y &\leq -4 \end{aligned}\right\}$$

(e)

$$\left.\begin{aligned} x + 2y &\leq 12 \\ x + 3y &\leq 12 \\ y &\leq 5 \\ 2x - 3y &\leq -4 \\ x - 4y &\leq 0 \\ x &\leq 4 \end{aligned}\right\}$$

(f)

$$\left.\begin{aligned} x - 4y &\le 3 \\ 2x - 3y &\le -3 \\ 2x - y &\le -1 \\ 6x - y &\le 11 \\ x &\le 2 \end{aligned}\right\}$$

(g)

$$\left.\begin{aligned} -x + 3y &\le 8 \\ x + 6y &\le 36 \\ -x - y &\le -2 \\ 5x - y &\le 16 \\ -y &\le 0 \\ -x &\le 1 \end{aligned}\right\}$$

(h)

$$\left.\begin{aligned} x + 3y &\le 5 \\ -2x - y &\le -5 \\ 4x - 3y &\le 5 \\ -3x - y &\le 6 \\ x - 5y &\le 8 \end{aligned}\right\}$$

(i)

$$\left.\begin{aligned} -x + 4y &\le 5 \\ -3x - 2y &\le 1 \\ 2x - 3y &\le 4 \\ 2x - y &\le -3 \\ y &\le 2 \end{aligned}\right\}$$

(j)

$$\left.\begin{aligned} -3x + 5y &\le -9 \\ 4x + y &\le 11 \\ 2x - y &\le -1 \\ -2x - 3y &\le -3 \\ -4x - 9y &\le 3 \end{aligned}\right\}$$

Exercise 1.3 *Find the set of solutions of the following systems of linear inequalities with help of Fourier-Motzkin elimination.*

(a)

$$\left.\begin{aligned} 3x_1 - 2x_2 + x_3 &\le 4 \\ -x_1 + 4x_2 + 2x_3 &\le 1 \\ 5x_2 + 4x_3 &\le 3 \\ x_1 - 3x_3 &\le 2 \\ 2x_1 - x_2 &\le 1 \end{aligned}\right\}$$

(b)

$$\left.\begin{aligned} 12x_1 + 20x_2 + 15x_3 &\leq 60 \\ -3x_1 - 5x_2 - 7x_3 &\leq -15 \\ -5x_1 - 4x_2 - 3x_3 &\leq -12 \\ -4x_1 - 13x_2 - 5x_3 &\leq -22 \end{aligned}\right\}$$

(c)

$$\left.\begin{aligned} -2x_1 + x_2 + 4x_3 &\leq 1 \\ 3x_2 + x_3 &\leq 4 \\ 3x_1 + 2x_2 - 6x_3 &\leq 12 \\ -x_1 + 2x_2 - x_3 &\leq 3 \end{aligned}\right\}$$

(d)

$$\left.\begin{aligned} -2x_1 - 8x_2 + 9x_3 &\leq 20 \\ 2x_1 + x_3 &\leq 4 \\ x_1 + 3x_2 - 4x_3 &\leq -7 \\ -2x_1 + 6x_2 - 5x_3 &\leq -22 \\ 3x_1 - x_2 &\leq 9 \end{aligned}\right\}$$

(e)

$$\left.\begin{aligned} -x_3 + 3x_4 &\leq 10 \\ 2x_3 + x_4 &\leq 8 \\ x_3 - x_4 &\leq 1 \\ -3x_3 - 2x_4 &\leq -3 \\ -x_1 - 2x_2 &\leq -9 \\ -7x_1 + 3x_2 &\leq -29 \\ 9x_1 + x_2 &\leq 64 \end{aligned}\right\}$$

2 Computed tomography

Exercise 2.1 *Let $R = [a, b] \times [c, d]$ be the picture region with an $m \times n$ uniform partition. Calculate the line integral of the simple phantom defined by the matrix A along the line l which is parallel to the vector $\underline{v}$ and passes through the point P.*

(a) $R = [1, 5] \times [2, 5]$, $m = 3$, $n = 4$,

$$A = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

and $\underline{v} = (3, 1)$, $P = (0, 3)$.

(b) $R = [-1,3] \times [3,6]$, $m = 3$, $n = 4$,

$$A = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{6} & 1 \\ 1 & 0 & \frac{2}{3} & \frac{5}{6} \\ \frac{1}{3} & 1 & 1 & 0 \end{pmatrix}$$

and $\underline{v} = (1,4)$, $P = (0,4)$.

(c) $R = [3,6] \times [1,5]$, $m = 4$, $n = 3$,

$$A = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & 1 & 0 \\ 0 & \frac{1}{6} & 1 & \frac{5}{6} \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

and $\underline{v} = (1,-2)$, $P = (4,4)$.

(d) $R = [2,6] \times [-1,3]$, $m = 4$, $n = 4$,

$$A = \begin{pmatrix} \frac{1}{3} & \frac{5}{6} & 0 & \frac{1}{12} \\ 1 & \frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{2} & 0 & 1 & \frac{5}{12} \end{pmatrix}$$

and $\underline{v} = (-3,1)$, $P = \left(2, \frac{3}{2}\right)$.

Exercise 2.2 *Let* $R = [0,4] \times [0,3]$ *be the picture region with an* $m \times n$ *uniform partition, where* $m = 3$ *and* $n = 4$*. Consider the following line set:*

- *The lines* l_1, l_2, l_3 *are parallel to the vector* $\underline{v}_1 = (1,0)$ *and passing through the points* $P_{1,1} = \left(0, \frac{5}{2}\right)$, $P_{1,2} = \left(0, \frac{3}{2}\right)$, $P_{1,3} = \left(0, \frac{1}{2}\right)$ *respectively.*
- *The lines* $l_4, l_5, l_6, l_7, l_8, l_9$ *are parallel to the vector* $\underline{v}_2 = (1,1)$ *and passing through the points* $P_{2,1} = (0,2)$, $P_{2,2} = (0,1)$, $P_{2,3} = (0,0)$, $P_{2,4} = (0,-1)$, $P_{2,5} = (0,-2)$, $P_{2,6} = (0,-3)$ *respectively.*
- *The lines* $l_{10}, l_{11}, l_{12}, l_{13}, l_{14}, l_{15}$ *are parallel to the vector* $\underline{v}_3 = (1,-1)$ *and passing through the points* $P_{3,1} = (0,6)$, $P_{3,2} = (0,5)$, $P_{3,3} = (0,4)$, $P_{3,4} = (0,3)$, $P_{3,5} = (0,2)$, $P_{3,6} = (0,1)$ *respectively.*

The line integrals of an unknown function f *along lines* l_k *for* $k = 1, 2, \dots, 15$ *are*

$$\begin{array}{llllll} m_1 = 3, & m_2 = 2, & m_3 = 2, & & & \\ m_4 = \sqrt{2}, & m_5 = 0, & m_6 = 2\sqrt{2}, & m_7 = 3\sqrt{2}, & m_8 = 0 & m_9 = \sqrt{2}, \\ m_{10} = \sqrt{2}, & m_{11} = \sqrt{2}, & m_{12} = 2\sqrt{2}, & m_{13} = 2\sqrt{2}, & m_{14} = \sqrt{2}, & m_{15} = 0. \end{array}$$

(a) *Find all the possible values $x_{i,j}$, such that the function g, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines l_k equal to m_k for all $k = 1, 2, \dots 15$.*

(b) *Which are the non-negative solutions?*

(c) *Find the non-negative solutions whose values are not larger than 1.*

(d) *Which are the binary solutions?*

Exercise 2.3 *Let $R = [0, 4] \times [0, 3]$ be the picture region with an $m \times n$ uniform partition, where $m = 3$ and $n = 4$. Consider the following line set:*

- *The lines l_1, l_2, l_3, l_4 are parallel to the vector $\underline{v}_1 = (0, 1)$ and passing through the points $P_{1,1} = \left(\frac{1}{2}, 0\right)$, $P_{1,2} = \left(\frac{3}{2}, 0\right)$, $P_{1,3} = \left(\frac{5}{2}, 0\right)$, $P_{1,4} = \left(\frac{7}{2}, 0\right)$ respectively.*
- *The lines l_5, l_6, l_7, l_8, l_9 are parallel to the vector $\underline{v}_2 = (1, 2)$ and passing through the points $P_{2,1} = (-1, 0)$, $P_{2,2} = (0, 0)$, $P_{2,3} = (1, 0)$, $P_{2,4} = (2, 0)$, $P_{2,5} = (3, 0)$ respectively.*
- *The lines $l_{10}, l_{11}, l_{12}, l_{13}$ are parallel to the vector $\underline{v}_3 = (2, 1)$ and passing through the points $P_{3,1} = (0, 2)$, $P_{3,2} = (0, 1)$, $P_{3,3} = (0, 0)$, $P_{3,4} = (0, -1)$ respectively.*

The line integrals of an unknown function f along lines l_k for $k = 1, 2, \dots, 13$ are

$$\begin{array}{lllll} m_1 = 1, & m_2 = 2, & m_3 = 1, & m_4 = 2, & \\ m_5 = 0, & m_6 = \sqrt{5}, & m_7 = \frac{\sqrt{5}}{2}, & m_8 = \frac{\sqrt{5}}{2}, & m_9 = \sqrt{5}, \\ m_{10} = \frac{\sqrt{5}}{2} & m_{11} = \frac{\sqrt{5}}{2}, & m_{12} = \frac{3\sqrt{5}}{2}, & m_{13} = \frac{\sqrt{5}}{2}. & \end{array}$$

(a) *Find all the possible values $x_{i,j}$, such that the function g, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines l_k equal to m_k for all $k = 1, 2, \dots 13$.*

(b) *Which are the non-negative solutions?*

(c) *Find the non-negative solutions whose values are not larger than 1.*

(d) *Which are the binary solutions?*

3 Discrete tomography

Exercise 3.1 *Is there a binary matrix of size 6×5 with the row sum vector R and column sum vector S? If yes find such matrix. Is the solution unique? If no, then give another matrix with same row sum vector and column sum vector.*

(a) $R = (4, 4, 2, 0, 3, 5)$, $S = (1, 2, 5, 7, 3)$.

(b) $R = (2, 3, 5, 0, 3, 5)$, $S = (3, 2, 5, 4, 3)$.

(c) $R = (4, 1, 2, 4, 3, 1)$, $S = (1, 5, 4, 5, 0)$.

(d) $R = (0, 2, 4, 3, 3, 1)$, $S = (4, 3, 1, 2, 3)$.

(e) $R = (4, 2, 3, 3, 1, 1)$, $S = (1, 6, 3, 0, 4)$.

(f) $R = (2, 2, 5, 3, 4, 0)$, $S = (5, 4, 4, 3, 0)$.