# Chapter 1

# Discrete Tomography: A Historical Overview

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*ABSTRACT In this chapter we introduce the topic of discrete tomography and give a brief historical survey of the relevant contributions. After discussing the nature of the basic theoretical problems (those of consistency, uniqueness, and reconstruction) that arise in discrete tomography, we give the details of the classical special case (namely, two-dimensional discrete sets — i.e., binary matrices — and two orthogonal projections) including a polynomial time reconstruction algorithm. We conclude the chapter with a summary of some of the applications of discrete tomography.*

## 1.1 Introduction

We assume that there is a domain, which may itself be discrete (such as a set of ordered pairs of integers) or continuous (such as Euclidean space). We further assume that there is an unknown function $f$ whose range is known to be a given discrete set (usually of real numbers). The problems of *discrete tomography*, as we perceive the field, have to do with determining $f$ (perhaps only partially, perhaps only approximately) from weighted sums over subsets of its domain in the discrete case and from weighted integrals over subspaces of its domain in the continuous case. In many applications these sums or integrals may be known only approximately. From this point of view, the most essential aspect of discrete tomography is that knowing the discrete range of $f$ may allow us to determine its value at points where without this knowledge it could not be determined. Discrete tomography is full of mathematically fascinating questions and it has many interesting applications. The name *discrete tomography* is due to Larry Shepp, who organized the first meeting devoted to the topic (in 1994).

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The reconstruction algorithms used in CT (computerized tomography, see, e.g., [1]) are derived from the not discrete (let us say, general) tomography model in which the range of $f$ is the real numbers. Such reconstruction algorithms are unlikely to be appropriate for discrete tomography. For example, when using such an algorithm we cannot expect to produce a two-valued function, not even in the case when the data are taken from a two-valued $f$.

Since discrete functions can be considered to be special cases of general functions, discrete tomography (DT) can be thought of as a special kind of CT: it seems natural to apply the results of CT to discrete functions. However, it turns out that DT needs its own theory to answer questions concerning consistency, existence and uniqueness. Another reason for investigating special discrete reconstruction methods is that, since $f$ is discrete, there is hope that it can be determined from less data than what are necessary for general functions. Accordingly, in DT the typical number of projections (a projection is a collection of line sums or line integrals for a set of lines which are either all parallel to each other or diverge from a single point) is two to four, which is much less than what is typically used in CT (a few hundred). General CT reconstruction methods cannot be used effectively if the number of projections is so small.

DT has its own mathematical theory based mostly on discrete mathematics. It has strong connections with combinatorics and geometry. In the rest of this section we give a brief discussion of the connections of DT to other fields of mathematics. The only intent of this discussion is to set DT into its mathematical historical context; for this reason we will use terms here without carefully defining them. The sections that follow are oriented more toward the specific material in this book; in those sections we will be careful to define all terms that form part of the discussion. (Some of these terms will already appear in the current section, but even in such cases we postpone the definitions until the following sections.)

DT has connections to the *analysis of functions.* Lorentz gave in 1949 [2] a necessary and sufficient condition for a function-pair to be the projections of a planar measurable set. This condition can be considered as the first consistency result of DT. He also found a condition on two orthogonal projections of a measurable set by which one can determine if there is no other measurable set with the same projections. He also showed that for any integer $n$ there are always two different bounded sets which have the same $n$ projections. For related results see the works of Rényi [3], Heppes [4] and Kellerer [5–7].

Many problems of DT were first discussed as *combinatorial problems* during the late 1950s and early 1960s. In 1957 Ryser [8] published a necessary and sufficient consistency condition for a pair of integral vectors being the row and column sum vectors of a (0,1)-matrix. (It is interesting that this consistency theorem is the same as Lorentz's result [2] specialized to the case of (0,1)-matrices. The connection between the general and discrete

cases is discussed in Chapter 5 by Kaneko and Huang in this book.) By giving a constructive proof of his theorem, Ryser provided the first reconstruction algorithm. He also recognized that the so-called *interchange* is the elementary operation by which any two (0,1)-matrices can be transformed into each other if they have the same row and column sums. In the same year Gale [9] proved the same consistency condition as Ryser, but applying it to flows in networks. In 1960 Ryser introduced the concept of the *structure matrix* [10], which is useful, for example, in leading to a new consistency condition [11, p. 82].

The discussion of the *geometric connections* of DT was started by Hammer (although it was foreshadowed by Jakob Steiner [12] in the 19th century), who in 1961 at the AMS Symposium on Convexity [13] raised the problem: when is a planar convex body uniquely determined from its projections? For projections along parallel lines, an answer is due to Giering [14], who proved that for any planar convex body there exist three projections which uniquely determine it. Gardner and McMullen [15] proved that it is possible to find four projections that will uniquely determine all planar convex bodies. (As a summary of the geometric results connected with tomography we can suggest the excellent book written by Gardner [16].) Although these results of convex geometry or, more exactly, geometric tomography, are about the reconstruction of convex bodies, there are corresponding results for discrete sets. For example, Lorentz [2] gave a method to construct, for any finite number of directions, distinct discrete sets having the same projections along these directions. That is, a finite number of projections are not generally sufficient to reconstruct discrete sets. However, if the number of points in the discrete set to be reconstructed is known, it is possible to find finitely many projections that will guarantee uniqueness [3,17]. A discrete analogue of Gardner and McMullen's theorem was obtained by Gardner and Gritzmann [18], who showed that convex lattice sets in $\mathbb{Z}^2$ ($\mathbb{Z}^d$ denotes the set of $d$-dimensional vectors of integers) are determined by certain prescribed sets of four lattice directions.

The interest in DT is well illustrated by the fact that since September 19, 1994, when the first meeting of the topic was held at DIMACS, Rutgers University, a five-day seminar [19] was held in Dagstuhl, Germany in January 1997, a Discrete Tomography Workshop was held in Szeged, Hungary, in August 1997 (some of its lectures were published in a Special Issue of the International Journal of Imaging Systems and Technology [20]), and a workshop of Discrete Tomography and Related Problems has been scheduled for 1999 in Chateau de Volkrange, France.

In the rest of this introductory chapter we roughly follow the structure of the book. In the first section we discuss the theoretical results of discrete tomography related to three basic problems: consistency, uniqueness and reconstruction. After discussing general theoretical results, the special case of reconstructing two-dimensional discrete sets (i.e., binary matrices) from two projections is presented. The problems of consistency, uniqueness

and reconstruction are discussed again because the results are much more powerful than in the general case of more than two projections. The applications of DT is the topic of the last section.

Discrete tomography is a relative young and actively studied field. It is therefore inevitable that its terminology has not as yet settled down. This is reflected in our book: the same concepts are given different names by the authors of the various chapters and each chapter introduces its own specific notation. The editors decided that they should not interfere with this aspect of the chapters, not only because they were too lazy to do so, but also because this way the readers are introduced to the full range of terminology and notation in the DT literature. This introductory chapter will have its own definitional and notational quirks, but some (by no means exhaustive) hints will be given to alternatives used below in the book.

## 1.2 Foundations and algorithms

### *1.2.1 Definitions, notations, and basic problems*

In this subsection we are working in $d$-dimensional Euclidean space. In such a space a *lattice* is defined as the set of all linear combinations with integer coefficients of a fixed set of $d$ linearly independent vectors. Since any such lattice is isomorphic to the integer lattice $\mathbb{Z}^d$ under a nonsingular linear transformation, in DT it is enough to study the case of the integer lattice $\mathbb{Z}^d$. We will be doing this for the rest of this chapter without further comment.

The finite subsets of $\mathbb{Z}^d$ will be called *lattice sets* or *discrete sets*. The so-called *lattice directions* are represented by any nonzero vectors of $\mathbb{Z}^d$. We are going to use a set of distinct lattice directions, $D = (v^{(1)}, \ldots, v^{(q)})$, $q \geq 2$ (here distinct means that there are no two vectors in $D$ which are parallel to each other). We say that a line $\ell$ in $d$-dimensional Euclidean space is a *lattice line* if it is parallel to a vector $v^{(k)} \in D$ and passes through at least one point in $\mathbb{Z}^d$. Let $\mathcal{L}^{(k)}$ denote the set of all lattice lines that are parallel to $v^{(k)} \in D$.

**Definition 1.1.** *Let $F \subset \mathbb{Z}^d$ be a lattice set. Its projection in direction $v^{(k)}$ is defined as the function $\mathcal{P}_F^{(k)} : \mathcal{L}^{(k)} \to \mathbb{N}_0$ (the set of nonnegative integers) by*

$$\mathcal{P}_F^{(k)}(\ell) = |F \cap \ell| = \sum_{x \in \ell} f(x) \tag{1.1}$$

*where $f$ denotes the characteristic function of the discrete set $F$.*

Unfortunately, there is no uniform terminology. There are authors using the names X-ray, marginal or line sum for this concept. For example, the notion of lattice directions is closely related to the notion of fundamental

direction vectors in Chapter 3 by Kong and Herman in this book, but there is a subtle, but important, difference in the definition of a lattice line above and of a grid line in that chapter.

Let $D = (v^{(1)}, \ldots, v^{(q)})$ be a set of distinct lattice directions. We say that two discrete sets $F$ and $F'$ are *tomographically equivalent* with respect to the directions $D$ if $\mathcal{P}_F^{(k)} = \mathcal{P}_{F'}^{(k)}$ for $k = 1, \ldots, q$. Let $\mathcal{E}$ be a class of finite sets in $\mathbb{Z}^d$. In our terminology the discrete set $F \in \mathcal{E}$ is *determined* by the projections parallel to $D$ in the class $\mathcal{E}$ if there is no tomographically equivalent other set with respect to the directions $D$ in the class $\mathcal{E}$.

Let $\mathcal{E}$ be a class of finite sets in $\mathbb{Z}^d$ and $D$ be a set of directions in $\mathbb{Z}^d$. Let, furthermore, $\mathcal{L} = (\mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(q)})$ $(q \geq 2)$ be the collection of the sets of lattice lines determined by $D$. We now introduce the problems of consistency, uniqueness and reconstruction for $\mathcal{E}$ and $\mathcal{L}$.

CONSISTENCY$(\mathcal{E}, \mathcal{L})$.

Given: *For $k = 1, \ldots, q$, a function $p^{(k)} : \mathcal{L}^{(k)} \to \mathbb{N}_0$ with finite support.*

Question: *Does there exist an $F \in \mathcal{E}$ such that $\mathcal{P}_F^{(k)} = p^{(k)}$ for $k = 1, \ldots, q$?*

For a discussion of this problem see Chapter 4 by Gardner and Gritzmann in this book. A basic result there is that CONSISTENCY$(\mathcal{E}, \mathcal{L})$ is NP-complete for $q \geq 3$. In the case of $q = 2$, which is discussed in detail in Subsection 1.2.2, the problem of CONSISTENCY$(\mathcal{E}, \mathcal{L})$ can be solved in polynomial time.

UNIQUENESS$(\mathcal{E}, \mathcal{L})$.

Given: *An $F \in \mathcal{E}$.*

Question: *Does there exist a different $F' \in \mathcal{E}$ such that $F$ and $F'$ are tomographically equivalent with respect to the directions of $D$?*

Three chapters of this book discuss UNIQUENESS$(\mathcal{E}, \mathcal{L})$. As an introduction to the problem of uniqueness and its connection with computational complexity see Chapter 4 by Gardner and Gritzmann. Special aspects of uniqueness are discussed in Chapter 2 by Fishburn and Shepp. They show that the additivity of the lattice sets, which is necessary and sufficient for the uniqueness in the case of $q = 2$, is sufficient but not necessary if $q \geq 3$ (independently from the dimension of the lattice). Kong and Herman prove in Chapter 3 that in the case of $q \geq 3$ the uniqueness of a discrete set cannot be decided simply by finding certain patterns of 0's and 1's in the discrete space. As it is known from Ryser [8], the situation is just the opposite if $q = 2$, because then the existence of a certain type of $2 \times 2$ submatrix is equivalent to nonuniqueness. It is interesting that all of these chapters

express clearly that the uniqueness of discrete sets from two projections is basically different from the case when there are more than two projections.

RECONSTRUCTION$(\mathcal{E}, \mathcal{L})$.

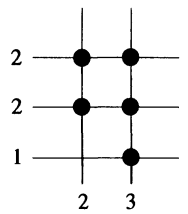
Given: *For $k = 1, \ldots, q$, a function $p^{(k)} : \mathcal{L}^{(k)} \to \mathbb{N}_0$ with finite support.*

Task: *Construct a finite set $F \in \mathcal{E}$ such that $\mathcal{P}_F^{(k)} = p^{(k)}$ for $k = 1, \ldots, q$.*

Suppose that there are given functions $p^{(k)} : \mathcal{L}^{(k)} \to \mathbb{N}_0$ having finite supports with cardinalities $m_k$ for $k = 1, \ldots, q$. Let $M = m_1 + \cdots + m_q$. It is clear that if $F$ is a finite set having projections $p^{(1)}, \ldots, p^{(q)}$ in the directions $v^{(1)}, \ldots, v^{(q)}$, respectively, then $F \subseteq G$, where $G$ consists of lattice points $z \in \mathbb{Z}^d$ for which $p^{(k)}(\ell) > 0$, where $\ell$ is the lattice line passing through $z$ in direction $v^{(k)}$, for each $k = 1, \ldots, q$. Because the function $p^{(k)}$ has finite support, $k = 1, \ldots, q$, $G$ is also finite, $|G| = N$ (say). Then the discrete reconstruction problem can be reformulated as the following linear feasibility problem:

$$\text{find} \quad Px = b, \quad \text{such that} \quad x \in \{0,1\}^N, \tag{1.2}$$

where $P \in \{0,1\}^{M \times N}$, $b \in \mathbb{N}_0^M$. The matrix $P$ describes the geometric relation between the points of $G$ and the lattice line $\ell$; that is, it specifies which points of $G$ are on a line $\ell$. Each equation in (1.2) corresponds to a line sum on a lattice line. The vector $x$ represents the set $G$. The nonnegative integral vector $b$ contains the values of the functions $p^{(k)}$, $k = 1, \ldots, q$ (see Fig. 1.1).




$$\begin{array}{llllllll} x_1 & +x_2 & & & & & = & 2 \\ & & x_3 & +x_4 & & & = & 2 \\ & & & & x_5 & +x_6 & = & 1 \\ x_1 & & +x_3 & & +x_5 & & = & 2 \\ & x_2 & & +x_4 & & +x_6 & = & 3 \end{array}$$

**FIGURE 1.1. A lattice set of $\mathbb{Z}^2$, its projections in the directions $(1,0)$ and $(0,1)$, and the corresponding linear equation system.**

In order to find a solution of the equation system in (1.2) (without the binary constrain $x \in \{0,1\}^N$) some of the iterative CT reconstruction algorithms (such as ART [21] or its versions [22]) can be used. Of course, they would give a not necessarily binary solution. However, they can be modified in order to get a binary solution. Such a binary ART, called BART, was suggested by Herman [23]. Recently this idea has been developed to a

binary steering schema (see Chapter 12 by Censor and Matej in this book) by which an iterative general CT reconstruction algorithm can be steered toward a binary solution. Yagle shows in Chapter 11 of this book how the equation system and also the integrality constraint of (1.2) can be included into a quadratic equation system and solved by algebraic methods.

Alternatively, one can transform (1.2) into the linear programming problem

$$\text{maximize} \quad |\sum_{j=1}^{N} x_j|, \quad \text{subject to} \quad Px = b \quad \text{and} \quad x \in \{0,1\}^N, \tag{1.3}$$

or, by relaxing the integrality constraint in (1.2) to the interval constraint $0 \leq x_j \leq 1$, for $j = 1, \ldots, N$, into

$$\text{maximize} \quad |\sum_{j=1}^{N} x_j|, \quad \text{subject to} \quad Px = b, \quad \text{and} \quad x \in [0,1]^N. \tag{1.4}$$

These linear programming problems can be solved by the simplex method or by using interior point methods [24].

Another reconstruction method can be deduced by the following approach [25]. After normalizing the equations in (1.2) the EM algorithm can be applied to minimize the nonlinear Kullback-Leibler information divergence criterion. For details, see Chapter 13 by Vardi and Zhang in this book.

If the number of projections is small, the system of linear equations in (1.2) is very undetermined and the number of solutions can be very large. In this case we can try to find only the solution(s) having some a priori property. This property can be given, for example, as a statistical property such as that the image to be reconstructed should have a high probability in a certain distribution. In Chapter 8 by Matej *et al.* in this book, a Metropolis algorithm is described for reconstructing discrete sets on a hexagonal grid from their three projections, when the probability of the occurrence of a discrete set follows some Gibbs distribution. The most probable solution is approached by iterative steps.

In DT we can reconstruct discrete sets of points where the points can have different weights; that is, the function to be reconstructed can have values other than 0 or 1. As it is shown in Chapter 6 by Kuba in this book, if the number of the possible values of these weights is two, then the binary reconstruction methods can be applied even if these two values are not known in advance (see also [26] about two-valued matrices). In the general case of multi-valued discrete images statistical methods, such as Bayesian maximum a posteriory (MAP) estimation, can be applied (see Chapter 10 by Frese, Bouman and Sauer in this book).

Another way of reconstructing discrete images is to use some CT method in the first step and then to postprocess the images using some assumed property. Such a method is described in Chapter 9 by Chan, Herman and Levitan in this book.

It is also possible to reconstruct parametrized objects such as polygons from a few (and noisy) projections. In this case the reconstruction can be considered as a parameter estimation problem, where the discrete set of the vertices of the polygon need to be recovered. The nonlinear problem can be solved by some optimization technique (e.g., maximum likelihood). Such a reconstruction method was suggested by Rossi and Willsky [27]. For the reconstruction of polyhedral shapes another method is presented in Chapter 14 by Muhammad-Djafari and Soussen in this book.

If the projections of the discrete object are defined not only on straight lines but on any path between the source and the detector, then we have the more complicated model of *diffuse tomography* [28]. A recursive algorithm for diffuse planar tomography is described in Chapter 20 by Patch in this book.

### *1.2.2 Reconstruction of binary matrices from two projections*

In this subsection we consider the classical special case of $m \times n$ binary matrices and two projections corresponding to row and column sums. We start with a discussion of the problem of consistency.

**Definition 1.2.** *Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be nonnegative integral vectors. The class of all binary matrices $A = (a_{ij})$ satisfying the equations*

$$\sum_{j=1}^{n} a_{ij} = r_i, \quad i = 1, \ldots, m, \tag{1.5}$$

$$\sum_{i=1}^{m} a_{ij} = s_j, \quad j = 1, \ldots, n, \tag{1.6}$$

*is denoted by $\mathfrak{A}(R, S)$. The vectors $R$ and $S$ are called the row and column sum vectors of any matrix $A \in \mathfrak{A}(R, S)$.*

**Definition 1.3.** *A pair $(R, S)$ of vectors is said to be compatible if there exist positive integers $m$ and $n$ such that*

*(i)* $R \in \mathbb{N}_0^m$ *and* $S \in \mathbb{N}_0^n$;

*(ii)* $r_i \leq n$, *for* $1 \leq i \leq m$, *and* $s_j \leq m$, *for* $1 \leq j \leq n$;

*(iii)* $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$.

Clearly, if $\mathfrak{A}(R,S)$ is not empty, then $(R,S)$ is compatible. In 1957 Ryser and Gale (independently of each other) gave a necessary and sufficient condition under which the class $\mathfrak{A}(R,S)$ is nonempty [8,9].

Consider the matrix $\bar{A}$ in which, for $i = 1,\ldots,m$, row $i$ consists of $r_i$ 1's followed by $n - r_i$ 0's. A matrix having this property will be called *maximal*. A maximal matrix $\bar{A}$ is uniquely determined by its row sum vector. Let its column sum vector be denoted by $\bar{S}$. Furthermore, let us denote the nonincreasing permutations of the elements of $R$ and $S$ by $R'$ and $S'$, respectively, that is, $r'_1 \geq r'_2 \geq \cdots \geq r'_m$ and $s'_1 \geq s'_2 \geq \cdots \geq s'_n$.

**Theorem 1.1.** *Let $R = (r_1,\ldots,r_m)$ and $S = (s_1,\ldots,s_n)$ be a pair of compatible vectors. The class $\mathfrak{A}(R,S)$ is nonempty if and only if*

$$\sum_{j=l}^{n} s'_j \geq \sum_{j=l}^{n} \bar{s}_j, \quad \text{for} \quad 2 \leq l \leq n. \tag{1.7}$$

**Proof:** Suppose that $\mathfrak{A}(R,S)$ contains a binary matrix $A$. Then the class $\mathfrak{A}(R,S')$ contains a binary matrix $A'$ constructed from $A$ by a suitable permutations of the columns. $\bar{A}$ can be obtained from $A'$ (if they are different at all) by shifting 1's to the left in the rows of $A'$. Therefore we have (1.7).

Now let us suppose that (1.7) is true for the vectors $R$ and $S$. We are going to construct a binary matrix $A$ by the following algorithm, whose output is illustrated in Fig. 1.2.

**Algorithm 1.1.**

*Input:* *a compatible pair of vectors $(R,S)$ satisfying (1.7);*

*Step 1.* *construct $S'$ from $S$ by permutation $\pi$;*

*Step 2.* *let $B = \bar{A}$ and $k = n$;*

*Step 3.* *while* $(k > 1)$,

{

*while* $(s'_k > \sum_{i=1}^{m} b_{ik})$,

{

*let* $j_0 = \max_{1 \leq i \leq m}\{j < k \mid b_{ij} = 1,\ b_{i,j+1} = \cdots = b_{ik} = 0\}$;

*let row $i_0$ be where such a $j_0$ was found;*

*set* $b_{i_0 j_0} = 0$ *and* $b_{i_0 k} = 1$ *(i.e., shift the 1 to the right)*

};

*reduce k by 1*

};

*Step 4. construct the matrix $A$ from $B$ by permutation $\pi^{-1}$ of the columns;*

*Output: matrix $A$.*

In order to prove that Algorithm 1.1 produces a matrix $A$ with row sum vector $R$ and column sum vector $S$, we first prove the following property of Step 3. Suppose that at the beginning of the execution of the code within the outer brackets of Step 3 the matrix $B$ can be written as $(C|D)$, where $C$ is a maximal $m \times k$ matrix and $D$ is an $m \times (n-k)$ matrix ($D$ is an empty matrix when $k = n$), such that

(i) the row sum vector of $B$ is $R$,

(ii) denoting the column sum vector of $B$ by $T = (t_1, \ldots, t_n)$ we have that

(a) $s'_j = t_j$, for $k+1 \leq j \leq n$,

(b) $\sum_{j=l}^{n} s'_j \geq \sum_{j=l}^{n} t_j$, for $2 \leq l \leq n$,

(c) $\sum_{j=1}^{k} s'_j = \sum_{j=1}^{k} t_j$.

Under these circumstances, at the end of the execution of the code within the outer brackets of Step 3 the matrix $B$ will have the same property with $k$ replaced by $k-1$.

We note that (i) above will certainly be satisfied, since the only type of change that is made to $B$ during Step 3 is the shifting of a 1 to the right (in other words, the interchanging in a row a 1 and a 0).

At the beginning of the execution of the code we have that (observe (a) and (b) above and the fact that $2 \leq k \leq n$)

$$\begin{aligned} s'_k &= \sum_{j=k}^{n} s'_j - \sum_{j=k+1}^{n} s'_j \\ &\geq \sum_{j=k}^{n} t_j - \sum_{j=k+1}^{n} t_j = t_k (= \sum_{i=1}^{m} b_{ik}). \end{aligned} \tag{1.8}$$

If $s'_k = t_k$, we just move the last column of $C$ to become the first column of $D$ and it is clear that (ii) will be satisfied with $k$ replaced by $k-1$.

Now suppose that $s'_k = t_k + d$, for some $d > 0$. We observe that there has to be at least $d$ values of $i$ ($1 \leq i \leq m$), such that $b_{ik} = 0$ and $b_{ij} = 1$ for some $j$, $1 \leq j < k$. This is because, by the maximality of $C$, the definition of $S'$ and (c) above we get

$$kt_1 \geq \sum_{j=1}^{k} t_j = \sum_{j=1}^{k} s'_j \geq k s'_k = k(t_k + d) \tag{1.9}$$

and so $t_1 \geq t_k + d$. This implies that the instructions within the inner while loop can indeed be executed $d$ times; each execution will set to 1 a single $b_{i_0 k}$, which is zero at the beginning of the execution of the while loop. Maximality of the matrix consisting of the first $k-1$ columns of $B$ is retained by the choice of the $j_0$ and $i_0$ in Step 3 of the algorithm. When the inner while loop is completed we clearly satisfy $s'_j = t_j$, for $k \leq j \leq n$ (i.e., (ii)(a) with $k-1$ in place of $k$).

Let $j_1$ be the maximal value of $j < k$ such that $t_j \geq t_k + d$ at the beginning of the execution of the code. From the way that the $j_0$ are selected in Step 3, it follows that there are $d$ values of $i$ such that at the beginning of the execution of the code $b_{ik} = 0$ and $b_{ij} = 1$ and at the end of the execution of the code $b_{ik} = 1$ and $b_{ij} = 0$ for some $j$, $j_1 \leq j < k$, and these are the only $2d$ locations in the matrix that change value. Hence, for $2 \leq l \leq j_1$, the value of $\sum_{j=l}^{n} t_j$ does not change as a result of the execution of the code and the validity of (ii)(b) is preserved. On the other hand, at the end of the execution of the code, $s'_j \geq t_j$, for $j_1 < j \leq n$. For $k \leq j \leq n$, this follows from the already shown fact that (ii)(a) holds at the end of the execution of the code with $k-1$ in place of $k$. For $j_1 < j < k$, it follows from the definitions of $S'$ and $j_1$ that at the beginning of the execution of the code $s'_j \geq s'_k = t_k + d > t_j$ and the value of $t_j$ is not increased during the execution of the code. Hence (ii)(b) holds at the end of the execution of the code also for the alternate range $j_1 < l \leq n$.

Trivially, for (ii)(c), the left-hand side is reduced by $s'_k$ (since $k$ is to be replaced by $k-1$) and the right-hand side is reduced by the original $t_k$ (for the same reason) but also by $d$ (corresponding to the $d$ 1's that have been shifted from a column $j < k$ into the $k$th column during the execution of the code). The preservation of (ii)(c) follows, since $s'_k = t_k + d$ at the beginning of the execution of the code.

Given this behavior of Step 3 it is now easy to complete the proof. First we note that the conditions assumed upon entering the code within the outer brackets of Step 3 are satisfied for the initial entry when $k = n$ with $C = B = \bar{A}$ and $D$ the empty matrix. In this case (ii)(b) is just (1.7) and (ii)(c) follows from the compatibility of $R$ and $S$. Repeated applications of the code will bring us to the end of execution for the case $k = 2$, at which time will have $s'_j = t_j$ for $1 \leq j \leq n$ (by (ii)(a) and (ii)(c) with $k = 1$), proving that at this time $B$ has row sum vector $R$ (by (i)) and column sum vector $S'$ and, hence, $A$ has row sum vector $R$ and column sum vector $S$. □

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 0 | 1 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 1 |
| 3 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| | 4 | 2 | 1 | 1 | 1 |

**FIGURE 1.2. The binary matrix reconstructed from row and column sums $R = (2,3,3,1)$ and $S = (4,2,1,1,1)$ by Algorithm 1.1.**

**Remark 1.1.**

Note that while the statement of Theorem 1.1 is such that its provides an answer to the consistency problem (for binary matrices from two special projections), the proof of the theorem contains Algorithm 1.1, which provides an answer to the reconstruction problem.

1. The computational complexity of Algorithm 1.1 is $O(nm+n\log n) = O(n\cdot(m+\log n))$. Step 3 can be implemented efficiently if the column indices of the last 1's of each row of $C$ are stored and made use of during the execution.

2. There are several versions of Algorithm 1.1 (see, for example, [8,9,29]) depending on which row $i_0$ is selected (when there are two or more rows available). Further alternatives can be found in [30–33].

3. The determination of the precise number of matrices in $\mathfrak{A}(R,S)$ is an open problem. Only lower bounds are known, see [34–38].

**Definition 1.4.** *For all index sets $I \subseteq \{1,\ldots,m\}$ and $J \subseteq \{1,\ldots,n\}$, let*

$$t(I,J) = |I|\cdot|J| + \sum_{i\notin I} r_i - \sum_{j\in J} s_j. \tag{1.10}$$

With the help of this function Ford and Fulkerson [11] obtained another necessary and sufficient condition for the nonemptyness of the class $\mathfrak{A}(R,S)$.

**Theorem 1.2.** *Let $R=(r_1,\ldots,r_m)$ and $S=(s_1,\ldots,s_n)$ be a pair of compatible vectors. The class $\mathfrak{A}(R,S)$ is nonempty if and only if*

$$t(I,J) \geq 0 \tag{1.11}$$

*for all $I \subseteq \{1,\ldots,m\}$ and $J \subseteq \{1,\ldots,n\}$.*

**Proof:** (Based on [39].) Suppose that there is a binary matrix $A \in \mathfrak{A}(R,S)$. Let $A[I,J]$ denote the submatrix of $A$ obtained from the rows and columns indexed by $I$ and $J$, respectively. Let, furthermore, $\bar{I} = \{1,\ldots,m\}\setminus I$ and $\bar{J} = \{1,\ldots,n\}\setminus J$. For any binary matrix $X$, let $\sigma_0(X)$ and $\sigma_1(X)$ denote the number of 0's and 1's in $X$, respectively. Then it is easy to see that, for any $I \subseteq \{1,\ldots,m\}$ and $J \subseteq \{1,\ldots,n\}$,

$$t(I,J) = \sigma_0(A[I,J]) + \sigma_1(A[\bar{I},\bar{J}]) \geq 0. \tag{1.12}$$

Now suppose that (1.11) holds. For $1 \leq k \leq n$, let $J$ be a column index set such that $|J| = k$ and let $I = \{i \mid r_i \geq k\}$. Then

$$\sum_{j\in J} s_j \leq |I|\cdot|J| + \sum_{i\notin I} r_i = |I|\cdot k + \sum_{i\notin I} r_i$$
$$= \sum_{i=1}^{m} \min\{r_i, k\} = \sum_{j=1}^{k} \bar{s}_j \tag{1.13}$$

and so, specifically,

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \bar{s}_j. \tag{1.14}$$

It follows from Theorem 1.1 that $\mathfrak{A}(R,S)$ is not empty. □

We now consider binary matrices when not only the row and column sum vectors are given but also certain elements of the matrix are prescribed to be 0 or 1. Fulkerson [40] gave a condition for the existence of a binary matrix having zero trace (i.e., $\sum_{i=1}^{n} a_{ii} = 0$). Anstee published results [41–43] connected with the reconstruction and existence of binary matrices with at most one prescribed 1 or 0 in each row/column. Here we consider the general problem.

**Definition 1.5.** *Let $Q$ and $P$ be binary matrices of size $m \times n$. We say that $Q$ covers $P$ if $p_{ij} \leq q_{ij}$ for $i = 1,\ldots,m$, $j = 1,\ldots,n$. We denote this relation as $P \leq Q$. We define*

$$\mathfrak{A}_P^Q(R,S) = \{A \mid P \leq A \leq Q,\ A \in \mathfrak{A}(R,S)\}. \tag{1.15}$$

Trivially, $\mathfrak{A}_P^Q(R,S) = \mathfrak{A}(R,S)$ if $P = (0)_{m\times n}$ and $Q = (1)_{m\times n}$. Using suitable binary matrices $P$ and $Q$ we can prescribe binary values to any position. The value 0 is prescribed to $(i,j)$ if $q_{ij} = 0$ and 1 is prescribed to $(i,j)$ if $p_{ij} = 1$.

If $A \in \mathfrak{A}_P^Q(R,S)$ then

$$A - P \in \mathfrak{A}_O^{Q-P}(R - R(P), S - S(P)) \tag{1.16}$$

where $O = (0)_{m\times n}$, and $R(P)$ and $S(P)$ denote the row and column sum vector of $P$, respectively. It follows that there is no loss of generality if we restrict our studies to the classes $\mathfrak{A}^Q(R,S) = \mathfrak{A}_O^Q(R,S)$, in which the position $(i,j)$ can be classified as *prescribed* (if $q_{ij} = 0$) or *free* (if $q_{ij} = 1$).

**Theorem 1.3.** *Let $R = (r_1,\ldots,r_m)$ and $S = (s_1,\ldots,s_n)$ be a pair of compatible vectors. The class $\mathfrak{A}^Q(R,S) \neq \emptyset$ if and only if*

$$\sum_{i\in I}\sum_{j\in J} q_{ij} \geq \max_{I,J}\{\sum_{i\in I} r_i - \sum_{j\notin J} s_j,\ \sum_{j\in J} s_j - \sum_{i\notin I} r_i\} \tag{1.17}$$

*for all* $I \subseteq \{1,\ldots,m\}$ *and* $J \subseteq \{1,\ldots,n\}$.

Theorem 1.3 can be deduced as a special case of two more general theorems by Kellerer ([5] on reconstruction of functions) and Mirsky ([44] on reconstruction of integral matrices).

We now consider the problem of uniqueness for the classical case of binary matrices and two projections.

**Definition 1.6.** *We say that a binary matrix* $A$ *is nonunique (with respect to its row and column sums) if there is a binary matrix* $B \neq A$ *having the same row and column sums as* $A$. *Otherwise,* $A$ *is unique.*

**Definition 1.7.** *A switching component of a binary matrix* $A$ *is a* $2 \times 2$ *submatrix of either of the following two forms:*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.18}$$

*A switching (operation) is a transformation of the elements of* $A$ *that changes a submatrix of type* $A_1$ *into type* $A_2$ *or vice versa (and leaves all other elements of* $A$ *unaltered).*

Ryser used the name *interchange* [8] for this transformation. In fact, it is called by many different names in the literature; for example, it is referred to as a rectangular 4-switch in Chapter 3 by Kong and Herman in this book. Clearly, switching does not modify the row and column sums. Accordingly, if $A$ has a switching component then it is nonunique. The reverse statement is also true. In fact the material in Section 3.1 provides us not only with the the following necessary and sufficient condition for uniqueness, but also with a proof of Ryser's Theorem [8].

**Theorem 1.4.** *A binary matrix is nonunique (with respect to its row and column sums) if and only if it has a switching component.*

**Theorem 1.5.** *(Ryser's Theorem) If* $A$ *and* $B$ *are two binary matrices in* $\mathfrak{A}(R,S)$ *then* $A$ *is transformable into* $B$ *by a finite number of switchings (using switching components).*

Now consider the special case of Theorem 1.1, in which (1.7) is replaced by

$$\sum_{j=l}^{n} s'_j = \sum_{j=l}^{n} \bar{s}_j, \quad \text{for} \quad 2 \leq l \leq n. \tag{1.19}$$

In view of the compatibility of $(R,S)$ (1.19) is equivalent to $S' = \bar{S}$. Consequently, no changes are made to the matrix $B$ during the execution of Step

3 in Algorithm 1.1 and so the $B$ of Step 4 is the same as the $B$ of Step 2; namely, $\bar{A}$. Since the 1's of the maximal $\bar{A}$ are in the leftmost positions in their rows, $\bar{A}$ has no switching component. It follows that the $A$ obtained by a permutation of columns of $\bar{A}$ also cannot have a switching component and is consequently (by Theorem 1.4) unique with respect to its row and column sums.

We have just proven that if $(R, S)$ satisfies the conditions of Theorem 1.1 with (1.7) replaced by (1.19), then there is a unique matrix with respect to $R$ and $S$. Conversely, if the binary matrix $A$ is unique then it has no switching component. Consider two columns of $A$, say $j_1$ and $j_2$. Let us suppose that $s_{j_1} \leq s_{j_2}$. If $a_{ij_1} = 1$ for some $i$ then $a_{ij_2} = 1$. (Otherwise, if $a_{ij_1} = 1$ and $a_{ij_2} = 0$, there is at least one row $i'$ such that $a_{i'j_1} = 0$ and $a_{i'j_2} = 1$, which contradicts to the assumption that $A$ has no switching component.) In other words, the 1's in column $j_1$ are in the rows in which there is also a 1 in column $j_2$. This means that if the columns of $A$ are permuted nonincreasingly then we get just the maximal matrix $\bar{A}$. Therefore, $S' = \bar{S}$ and so (1.19) is true.

From the discussion of the previous two paragraphs it follows that if $A$ is a unique binary matrix, then it can be recovered from its row sum $R$ and column sum $S$ using Algorithm 1.1 without Step 3. That is, we can construct the maximal matrix $\bar{A}$ from $R$ and then recover $A$ by a permutation $\pi^{-1}$ at the columns of $\bar{A}$ (where $\pi$ is the permutation that produces $S'$ from $S$). This observation leads us to a remarkable property of unique binary matrices, which is a consequence of the following easily proved result on maximal matrices.

**Lemma 1.1.** *If $A$ is a maximal binary matrix, then*

$$a_{ij} = 1 \Longleftrightarrow s_j \geq |\{k \mid r_k \geq r_i\}|. \tag{1.20}$$

Now observe that for a unique binary matrix $A$ in $\mathfrak{A}(R, S)$, the maximal matrix $\bar{A}$ is the unique element of $\mathfrak{A}(R, \bar{S}) = \mathfrak{A}(R, S')$. Since $A$ is obtained from $\bar{A}$ by the permutation $\pi^{-1}$ of the columns, it follows that for a unique binary matrix $A$ in $\mathfrak{A}(R, S)$, (1.20) holds (even if $A$ is not maximal). This leads to yet another characterization of uniqueness.

**Definition 1.8.** *An $m \times n$ binary matrix $A = [a_{ij}]$ is additive if there are vectors $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $a_{ij} = 1$ if and only if $x_i + y_j \geq 0$.*

**Theorem 1.6.** *A binary matrix is unique if and only if it is additive.*

**Proof:** Let $A$ be an $m \times n$ binary matrix with row and column sum vectors $R$ and $S$. If $A$ is unique, then it satisfies (1.20). This implies that $A$ is additive with respect to the vectors $x_i = -|\{k \mid r_k \geq r_i\}|$, $i = 1, \ldots, m$, and $y_j = s_j$,

$j = 1, \ldots, n$.

Now let us suppose that $A$ is additive with respect to the vectors $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$. Let $B \in \mathfrak{A}(R, S)$. Consider the function

$$\mathcal{K}(A, B) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j)(a_{ij} - b_{ij}). \tag{1.21}$$

From Definition 1.8 we see that each term in the sum on the right hand side of (1.21) is nonnegative. Furthermore,

$$\begin{aligned} \mathcal{K}(A, B) &= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} (a_{ij} - b_{ij}) + \sum_{j=1}^{n} y_j \sum_{i=1}^{m} (a_{ij} - b_{ij}) \\ &= \sum_{i=1}^{m} x_i (r_i - r_i) + \sum_{j=1}^{n} y_j (s_j - s_j) = 0. \end{aligned} \tag{1.22}$$

This implies that each term in the sum on the right-hand side of (1.21) is in fact zero. Together with Definition 1.8 this implies that if $a_{ij} = 0$ then $b_{ij} = 0$. However, $A$ and $B$ have the same number of 0's, therefore $A = B$. That is, $A$ is unique. □

As a summary of uniqueness for a nonempty class $\mathfrak{A}(R, S)$, we have the following.

**Theorem 1.7.** *Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be vectors of nonnegative integers such that there is a binary matrix $A \in \mathfrak{A}(R, S)$. The following conditions are equivalent:*

*(1) $A$ is unique with respect to $R$ and $S$;*

*(2) $A$ has no switching component;*

*(3) (1.19) is satisfied;*

*(4) $A$ is additive.*

Condition (2) was found by Ryser [8]. It seems that (1.19) as a necessary and sufficient condition on uniqueness was first published by Wang [33]. The condition given by (1.20) can be considered to be a discrete version of that given in [45]. Additivity was introduced by Fishburn *et al.* [46] in a more general way. Further combinatorial results about the class $\mathfrak{A}(R, S)$ are in [47,48]. The cardinality of the class $\mathfrak{A}_P^Q(R, S)$ is discussed in [35,38].

For the class $\mathfrak{A}(R, S)$ switching components are used to decide the question of uniqueness. For the class $\mathfrak{A}^Q(R, S)$ *switching chains* (a generalization of switching components) can be used for the same purpose.

**Definition 1.9.** *Let $A \in \mathfrak{A}^Q(R,S)$. A switching chain is a finite sequence $(i_1,j_1),(i_1,j_2),(i_2,j_2),(i_2,j_3),\ldots,(i_k,j_k),(i_k,j_1)$ of free positions of the matrix $A$ such that*

$$\begin{aligned} a_{i_1j_1} = a_{i_2j_2} = \quad \cdots \quad &= a_{i_kj_k} = \\ 1 - a_{i_1j_2} = 1 - a_{i_2j_3} = \quad \cdots \quad &= 1 - a_{i_{k-1}j_k} = 1 - a_{i_kj_1} \end{aligned} \tag{1.23}$$

*($k \geq 2$). The corresponding switching (operation) is defined as changing the 0's and the 1's at all positions in the switching chain.*

It is clear that switching in a matrix $A$ does not change the row and column sums of $A$. As an example, see the following matrices generated from each other by switching (the prescribed elements are denoted by $x$).

$$\begin{pmatrix} 1 & x & 0 \\ x & 0 & 1 \\ 0 & 1 & x \end{pmatrix} \qquad \begin{pmatrix} 0 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & x \end{pmatrix} \tag{1.24}$$

The following theorems can be proven in a similar way as for the class $\mathfrak{A}(R,S)$ (see [49]).

**Theorem 1.8.** *A binary matrix with prescribed values is nonunique if and only if it has a switching chain.*

**Theorem 1.9.** *Let the binary matrices $A, B \in \mathfrak{A}^Q(R,S)$. Then $A$ is transformable into $B$ by a finite number of switchings (using switching chains).*

We now consider the possible values of a matrix element $a_{ij}$ in the class $\mathfrak{A}(R,S)$. The positions can be classified into one of three sets as follows.

**Definition 1.10.** *Let $\mathfrak{A} = \mathfrak{A}(R,S)$ be a nonempty class. The position $(i,j)$ is variant if there are matrices $A, B \in \mathfrak{A}$ such that $a_{ij} = 1 - b_{ij}$. A position $(i,j)$ is an invariant 0 or an invariant 1 if $a_{ij} = 0$ or $a_{ij} = 1$ for all $A \in \mathfrak{A}$, respectively. The sets of variant, invariant 0 and invariant 1 positions of the class $\mathfrak{A}$ are denoted by $V(\mathfrak{A})$, $I^{(0)}(\mathfrak{A})$ and $I^{(1)}(\mathfrak{A})$, respectively, and $(V(\mathfrak{A}), I^{(0)}(\mathfrak{A}), I^{(1)}(\mathfrak{A}))$ is called the structure of the class $\mathfrak{A}$.*

For example, the positions of a switching component of a binary matrix $A \in \mathfrak{A}$ are in $V(\mathfrak{A})$. The structure of $\mathfrak{A}(R,S)$ shows which part of the discrete space is ambiguous/unambiguous with respect to the row and column sums. Without loss of generality we present structure results for classes having nonincreasing row and column sum vectors.

**Definition 1.11.** *A class $\mathfrak{A}' = \mathfrak{A}'(R',S')$ is called normalized if the elements of the vectors $R' = (r'_1,\ldots,r'_m)$ and $S' = (s'_1,\ldots,s'_n)$ are ordered as $r'_1 \geq r'_2 \geq$*

$\cdots \geq r'_m$ *and* $s'_1 \geq s'_2 \geq \cdots \geq s'_n$.

*In a normalized class* $\mathfrak{A}' = \mathfrak{A}'(R', S')$ *we define the* $(m+1) \times (n+1)$ *structure matrix [10]* $T = (t_{kl})$ *by*

$$\begin{aligned} t_{kl} &= \min\{t(I,J) \mid |I| = k,\ |J| = l\} \\ &= k \times l + \sum_{i>k} r'_i - \sum_{j \leq l} s'_j \end{aligned} \tag{1.25}$$

*for all* $k = 0, 1, \ldots, m$ *and* $l = 0, 1, \ldots, n$.

Rephrasing Theorem 1.2 we can say that a normalized class $\mathfrak{A}'$ is not empty if and only if the structure matrix $T$ of $\mathfrak{A}'$ has no negative elements. If $A \in \mathfrak{A}'$, then from (1.12) we have that

$$\begin{aligned} t_{kl} &= \sigma_0(A[\{1, \ldots, k\}, \{1, \ldots, l\}]) + \\ &\quad \sigma_1(A[\{k+1, \ldots, m\}, \{l+1, \ldots, n\}]) \end{aligned} \tag{1.26}$$

for any $k = 0, 1, \ldots, m$, $l = 0, 1, \ldots, n$. It follows that if $t_{kl} = 0$ then

$$\begin{aligned} \{1, \ldots, k\} \times \{1, \ldots, l\} &\subseteq I^{(1)}(\mathfrak{A}'), \\ \{k+1, \ldots, m\} \times \{l+1, \ldots, n\} &\subseteq I^{(0)}(\mathfrak{A}'). \end{aligned} \tag{1.27}$$

We are going to show that the invariant 1 and invariant 0 sets are unions of such discrete rectangles.

**Lemma 1.2.** *Let* $\mathfrak{A}' = \mathfrak{A}(R', S')$ *be a normalized class.*

*(i) If there is a matrix* $A$ *in the class* $\mathfrak{A}'$ *such that* $a_{ij} = 0$ *and* $a_{ij'} = 1$ *for some* $1 \leq i \leq m$ *and* $1 \leq j < j' \leq n$, *then both* $(i,j)$ *and* $(i,j')$ *are variant positions of a switching component in* $A$.

*(ii) If there is a matrix* $A$ *in the class* $\mathfrak{A}'$ *such that* $a_{ij} = 0$ *and* $a_{i'j} = 1$ *for some* $1 \leq i < i' \leq m$ *and* $1 \leq j \leq n$, *then both* $(i,j)$ *and* $(i',j)$ *are variant positions of a switching component in* $A$.

**Proof:** Statement (i) is true because $s'_j \geq s'_{j'}$ in a normalized class and therefore there is a row $i'$ such that $a_{i'j} = 1$ and $a_{i'j'} = 0$. Part (ii) can be proven analogously. □

Now let $i$ be any row that contains a variant position and let $j$ and $j'$ be the column indices of the leftmost and rightmost variant positions in row $i$. It follows that $(i,k)$ is a variant position for $j \leq k \leq j'$. For suppose otherwise. If $(i,k)$ is an invariant 1 position then for some matrix $A$ in the

class $a_{ij} = 0$ and $a_{ik} = 1$, which implies that $(i, k)$ is a variant position (by Lemma 1.2 (i)). A similar argument holds if $(i, k)$ is an invariant 0 position. Therefore, the variant positions follow each other in the rows (analogously, also in columns) consecutively.

Furthermore, also on the base of Lemma 1.2 (i), it is easy to show that in the rows of the normalized class the invariant 1, variant and invariant 0 positions (if any) follow in this order from left to right. Similarly, on the base of Lemma 1.2 (ii), it can be proven that in the columns of the normalized class the invariant 1, variant and invariant 0 positions (if any) follow in this order from top to bottom.

In the proof of the following lemma we will make repeated use of the following trivial consequence of Ryser's Theorem (Theorem 1.5). Let $\mathfrak{A} = \mathfrak{A}(R, S)$. If $(i, j) \in V(\mathfrak{A})$, then for any matrix $A \in \mathfrak{A}$, there is a row $i'$ and column $j'$ such that $(i', j) \in V(\mathfrak{A})$, $(i, j') \in V(\mathfrak{A})$ and $a_{ij} = 1 - a_{i'j} = 1 - a_{ij'}$.

**Lemma 1.3.** *Let $\mathfrak{A}' = \mathfrak{A}(R', S')$ be a normalized class, let $i_1$ and $i_2$ be two rows which contain variant positions such that $1 \leq i_1 < i_2 \leq m$ and let $[j_1, j_1']$ and $[j_2, j_2']$ be the corresponding ranges of the (consecutive) variant positions. Then*

$$\text{either} \quad j_1 > j_2' \quad \text{or} \quad \text{both} \quad j_1 = j_2 \quad \text{and} \quad j_1' = j_2'. \tag{1.28}$$

**Proof:** We note that $j_1 < j_2$ is impossible, for then we would have the invariant 1 position $(i_2, j_1)$ below the variant position $(i_1, j_1)$. We complete the proof by showing that the assumption $j_2 < j_1$ implies that $j_1 > j_2'$. (One can prove similarly that $j_1' \neq j_2'$ also implies that $j_1 > j_2'$.)

Since $(i_1, j_1)$ is a variant position, there is an $A \in \mathfrak{A}'$ for which

$$a_{i_1 j_1} = 0. \tag{1.29}$$

Consider the first $j_1 - 1$ columns of $A$. If $A[\{1, \ldots, m\}, \{1, \ldots, j_1 - 1\}]$ contains only 1's then all positions of $\{1, \ldots, m\} \times \{1, \ldots, j_1 - 1\}$ are invariant 1's (because $t_{m, j_1 - 1} = 0$ and (1.27)), which contradicts the assumption that $j_2 < j_1$. Therefore, there is a 0 somewhere in the first $j_1 - 1$ columns of $A$. Let $i$ be the index of the uppermost row containing a 0 in the first $j_1 - 1$ columns of $A$. Then

$$\sigma_0(A[\{1, \ldots, i-1\}, \{1, \ldots, j_1 - 1\}]) = 0. \tag{1.30}$$

Furthermore, $i > i_1$, because there are only invariant 1's to the left of $j_1$, and so there can be only invariant 1's above $(i_1, 1), \ldots, (i_1, j_1 - 1)$.

Let $j$ be a column such that

$$a_{ij} = 0. \tag{1.31}$$

Clearly

$$a_{i_1 j} = 1. \tag{1.32}$$

Then $a_{ij_1} = 0$, for otherwise $(i_1, j), (i_1, j_1), (i, j)$, and $(i, j_1)$ would constitute a switching component (see (1.29), (1.31), and (1.32)). Furthermore, if $a_{il} = 1$ for some $l \in (j_1, \ldots, n]$, then on the basis of Lemma 1.2 (i) we can perform a switching on elements $a_{ij_1} = 0$ and $a_{il} = 1$ (and the other two in these columns) getting a new binary matrix $B \in \mathfrak{A}'$ (without altering $a_{i_1 j}$ and $a_{i_1 j_1}$). But in this case $b_{i_1 j} = 1$, $b_{i_1 j_1} = 0$, $b_{ij} = 0$, and $b_{ij_1} = 1$, so these four elements constitute a switching component, which is in contradiction with $(i_1, j)$ being an invariant 1 position. Therefore, $a_{il} = 0$ for $l = j_1, \ldots, n$. Similarly, $a_{kj_1} = 0$ for $k = i, \ldots, m$ (on the basis of Lemma 1.2 (ii)).

Furthermore, for the same reason, all elements of $A$ in the rectangle $\{i, \ldots, m\} \times \{j_1, \ldots, n\}$ are 0, that is

$$\sigma_1(A[\{i, \ldots, m\}, \{j_1, \ldots, n\}]) = 0. \tag{1.33}$$

Summarizing (1.30) and (1.33), we get $t_{i-1, j_1 - 1} = 0$. This implies that the rectangles $\{1, \ldots, i-1\} \times \{1, \ldots, j_1 - 1\}$ and $\{i, \ldots, m\} \times \{j_1, \ldots, n\}$ contain only invariant 1 and invariant 0 positions, respectively. By the assumption that $j_2 < j_1$, this implies that $i_2 \geq i$. That in turn implies that $j_1 > j_2'$, as needed to complete the proof. □

**Theorem 1.10.** *The variant set of a normalized class $\mathfrak{A}' = \mathfrak{A}(R', S') \neq \emptyset$, can always be written as*

$$V(\mathfrak{A}') = \bigcup_{q=1}^{p} I_q \times J_q \tag{1.34}$$

*($p = 0$ if there are no variant elements), where*

$$I_q = \{i_q, \ldots, i_q'\}, \quad 1 \leq i_1 < i_1' < i_2 < i_2' < \cdots < i_p < i_p' \leq m, \tag{1.35}$$

$$J_q = \{j_q, \ldots, j_q'\}, \quad 1 \leq j_p < j_p' < j_{p-1} < j_{p-1}' < \cdots < j_1 < j_1' \leq n. \tag{1.36}$$

**Proof:** We define $i_q$, $i_q'$, $j_q$, $j_q'$ inductively as follows. We set $i_0' = 0$. Assume that $i_q'$ is already defined. If there is no $i > i_q'$ such that row $i$ contains a variant position, then $p = q$ and we are done. Otherwise, let $i_{q+1}$ be the smallest such $i$ and $[j_{q+1}, j_{q+1}']$ be the associated range of the consecutive variant positions. (Note that $j_{q+1} < j_{q+1}'$ due to the consequence of Ryser's Theorem as stated above in Lemma 1.3.) Let $i_{q+1}'$ be the maximal value of $i$ such that the associated range of consecutive variant positions is still $[j_{q+1}, j_{q+1}']$. (By the same reasoning involving Ryser's Theorem $i_{q+1} < i_{q+1}'$. Also, for all rows $i$ such that $i_{q+1} \leq i \leq i_{q+1}'$, we must have that the range of variant positions is $[j_{q+1}, j_{q+1}']$.) By the properties stated above and in Lemma 1.3, we must also have that $j_{q+1}' < j_q$. □

**Remark 1.2.** *Since the rows/columns of the variant rectangles are between the invariant 1's and 0's, the invariant 1 and invariant 0 sets of the class $\mathfrak{A}'$ are the unions of rectangles of the form (1.27).*

We illustrate the material on the structure of a class in Fig. 1.3 for $R = (4,4,10,13,11,7,10,9,1)$ and $S = (1,7,4,7,9,8,5,7,7,6,1,3,4)$. On the left of the figure is an element of $\mathfrak{A}(R',S')$ with the variant positions shaded. It is trivial to check that the shaded positions are indeed variant (all values within either shaded box can be altered by switchings using switching components entirely within that box). The question arises, how do we know that none of the unshaded positions are variant?

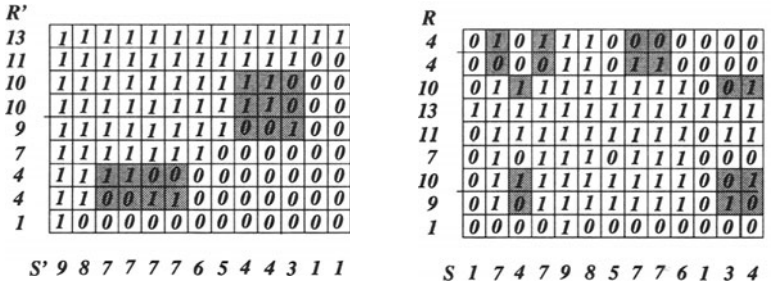



**FIGURE 1.3. Illustration of the structure of the class $\mathfrak{A}$. Right: The row and column sum vectors, $R$ and $S$, and the structure of the class $\mathfrak{A}(R,S)$. The variant positions are shaded. Left: The structure of the normalized class $\mathfrak{A}(R',S')$.**

This is, in fact, a consequence of some remarkable general results in [50]. Here we state without proof the applicable special case of Theorem 2.6 (i) of [50]. As it will be easily noted, this result is close in spirit to Theorem 1.6. (In fact, Theorem 1.6 is derived in [50] as a corollary of the results presented there.) Another closely related theorem is stated and proved in Chapter 8 by Matej, Vardi, Herman and Vardi.

**Definition 1.12.** *Let $A = [a_{ij}]$ be an $m \times n$ binary matrix. A pair of vectors $X = (x_1,\ldots,x_m) \in \mathbb{R}^m$ and $Y = (y_1,\ldots,y_n) \in \mathbb{R}^n$ is said to be compatible with $A$ if, for $i = 1,\ldots,m$ and $j = 1,\ldots,n$,*

$$x_i + y_j = \begin{cases} \geq 0, & \text{if } a_{ij} = 1, \\ \leq 0, & \text{if } a_{ij} = 0. \end{cases} \tag{1.37}$$

**Theorem 1.11.** *Let $R = (r_1,\ldots,r_m)$ and $S = (s_1,\ldots,s_n)$ be vectors of nonnegative integers such that there is a binary matrix $A \in \mathfrak{A}(R,S)$. Then, for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, $(i,j)$ is not a variant position if and only if there exists a pair of vectors $X = (x_1,\ldots,x_m)$ and $Y = (y_1,\ldots,y_n)$ which is compatible with $A$ such that $x_i + y_j \neq 0$.*

Let $X = (11,9,8,8,8,8,7,5,4,4,4,2,2)$ and $Y = (-1,-3,-4,-4,-4,-6,-8,-8,-10)$. Then it is easy to check that this pair of vectors is compatible with the matrix on the left of Fig 1.3 and that, for all the unshaded

positions $(i,j)$, $x_i + y_j \neq 0$. Theorem 1.11 therefore implies that the unshaded positions are in fact not variant positions. Once we have determined the structure of the normalized class $\mathfrak{A}(R',S')$ we can, by performing the reverse permutation of the rows and columns, obtain the structure of the original class $\mathfrak{A}(R,S)$; this is illustrated on the right of Fig 1.3.

The structure of the class of binary matrices was studied first by Ryser [51,52] and Haber [53]. Our summary is based on [54]. We remark that the structure of the class $\mathfrak{A}^Q(R,S)$ is similar to the structure of $\mathfrak{A}(R,S)$ as it was shown in [50]. The same paper describes an algorithm for producing the structure of a nonempty class based on a sample matrix from it; see also [55]. (Such algorithms can be used for finding the vectors $X$ and $Y$, which we seemed to have pulled out of a hat in the previous paragraph.)

We complete this subsection by considering the problem of reconstruction for the classical case of binary matrices and two projections. As we saw, for a nonempty class $\mathfrak{A}(R,S)$, Algorithm 1.1 constructs a solution in polynomial time.

If we happen to know that there is a unique binary matrix $A$ in $\mathfrak{A}(R,S)$, then we also know that this matrix must satisfy (1.20). This knowledge leads us to the following reconstruction algorithm, which in practice is likely to be much better than Algorithm 1.1. Its output is illustrated in Fig. 1.4. (Algorithms for reconstructing unique binary matrices were described also in [31,56,57].)

**Algorithm 1.2.**

*Input:* *a compatible pair of vectors,* $(R,S)$, *satisfying (1.19);*

*Step 1.* $A = O$; *(zero matrix)*

*Step 2.* *find* $i_1, i_2, \ldots, i_m$ *such that* $r_{i_1} \geq r_{i_2} \geq \cdots \geq r_{i_m}$;

*Step 3.* *for* $j = 1$ *to* $n$,
*for* $k = 1$ *to* $s_j$,
$a_{i_k j} = 1$;

*Output: matrix* $A$.

In certain cases the number of binary matrices having given row and column sums can be very high. For example, if $R = S = (1,\ldots,1)$ $(m = n)$ then $|\mathfrak{A}(R,S)| = n!$. For such reasons it is interesting to study the problem of reconstructing special binary matrices having some special property. Using such a property during the reconstruction, we can hope to reduce the number of possible solutions.

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 1 & 0 \\ 5 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ & & & & & \\ & 1 & 3 & 2 & 4 & 1 \end{array}$$

**FIGURE 1.4. Reconstruction of a unique binary matrix.**

The most frequently used properties are geometrical. For example, let us suppose that the 1's of the binary matrix to be reconstructed follow each other consecutively in the rows and columns.

**Definition 1.13.** *A binary matrix is h-convex (respectively, v-convex) if in the rows (respectively, columns) the 1's follow each other consecutively. If a binary matrix is both h- and v-convex then it is hv-convex (see Fig. 1.5). The class of h-convex, v-convex, and hv-convex binary matrices will be denoted by* $(h)$, $(v)$, *and* $(hv)$, *respectively.*

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**FIGURE 1.5. Examples of $h$-convex, $v$-convex, and $hv$-convex binary matrices.**

Kuba published an algorithm [58] for reconstructing $hv$-convex binary matrices from their row and column sums.

**Definition 1.14.** *The neighbors of the position* $(i,j)$ *are the positions* $(i-1,j)$, $(i,j-1)$, $(i,j+1)$, *and* $(i+1,j+1)$. *We say that two positions of 1's,* $(i,j)$ *and* $(k,l)$ *are connected if there is a sequence of positions of 1's*

$$(i,j) = (i_1,j_1),(i_2,j_2),\ldots,(i_{t-1},j_{t-1}),(i_t,j_t) = (k,l) \tag{1.38}$$

*(*$t \geq 2$*) such that* $(i_{u+1},j_{u+1})$ *is one of the neighbors of* $(i_u,j_u)$ *for all* $u = 1,\ldots,t-1$. *A polyomino is a binary matrix in which every position of 1 is connected to every other position of 1 (see, for example, Fig. 1.6). The class of polyominoes is denoted by* $(p)$.

The concept of polyomino is well-known in many other fields but it is usually given a different name. For example, in the picture processing literature [59] a polyomino would be called a digital picture with two levels in which the set of white pixels is 4-connected and in the more general theory

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

**FIGURE 1.6. A polyomino.**

of digital spaces [60] it would be called a finite binary picture over $(\mathbb{Z}^2, \omega_2)$ in which the set of 1-spels is $\omega_2$-connected.

Specifically, the reconstruction theory of the class of *hv*-convex polyominoes is well developed. There are upper and lower bounds to the maximum number of *hv*-convex polyominoes having given row and column sum vectors $R$ and $S$ [61]. Reconstruction methods [62–64] have been suggested for this class of binary matrices. A newer method is given in Chapter 7 by Del Lungo and Nivat in this book. An important result for this class of binary matrices is that the computational complexity of the reconstruction problem for *hv*-convex polyominoes is polynomial [62], but for the classes $(p)$, $(h)$, $(v)$, $(p,h)$ (i.e., *h*-convex polyominoes), $(p,v)$ (i.e., *v*-convex polyominoes), and $(hv)$ the problem is NP-complete [65].

## 1.3 Applications

### *1.3.1 Data compression, data coding, and image processing*

Projections can be considered as an encoding of the object. Data coding is interesting from the viewpoints of ***data security*** and ***data compression.*** Knowing that the RECONSTRUCTION$(\mathcal{E}, \mathcal{L})$ problem is NP-complete if the number of projections is greater than 2, encoding via projections can ensure some security of the data [66].

Let us now consider ***data compression.*** Let $F$ be a discrete set in $\mathbb{Z}^d$. To store an $F$ of size $n^d$ we need $n^d$ number of bits, while its $q$ projections with size $n^{d-1}$ need $q \cdot \log_2 n \cdot n^{d-1}$ bits for storage. Therefore, the data compression ratio is

$$\frac{n^d}{q \cdot \log_2 n \cdot n^{d-1}} = \frac{n}{q \cdot \log_2 n} \tag{1.39}$$

independently of the dimension $d$.

From Section 1.2 it is clear that, in general, some information is lost when only the projections are available instead of the discrete set (or, generally, discrete image). However, as Shilferstein and Chien pointed out in [67], several image processing operators have analogous operators on their projections. Further examples are given in [68] for image registration and in [69] for thinning of unique binary images, using only their projections in both cases.

The projections can be used also in image databases. Representing the components (symbols) of the images by their (symbolic) projections, database operations such as spatial reasoning, visualization and browsing can be performed (see Chapter 21 by Chang in this book).

### *1.3.2 Electron microscopy*

Images produced by transmission microscopes can be considered to be projections of the object to be studied. The electron beam transmitted through the specimen can be used to estimate line integrals. If the specimen is composed of a number of homogeneous parts, then DT can be applied for determining the spatial structure. Where only a few noisy projections can be taken from a limited range angles, DT seems to be the only way to reconstruct good quality images.

There are several types of electron microscopy to which the methods of DT had been applied. One of the first experiments was performed by Crew and Crew [70]. They suggested a heuristic discrete algorithm to reconstruct hemoglobin molecules from 3 and 4 projections.

Due to the introduction of a technique [71, 72] called QUANTITEM (QUantitative ANalysis of The Information from Transmission Electron Microscopy), based on high-resolution electron microscopy, it is possible to measure the projections of atomic structures in crystals. The problem of reconstructing discrete sets from their projections for the determination of atomic structures from QUANTITEM data motivated the Mini-Symposium on Discrete Tomography at DIMACS in 1994. Some of the crystalline phantoms designed for testing new reconstruction algorithms are used in a number of chapters of this book as well.

A description of biological problems, which are potentially solvable by electron microscopy and DT, is given in Chapter 18 by Carazo *et al.*

### *1.3.3 Biplane angiography*

For the visualization of a cardiac ventricle it is a standard procedure to inject Roentgen contrast agent into it and to take X-ray images. If we assume that the distribution of the dye is homogeneous and has unit absorption, then we have a binary object consisting of two kinds of points: points of 1's and 0's depending on the presence or absence of the dye. Usually, two orthogonal projections of the ventricle are collected with a conventional biplane X-ray system. The aim is to determine the three-dimensional structure. In practice it is sufficient to reconstruct the two-dimensional cross sections. This problem can be stated as the reconstruction of a binary matrix $A = (a_{ij})_{m \times n}$ from its row and column sum vectors. In general, there is not a unique solution to this problem, as it was shown in Subsection 1.2.2. One way to resolve this ambiguity is to reformulate the reconstruction problem as an optimization problem:

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} a_{ij}, \tag{1.40}$$

under the constraints of (1.5) and (1.6) where the elements of the matrix $C = (c_{ij})_{m \times n}$ represent the costs of assigning the value 1 to the element $a_{ij}$. With careful selection of $C$ we can hope to get a useful solution [73]. For example, we can use the fact that the successive slices are similar to each other in a ventricular structure. Accordingly, if a section is already reconstructed then we choose $C$ such that the binary matrix of the next section is similar to the previous one. Slump and Gerbrands [74] suggested to use a minimum cost capacitated network flow algorithm to find such an optimal solution. A reconstruction program has been developed to determine the dynamic 3D shape of the left or right heart chamber [75]. A summary of this problem is given in Chapter 17 by Onnasch and Prause in this book.

The first paper discussing the application of binary tomography in cardioangiography seems to be due to Chang and Chow [76]. They reconstructed a clay model of a dog heart from two projections estimated from digitized X-ray films. In order to reduce the ambiguity of the problem they supposed that the cross sections of the heart are convex and symmetric with respect to two orthogonal axes. For a very recent work (which applies the approach of Chapter 8 by Matej, Vardi, Herman, and Vardi in this book to cardioangiography) see [77].

Experiments show that the reconstruction of blood vessels is possible from so-called cone beam projections [78].

### *1.3.4 Computerized tomography*

As it was mentioned in Section 1.1, the methods used in CT are not suitable to reconstruct discrete functions from a few projections. If the number of projections is large enough, then CT is able to generate images that are near to the ideal discrete function. If we know the range of the discrete function to be reconstructed, then by the methods of DT we can hope to get images with better resolution and accuracy; see Chapter 15 by Browne and Koshy in this book, where they write about the technical challenges in DT for CT-assisted engineering and manufacturing.

A possible solution in this direction is given in Chapter 9 by Chan, Herman, and Levitan. Their idea for getting better quality images is a two-step procedure. In the first step a conventional reconstruction technique is used to create PET (positron emission tomography) images. Then these images are used as initial values to an iterative DT method.

## 1.4 Conclusion

In this chapter we have given a brief historical overview of some of the foundations of, algorithms for, and applications of discrete tomography. To get a more complete coverage of this field, read on!

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## References

[1] G. T. Herman, *Image Reconstruction from Projections: The Fundamentals of Computerized Tomography*, (Academic Press, New York), 1980.

[2] G. G. Lorentz, "A problem of plane measure," *Amer. J. Math.* **71**, 417-426 (1949).

[3] A. Rényi, "On projections of probability distributions," *Acta Math. Acad. Sci. Hung.* **3**, 131-142 (1952).

[4] A. Heppes, "On the determination of probability distributions of more dimensions by their projections," *Acta Math. Acad. Sci. Hung.* **7**, 403-410 (1956).

[5] H. Kellerer, "Funktionen auf Produkträumen mit vorgegebenen Marginal-Funktionen," *Math. Ann.* **144**, 323-344 (1961).

[6] H. Kellerer, "Masstheoretische Marginalprobleme," *Math. Ann.* **153**, 168-198 (1964).

[7] H. Kellerer, "Marginalprobleme für Funktionen," *Math. Ann.* **154**, 147-150 (1964).

[8] H. J. Ryser, "Combinatorial properties of matrices of zeros and ones," *Canad. J. Math.* **9**, 371-377 (1957).

[9] D. Gale, "A theorem on flows in networks," *Pacific J. Math.* **7**, 1073-1082 (1957).

[10] H. J, Ryser, "Traces of matrices of zeros and ones," *Canad. J. Math.* **12**, 463-476 (1960).

[11] L. R. Ford, Jr. and D. R. Fulkerson, *Flows in Networks* (Princeton University Press, Princeton, NJ), 1962.

[12] J. Steiner, "Einfache Beweis der isoperimetrischen Hauptsätze," *J. reine angew. Math.* **18**, 289-296 (1838).

[13] P. C. Hammer, "Problem 2," In *Proc. Symp. Pure Math., vol. VII: Convexity* (Amer. Math. Soc., Providence, RI), pp. 498-499, 1963.

[14] O. Giering, "Bestimmung von Eibereichen und Eikörpern durch Steiner-Symmetrisierungen," *Sitzungsberichten Bayer. Akad. Wiss. München, Math.-Nat. Kl.*, pp. 225-253 (1962).

[15] R. J. Gardner and P. McMullen, "On Hammer's X-ray problem," *J. London Math. Soc.* **21**, 171-175 (1980).

[16] R. J. Gardner, *Geometric Tomography* (Cambridge University Press, Cambridge, UK), 1995.

[17] G. Bianchi and M. Longinetti, "Reconstructing plane sets from projections," *Discrete Comp. Geom.* **5**, 223-242 (1990).

[18] R. J. Gardner and P. Gritzmann, "Discrete tomography: Determination of finite sets by X-rays," *Trans. Amer. Math. Soc.* **349**, 2271-2295 (1997).

[19] P. Gritzmann and M. Nivat (Eds.), *Discrete Tomography: Algorithms and Complexity*, (Dagstuhl-Seminar-Report 165, Dagstuhl, Germany), 1997.

[20] G. T. Herman and A. Kuba (Editors), *Discrete Tomography.* Special Issue of Intern. J. of Imaging Systems and Techn., Vol. 9, No. 2/3, 1998.

[21] R. Gordon, R. Bender, and G. T. Herman, "Algebraic reconstruction techniques (ART) for three-dimensional electron microscopy and X-ray photography," *J. Theor. Biol.* **29**, 471-481 (1970).

[22] G. T. Herman, A. Lent, and S. W. Rowland, "ART: Mathematics and applications," *J. Theor. Biol.* **42**, 1-32 (1973).

[23] G. T. Herman, "Reconstruction of binary patterns from a few projections," In A. Günther, B. Levrat and H. Lipps, *Intern, Computing Symposium 1973*, (North-Holland Publ. Co., Amsterdam), pp. 371-378, 1974.

[24] P. Gritzmann, D. Prangenberg, S. de Vries, and M. Wiegelmann, "Success and failure of certain reconstruction and uniqueness algorithms in discrete tomography," *Intern. J. of Imaging Systems and Techn.* **9**, 101-109 (1998).

[25] Y. Vardi and D. Lee, "The discrete Radon transform and its approximate inversion via the EM algorithm," *Intern. J. Imaging Systems and Techn.* **9**, 155-173 (1998).

[26] J. H. B. Kemperman and A. Kuba, "Reconstruction of two-valued matrices from their two projections," *Intern. J. of Imaging Systems and Techn.* **9**, 110-117 (1998).

[27] D. Rossi and A. Willsky, "Reconstruction from projections based on detection and estimation of objects: Performance analysis and robustness analysis," *IEEE Trans. Acoust. Speech Signal Process.* **32**, 886-906 (1984).

[28] J. Singer, F. A. Grünbaum, P. Kohn, and J. Zubelli, "Image reconstruction of the interior of bodies that diffuse radiation," *Science* **248**, 990-993 (1990).

[29] D. R. Fulkerson and H. J. Ryser, "Width sequences for special classes of (0,1)-matrices," *Canad. J. Math.* **15**, 371-396 (1963).

[30] S.-K. Chang and Y. R. Wang, "Three-dimensional reconstruction from orthogonal projections," *Pattern Recognition* **7**, 167-176 (1975).

[31] S.-K. Chang, "The reconstruction of binary patterns from their projections," *Commun. ACM* **14**, 21-25 (1971).

[32] S.-K. Chang, "Algorithm 445. Binary pattern reconstruction from projections," *Commun. ACM* **16**, 185-186 (1973).

[33] Y. R. Wang, "Characterization of binary patterns and their projections," *IEEE Trans. Computers* **C-24**, 1032-1035 (1975).

[34] W. Wandi, "The class $\mathfrak{A}(R,S)$ of (0,1)-matrices," *Discrete Math.* **39**, 301-305 (1982).

[35] W. Honghui, "Structure and cardinality of the class $\mathfrak{A}(R,S)$ of (0,1)-matrices," *J. Math. Research and Exposition* **4**, 87-93 (1984).

[36] S. Jiayu, "On a guess about the cardinality of the class $\mathfrak{A}(R,S)$ of (0,1)-matrices," *J. Tongji Univ.* **14**, 52-55 (1986).

[37] W. Xiaohong, "The cardinality of the class $\mathfrak{A}(R,S)$ of (0,1)-matrices," *J. Sichuan Univ. Nat. Sci. Edition* **4**, 95-99 (1986).

[38] W. Xiaohong, "A necessary and sufficient condition for $|\mathfrak{A}_P^Q(R,S)|$ to equal its lower bound," *J. Sichuan Univ. Nat. Sci. Edition* **24**, 136-143 (1987).

[39] L. Mirsky, *Transversal Theory* (Academic Press, New York), 1971.

[40] D. R. Fulkerson, "Zero-one matrices with zero trace," *Pacific J. Math.* **10**, 831-836 (1960).

[41] R. P. Anstee, "Properties of a class of (0,1)-matrices covering a given matrix," *Canad. J. Math.* **34**, 438-453 (1982).

[42] R. P. Anstee, "Triangular (0,1)-matrices with prescribed row and column sums," *Discrete Math.* **40**, 1-10 (1982).

[43] R. P. Anstee, "The network flows approach for matrices with given row and column sums," *Discrete Math.* **44**, 125-138 (1983).

[44] L. Mirsky, "Combinatorial theorems and integral matrices," *J. Combin. Theor.* **5**, 30-44 (1968).

[45] A. Kuba and A. Volčič, "Characterization of measurable plane sets which are reconstructable from their two projections," *Inverse Problems* **4**, 513-527 (1988).

[46] P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp, "Sets uniquely determined by projections on axes II. Discrete case," *Discrete Math.* **91**, 149-159 (1991).

[47] H. J. Ryser, *Combinatorial Mathematics* (The Math. Assoc. Amer., Washington, DC), 1963.

[48] R. A. Brualdi, "Matrices of zeros and ones with fixed row and column sum vectors," *Lin. Algebra and Its Appl.* **33**, 159-231 (1980).

[49] A. Kuba, "Reconstruction of unique binary matrices with prescribed elements," *Acta Cybern.* **12**, 57-70 (1995).

[50] R. Aharoni, G. T. Herman, and A. Kuba, "Binary vectors partially determined by linear equation systems," *Discrete Math.* **171**, 1-16 (1997).

[51] H. J. Ryser, "The term rank of a matrix," *Canad. J. Math* **10**, 57-65 (1958).

[52] H. J. Ryser, "Matrices of zeros and ones," *Bull. Amer. Math. Soc.* **66**, 442-464 (1960).

[53] M. Haber, "Term rank of 0, 1 matrices," *Rend. Sem. Mat. Padova* **30**, 24-51 (1960).

[54] A. Kuba, "Determination of the structure of the class $\mathfrak{A}(R,S)$ of (0,1)-matrices," *Acta Cybernetica* **9**, 121-132 (1989).

[55] W. Y. C. Chen, "Integral matrices with given row and column sums," *J. Combin. Theory, Ser. A* **61**, 153-172 (1992).

[56] S.-K. Chang and G. L. Shelton, "Two algorithms for multiple-view binary pattern reconstruction," *IEEE Trans. Systems, Man, Cybernetics* **SMC-1**, 90-94 (1971).

[57] L. Huang, "The reconstruction of uniquely determined plane sets from two projections in discrete case," *Preprint Series* **UTMS 95-29**, *Univ. of Tokyo* (1995).

[58] A. Kuba, "The reconstruction of two-directionally connected binary patterns from their two orthogonal projections," *Comp. Vision, Graphics, Image Proc.* **27**, 249-265 (1984).

[59] A. Rosenfeld and A. C.Kak, "Digital Picture Processing," (Academic Press, New York) 1976.

[60] G. T. Herman, "Geometry of Digital Spaces," (Birkhäuser, Boston) 1998.

[61] A. Del Lungo, M. Nivat and R. Pinzani, "The number of convex polyominoes reconstructible from their orthogonal projections," *Discrete Math.* **157**, 65-78 (1996).

[62] E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, "Reconstructing convex polyominoes from horizontal and vertical projections," *Theor. Comput. Sci.* **155**, 321-347 (1996).

[63] E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, "Medians of polyominoes: A property for the reconstruction," *Int. J. Imaging Systems and Techn.* **9**, 69-77 (1998).

[64] M. Chrobak and C. Dürr, "Reconstructing hv-convex polyominoes from orthogonal projections," (Preprint, Dept. of Computer Science, Univ. of California, Riverside, CA), 1998.

[65] G. J. Woeginger, "The reconstruction of polyominoes from their orthogonal projections," (Techn. Rep. F003, TU-Graz), 1996.

[66] R. W. Irving and M. R. Jerrum, "Three-dimensional statistical data security problems," *SIAM J. Comput.* **23**, 170-184 (1994).

[67] A. R. Shliferstein and Y. T. Chien, "Some properties of image-processing operations on projection sets obtained from digital pictures," *IEEE Trans. Comput.* **C-26**, 958-970 (1977).

[68] Z. Mao and R. N. Strickland, "Image sequence processing for target estimation in forward-looking infrared imagery," *Optical Engrg.* **27**, 541-549 (1988).

[69] A. Fazekas, G. T. Herman, and A. Matej, "On processing binary pictures via their projections," *Int. J. Imaging Systems and Techn.* **9**, 99-100 (1998).

[70] A. V. Crewe and D. A. Crewe, "Inexact reconstruction: Some improvements," *Ultramicroscopy* **16**, 33-40 (1985).

[71] P. Schwander, C. Kisielowski, M. Seibt, F. H. Baumann, Y. Kim, and A. Ourmazd, "Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy," *Phys. Rev. Letters* **71**, 4150-4153 (1993).

[72] C. Kisielowski, P. Schwander, F. H. Baumann, M. Seibt, Y. Kim, and A. Ourmazd, "An approach to quantitative high-resolution electron microscopy of crystalline materials," *Ultramicroscopy* **58**, 131-155 (1995).

[73] D. G. W. Onnasch, and P. H. Heintzen, "A new approach for the reconstruction of the right or left ventricular form from biplane angiocardiographic recordings," In *Conf. Comp. Card. 1976*, (IEEE Comp. Soc. Press, Washington), pp. 67-73, 1976.

[74] C. H. Slump and J. J. Gerbrands, "A network flow approach to reconstruction of the left ventricle from two projections," *Comp. Graphics and Image Proc.* **18**, 18-36 (1982).

[75] G. P. M. Prause and D. G. W. Onnasch, "Binary reconstruction of the heart chambers from biplane angiographic image sequences," *IEEE Trans. Medical Imaging* **MI-15**, 532-546 (1996).

[76] S.-K. Chang and C. K. Chow, "The reconstruction of three-dimensional objects from two orthogonal projections and its application to cardiac cineangiography," *IEEE Trans. Comput.* **C-22**, 18-28 (1973).

[77] B. M. Carvalho, G. T. Herman, S. Matej, C. Salzberg, and E. Vardi, "Binary tomography for triplane cardiography," In A. Kuba, M. Samal, A. Todd-Pokropek, *Information Processing in Medical Imaging Conference 1999* (Springer-Verlag, Berlin), to be published.

[78] N. Robert, F. Peyrin, and M. J. Yaffe, "Binary vascular reconstruction from a limited number of cone beam projections," *Med. Phys.* **21**, 1839-1851 (1994).