Co-funded by the Erasmus+ Programme of the European Union

This project has been funded with support from the European Commission. The author is solely responsible for this publication (communication) and the Commission accepts no responsibility for any use may be made of the information contained therein.

Project nº: 2019-1-TR01-KA203-077194

Project title: "ThinkBS - Promoting Deep and Wide Thinking / Early Dual Degrees in Basic Sciences"

Summer School - Debrecen

Time: July 4, Monday, 2022 - July 15, Friday

Online classes on workdays of July 4-8, 9:00 -11:30 AM, and 1:00 – 3:30 PM

Project work and presentation on the week July 11-15

‘Mathematics of Technical Diagnostics’ –

a practical approach

(application of Mathematics)

Course material prepared in the frame of the project is

Vibration Signal Analysis for Machinery Condition Monitoring

by Imre Kocsis and Krisztián Deák

University of Debrecen Faculty of Engineering, 2022

e-learning: tovabbkepzes.unideb.hu

www.eng.unideb.hu

bachelor programmes	master programmes
Mechanical Engineering	Mechanical Engineering
Mechatronics Engineering	Mechatronics Engineering
Vehicle Engineering	Professional Pilot
Civil Engineering	Urban System Engineering
Architecture	Building Service Engineering
Engineering Management	Engineering Management
Environmental Engineering	Environmental Engineering

Content of the course material

Part I - Imre Kocsis

1. Trigonometric and Exponential Functions
2. Statistical Analysis of Vibration Signals
3. Hilbert Spaces, Orthogonality, Similarity of Functions
4. Orthonormal Systems, Fourier Series, Trigonometric System
5. Exponential System, Vibration Spectrum
6. Continuous Fourier Transform, Discrete Fourier Transform, FFT

Part II - Krisztián Deák

7. Cepstrum Analysis, Envelope Analysis
8. Continuous and Discrete Wavelet Transform
9. MRA, Scalogram
10. Wavelet Transforms in Machine Fault Diagnostics
11. Digital Filters, FIR, IIR
12. Digital Filter Design

Content of this short course (Part I)

– Fields and goals and of machinery diagnostics
– Some mathematical tools used in vibration diagnostics (Fourier theory first of all)
– About an industrial condition monitoring system (SPM), case studies

About machinery (technical) diagnostics

Machinery diagnostics is a fundamental tool of predictive maintenance.

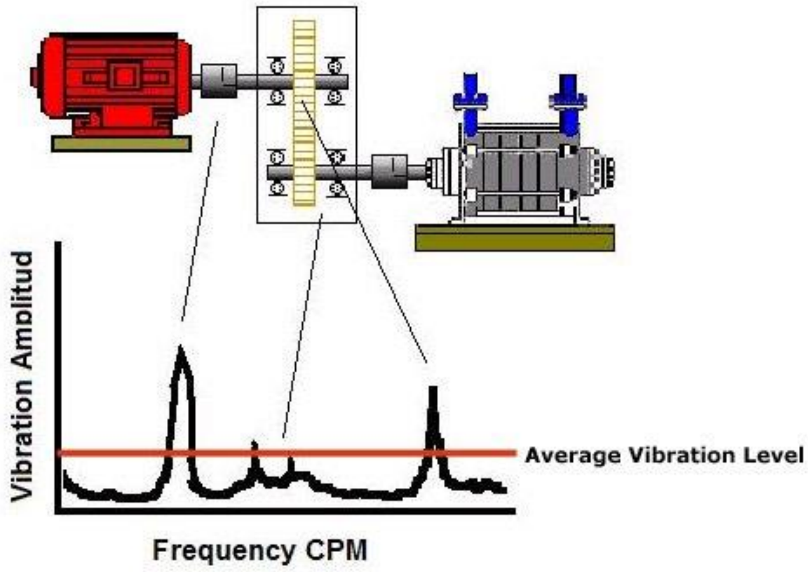
It provides data about the current condition of machine and process elements for maintenance decisions.

The goal of predictive maintenance is to provide the data required

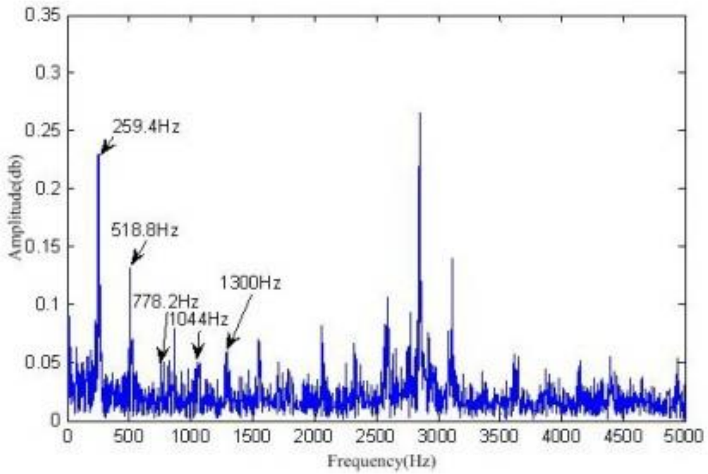
– to ensure the maximum interval between repairs and
– to minimize the number and cost of unscheduled outages created by failures.

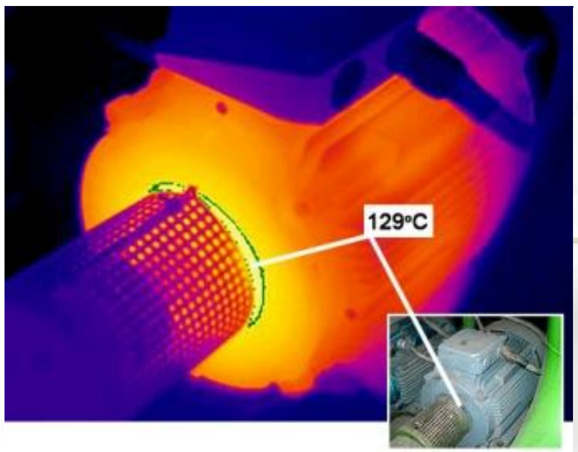
Techniques normally used for predictive maintenance are

– vibration diagnostics,
– acoustics (mainly ultrasonic),
– thermography,
– tribology (wear particle analysis),
– process parameter monitoring.



Most predictive maintenance programs use vibration analysis as the primary tool.





Symptoms

The first step of condition monitoring is to find connection between faults and measurable symptoms generated by the failures investigated.

In the field of vibration monitoring symptoms can be detected with the analysis of the vibration signal and its transforms.

Since nowadays mainly digital measurement systems are used, sampled signals are available for the analysis.

Some symptoms appear in the time-domain (e.g. in velocity-time function) others can be revealed from the frequency spectrum (frequency-domain analysis) or from other transforms of the signal.

Typical symptoms in vibration diagnostics

Symptoms in the 'time-domain'

Certain types of mechanical damage of rotating parts imply the change of some statistical parameters in time, such as

– mean, standard deviation,
– RMS, peak value,
– skewness, kurtosis

of the vibration velocity or acceleration data in the sampled signal.

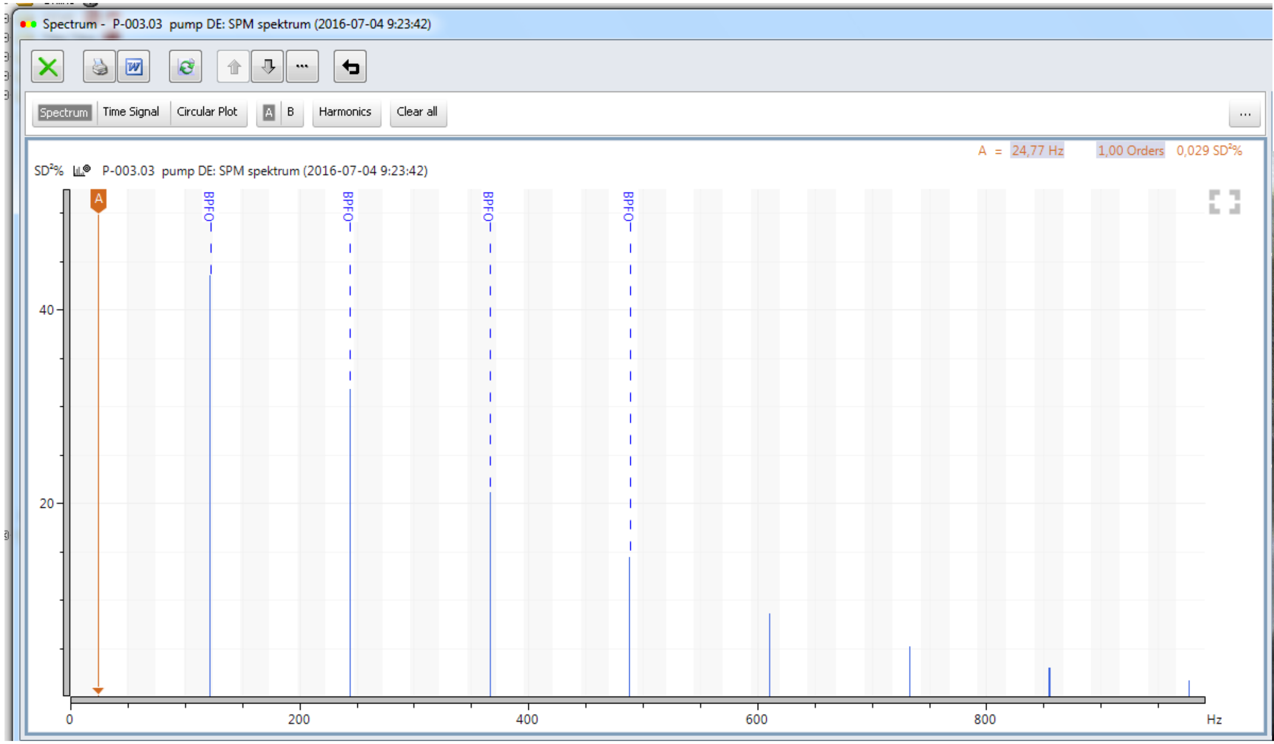
The changed shape of the probability density function of the vibration velocity or acceleration data can be an indicator of failures.

E.g. the level of shock pulses generated by a healthy ball bearing follows normal distribution, the appearance of damage in the bearing results in the change of probability density function.

Symptoms in the 'frequency-domain'

The generated vibrations have special frequencies depending on the rotational speed and the type of the rotating component.

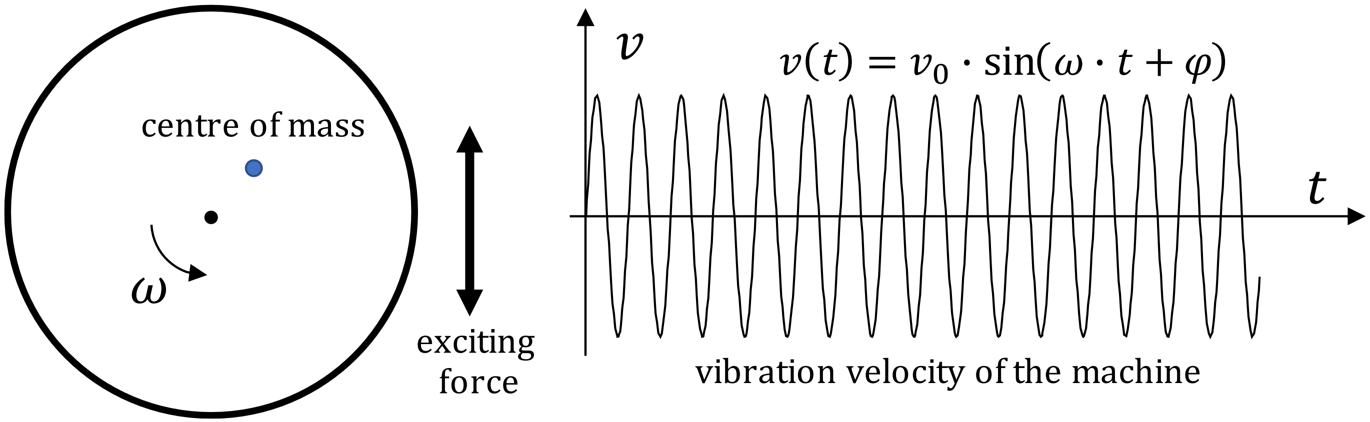
The majority of failures generates a group of spectrum lines (patterns characteristics to the failures).



The main sources of machine vibrations

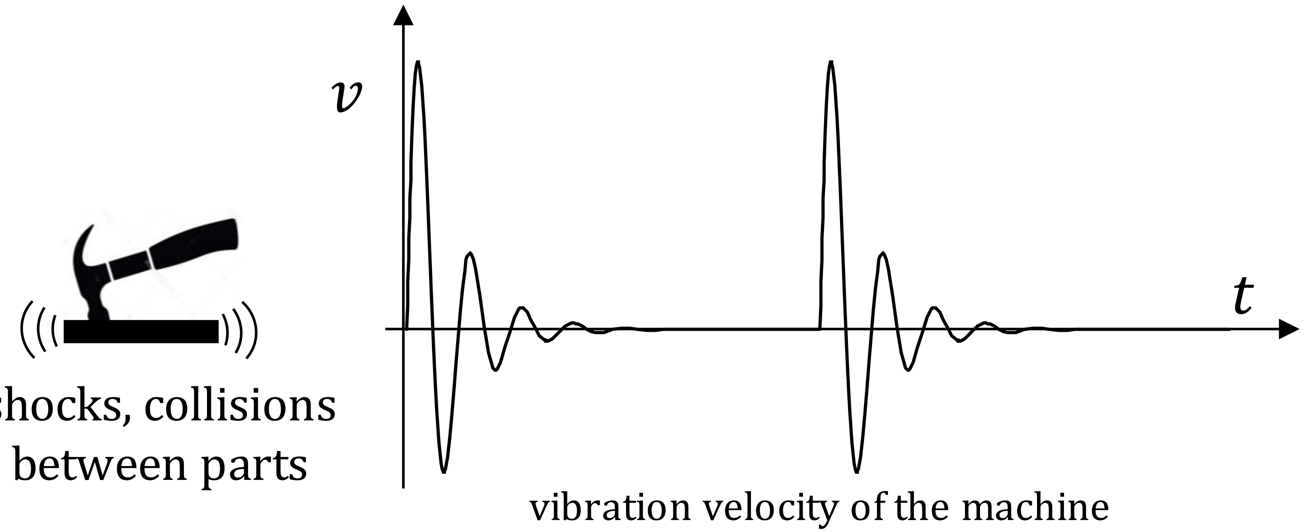
Harmonic vibrations

(generated by rotating parts)



Shock pulses

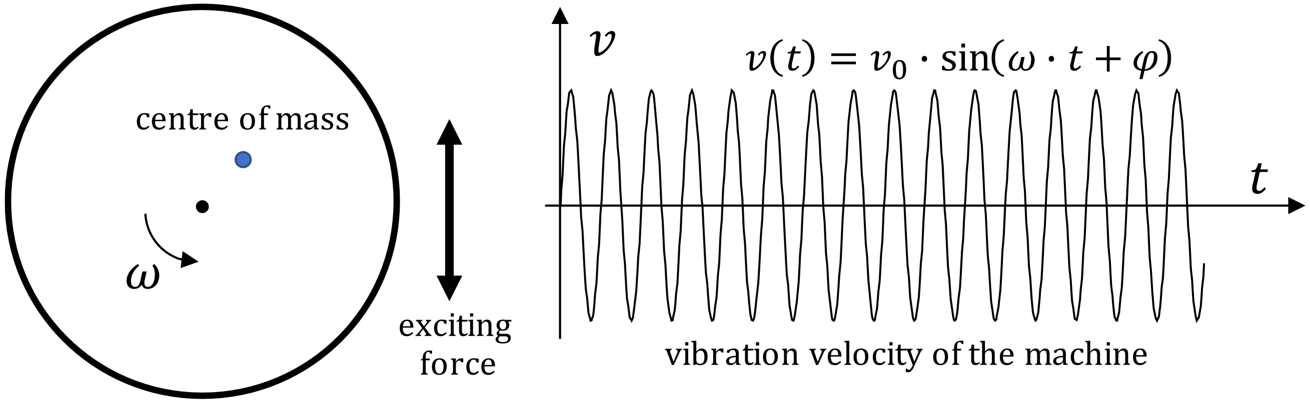
(generated by shocks and collisions between parts)



Harmonic vibrations generated by rotating parts with failures

Many types of mechanical failures of rotating parts generate periodic, nearly harmonic vibrations, for example:

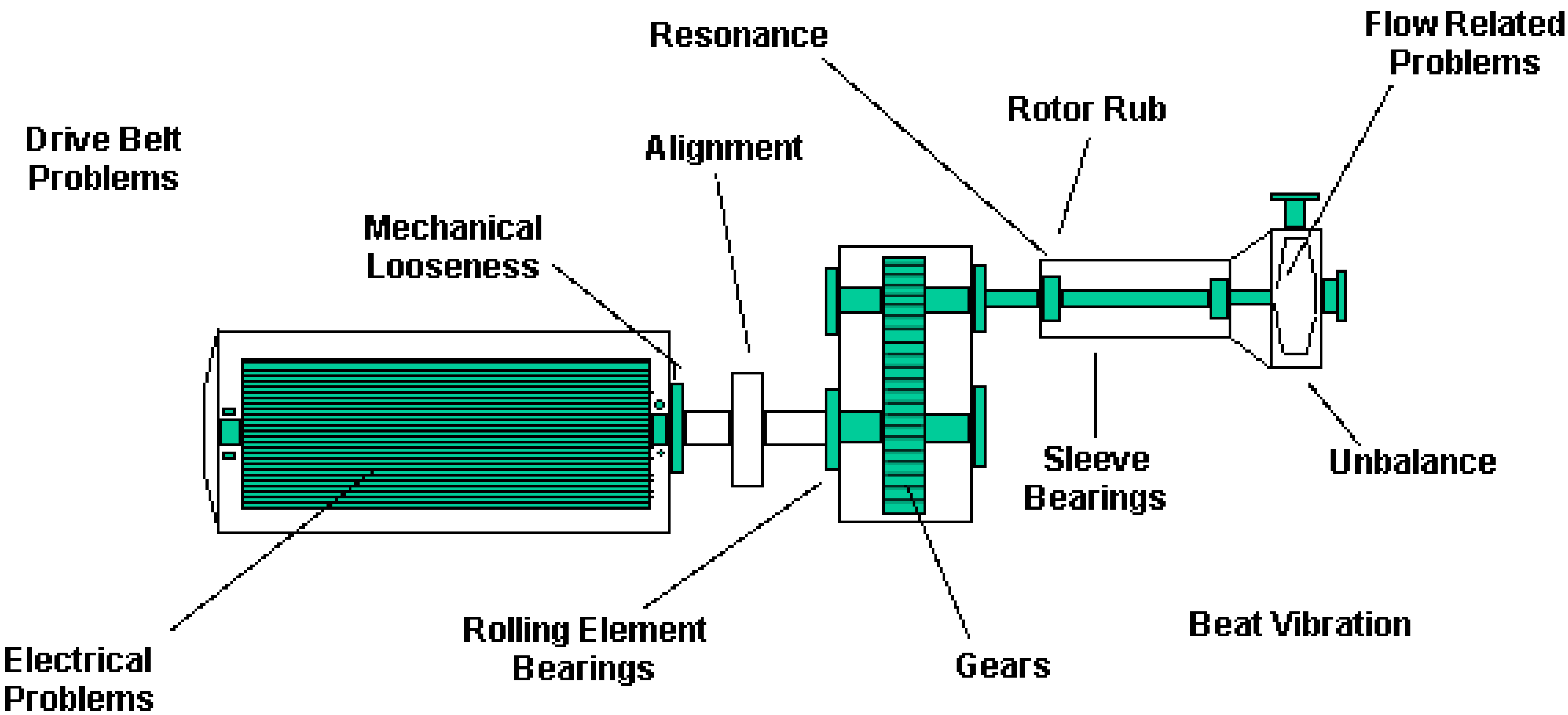
- unbalance,
- angular or parallel misalignment of shafts (at couplings)
- bended shafts



The generated vibrations have special frequencies depending on the rotational speed and the type of the rotating component.

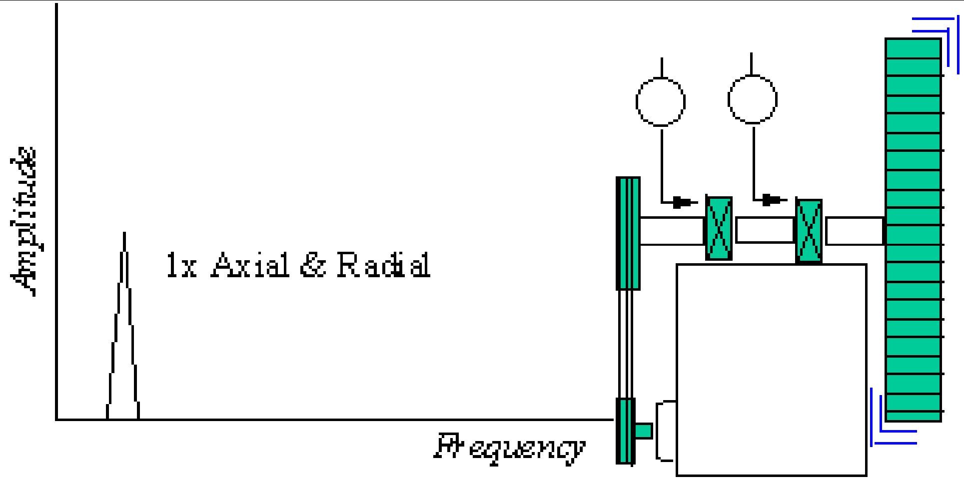
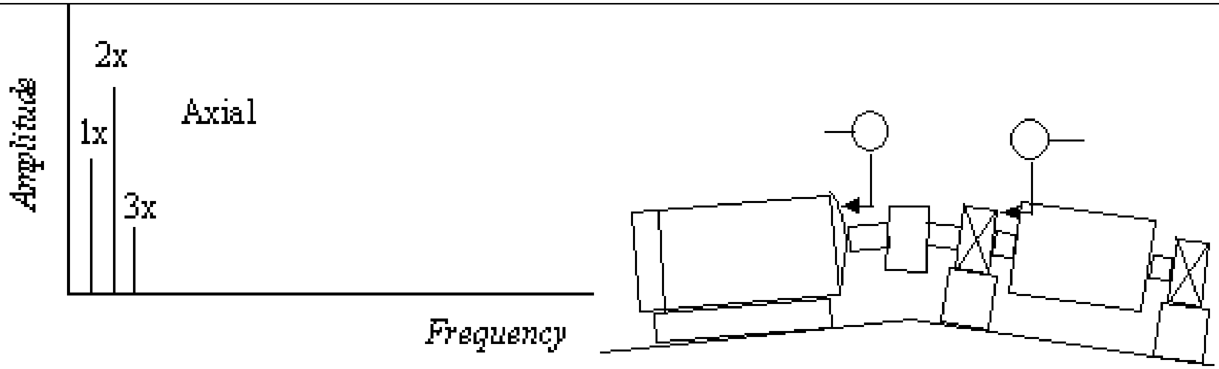
The phase shift of vibration signals coming from different sources can be informative in certain cases.

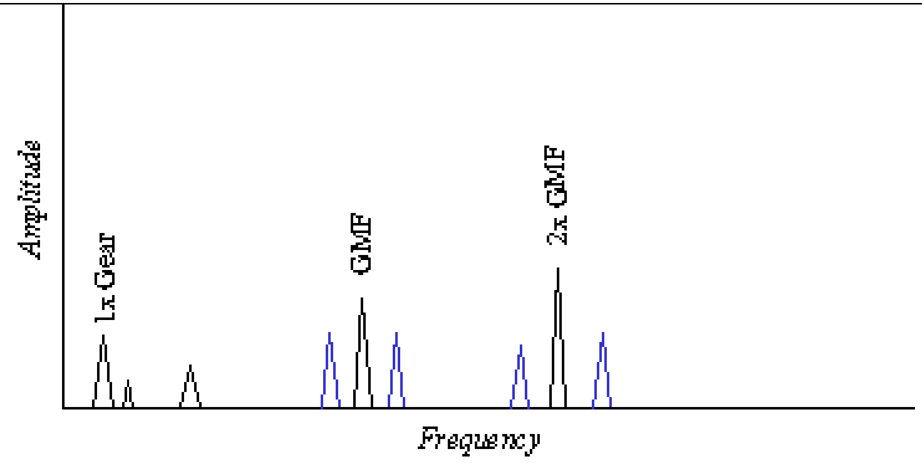
Characteristic frequency symptoms at different parts of a drive chain

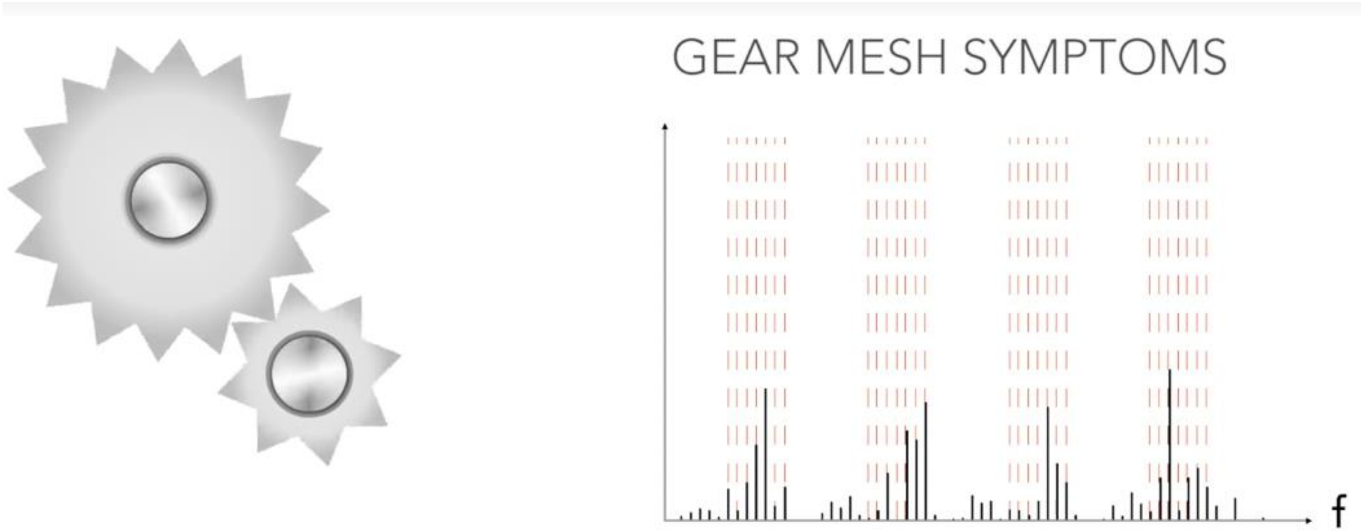


source: David Stevens http://www.vibanalysis.co.uk/vibanalysis/index.htm

The ‘basic frequency’ is the rotational speed of the shaft expressed in $\left[\frac{rev}{s}\right] = [Hz]$

type of the failure	order	symptom in the spectrum
unbalance of a rotor	1×	
angular misalignment of shafts at a coupling	2× (1×,3×)	

<table>
<tr>
<td>gear fault</td>
<td>a group of frequencies</td>
<td>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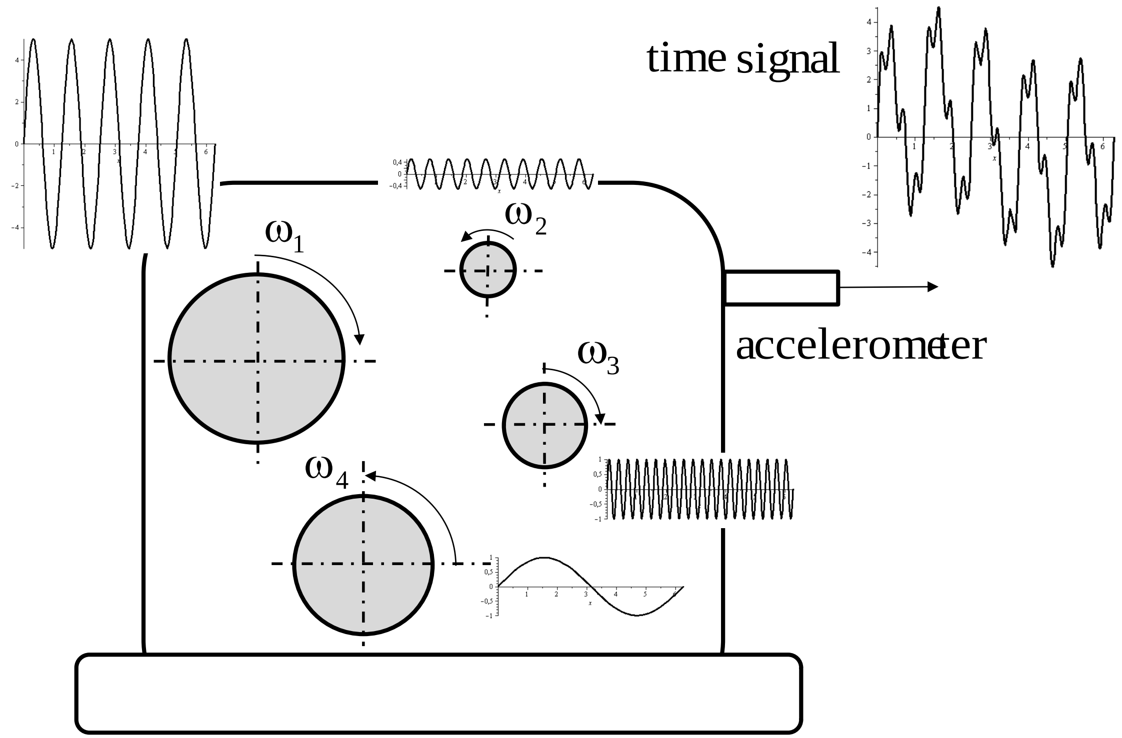
Since the majority of failures generates a group of spectrum lines (characteristic patterns), in many cases, pattern recognition is required rather than the detection of a certain frequency value.

Measurement of harmonic vibrations

Connecting a common accelerometer the superposition of harmonic vibrations generated by rotating parts can be measured.

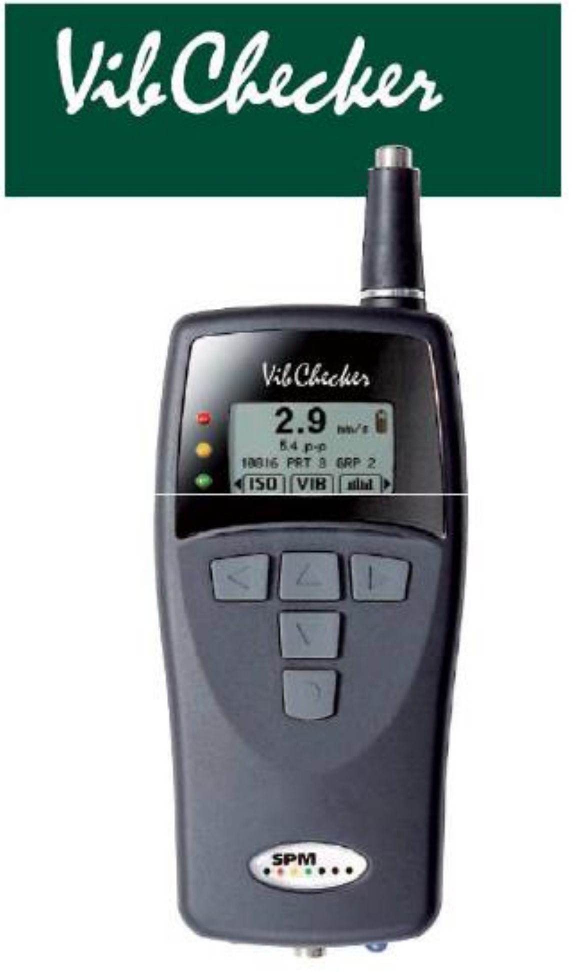


The vibration spectrum provided by the Fourier analysis shows the frequencies appearing in the vibration signal and the magnitudes belonging to them.

Based on these data the problematic components and the severity of the failures can be identified.

Commonly used basic quantities in vibration analysis

- vibration displacement in $[\mu m]$
- vibration velocity in $\left[\frac{mm}{s}\right]$
- vibration acceleration in $\left[\frac{m}{s^2}\right]$ or $[g]$

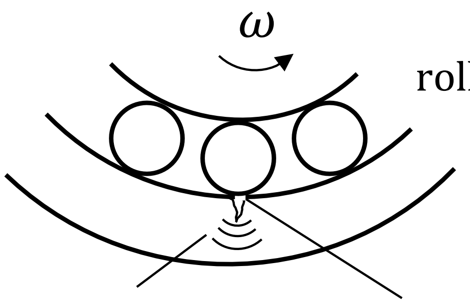


Shock pulses generated by defective rotating parts

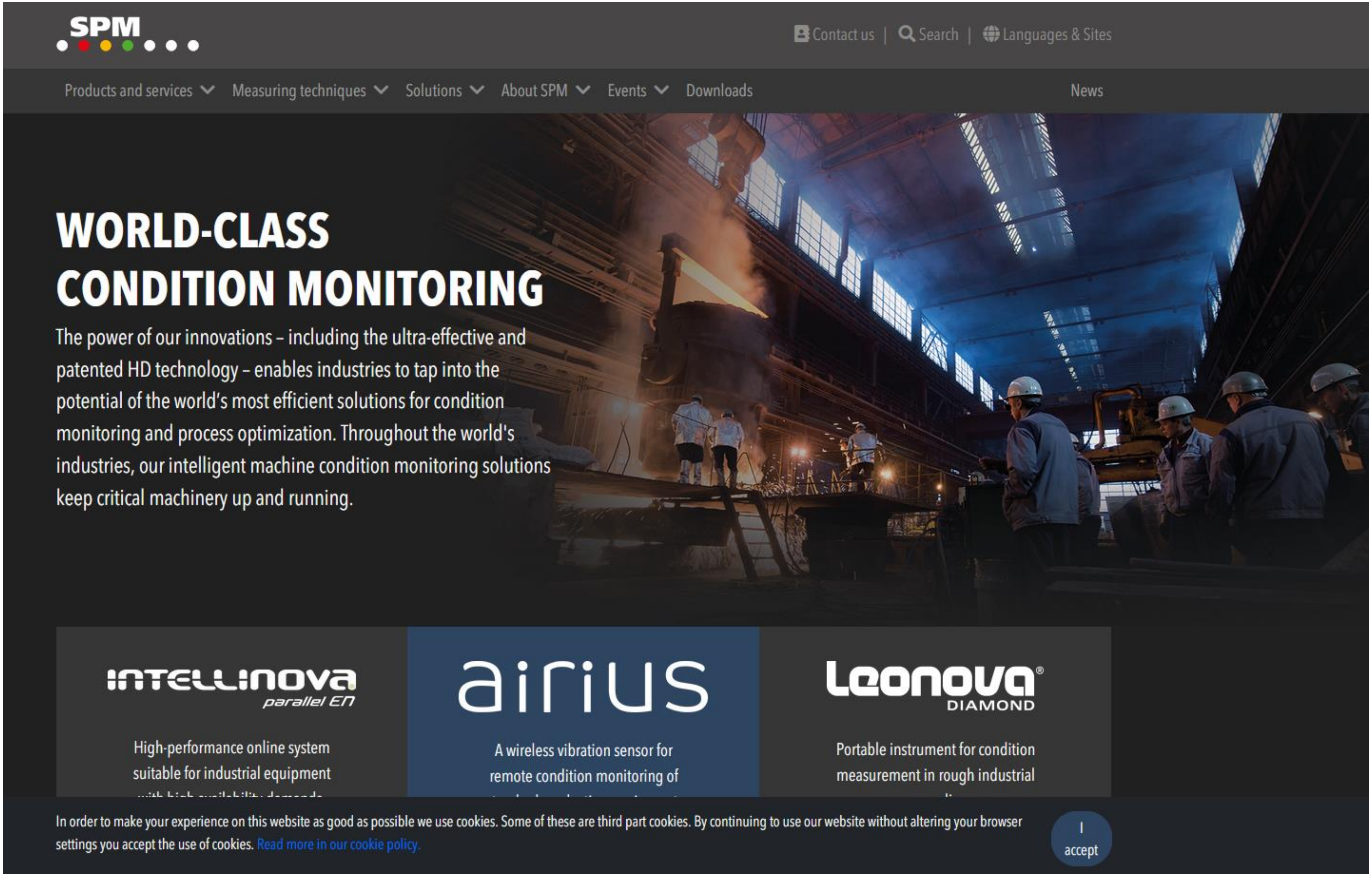
Shock pulses are non-periodic transient waves in the time signal.

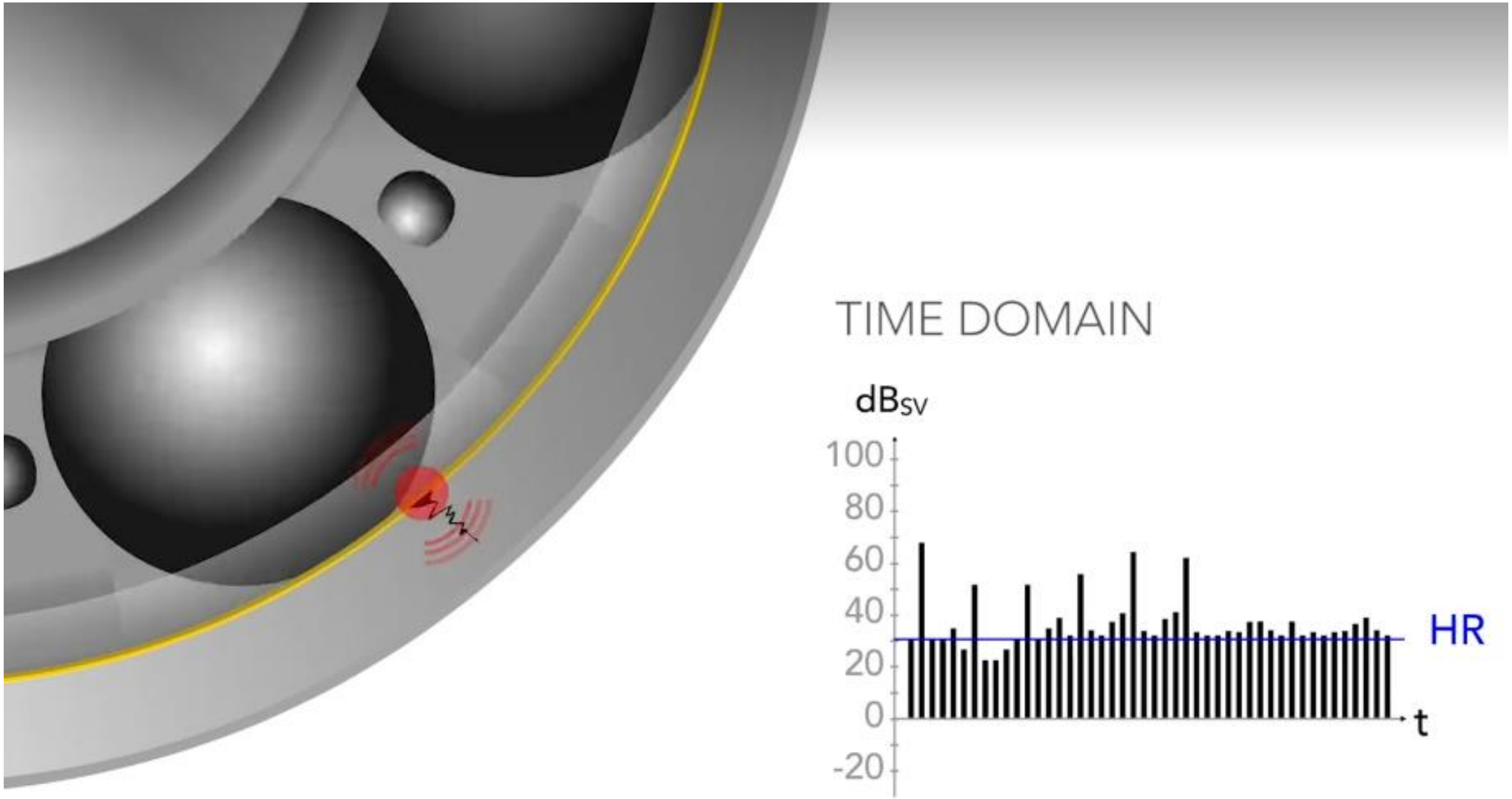
Some important types of failures cause low-energy transient vibrations rather than high-energy periodic vibrations.

The most important examples are bearing and gear failures.

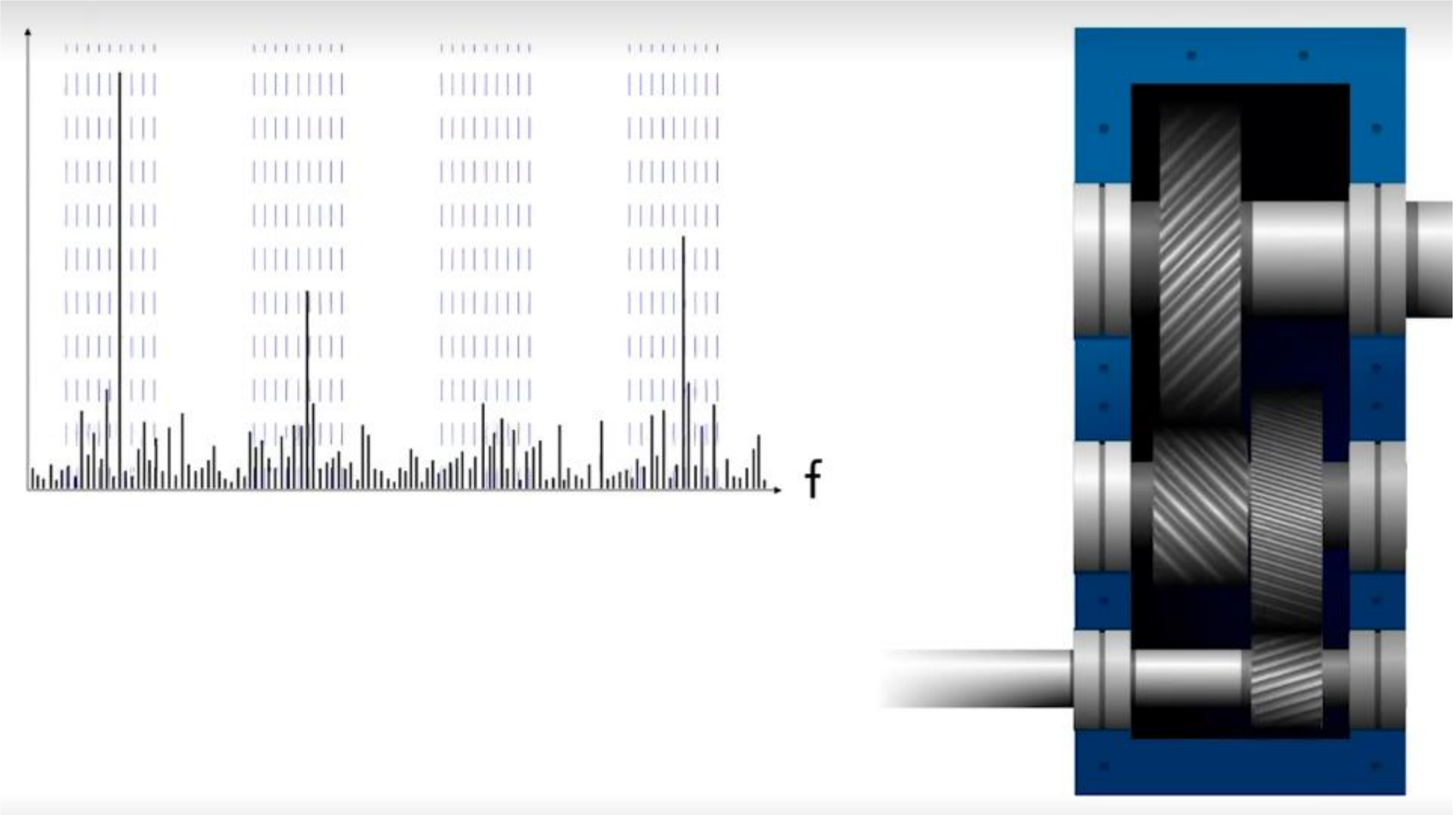


Animation: **spminstrument.com**





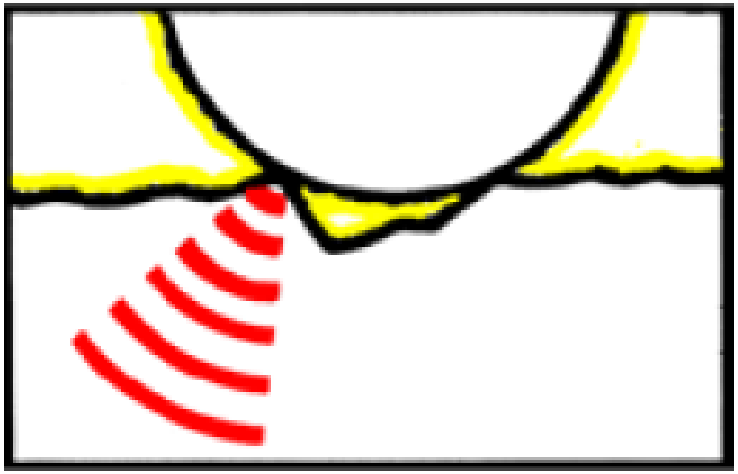
source: spminstrument.com



source: spminstrument.com

Measurement of shock pulses

Measurement of low-energy shock pulses requires special transducers and signal processing methods.



Since bearings hold shafts and all connected parts, they are crucial machine elements, detection bearing faults is an important task in diagnostics.

To be able to capture the low-energy shock pulses generated by bearing surface faults the 'shoch pulse transducer' must be mounted close to the load zone of the bearing and must be fixed properly.

Fourier theory – a fundamental tool in vibration diagnostics

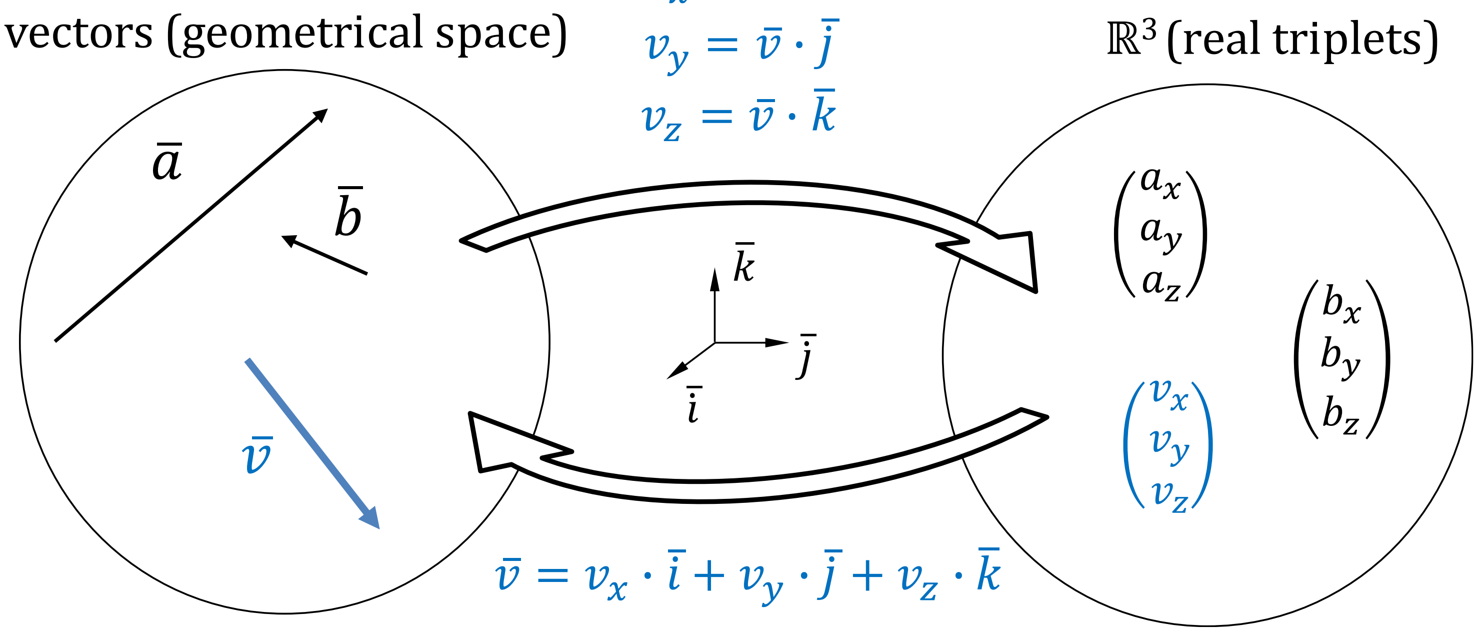
- The idea of the decomposition
- Decomposition of functions
- Hilbert spaces, orthogonality, similarity
- Fourier series
- The trigonometric system, the exponential system
- Fourier transform
- Discrete Fourier transform
- Fast Fourier transform

Decomposition with respect to an orthonormal system

Goal: to transfer the 'problem' into a space where its 'treatment' is easier

The simplest example: decomposition of vectors (in $\mathbb{R}^3$) with respect to the orthonormal basis $\{\bar{i}, \bar{j}, \bar{k}\}$

$$v_x = \bar{v} \cdot \bar{i}$$
$$v_y = \bar{v} \cdot \bar{j}$$
$$v_z = \bar{v} \cdot \bar{k}$$



$$\bar{v} = v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k}$$

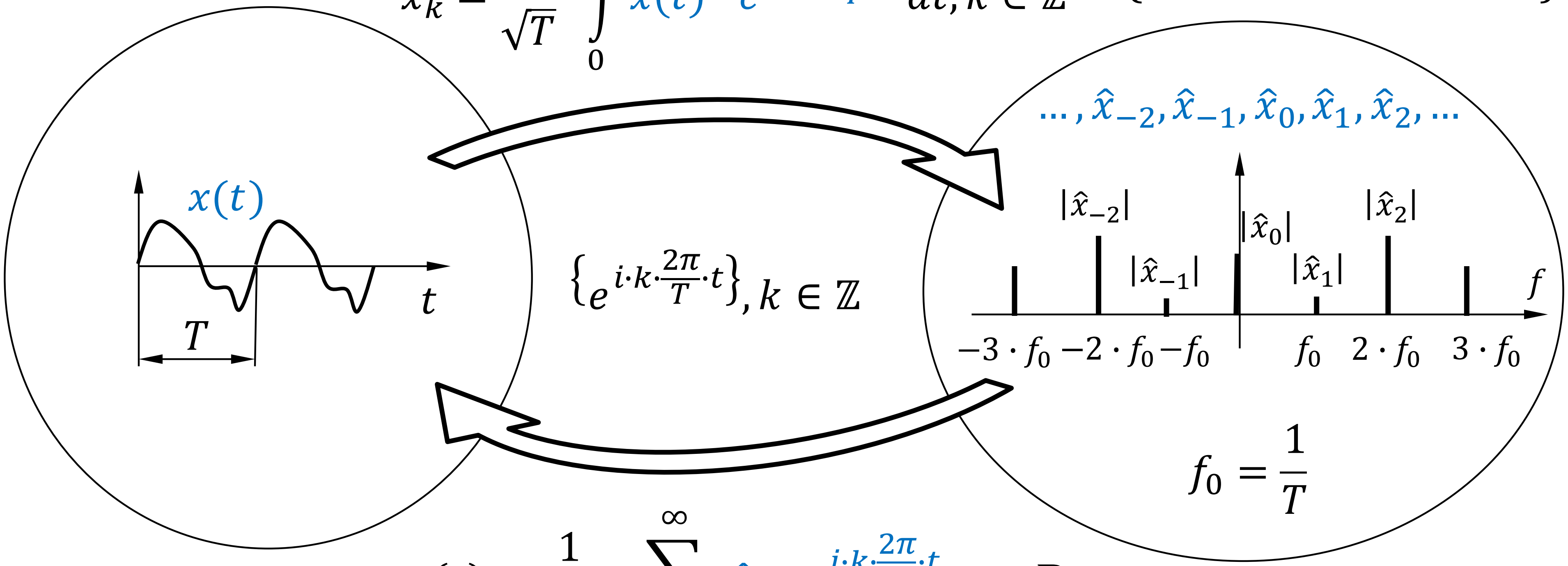
E.g. angle of vectors $\bar{a}$ and $\bar{b}$ is $\varphi = \arccos \dfrac{a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$

Decomposition of periodic functions

periodic signals

discrete complex spectrum (Fourier coefficients)

$$\hat{x}_k = \frac{1}{\sqrt{T}} \cdot \int_0^T x(t) \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\, dt, k \in \mathbb{Z}$$



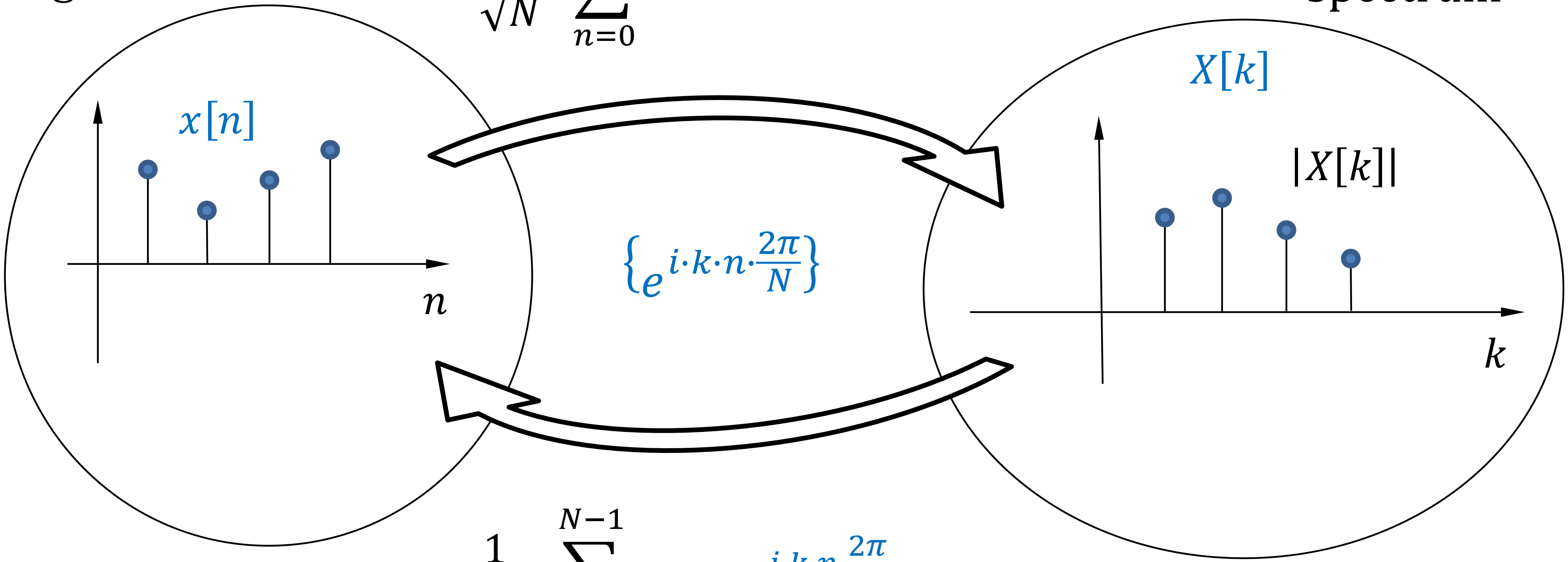
$$x(t) = \frac{1}{\sqrt{T}} \cdot \sum_{k=-\infty}^{\infty} \hat{x}_k \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}, t \in \mathbb{R}$$

Decomposition of sampled signals

sampled signal

$$X[k] = \frac{1}{\sqrt{N}} \cdot \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}, k = 0, \dots, N-1$$

discrete spectrum



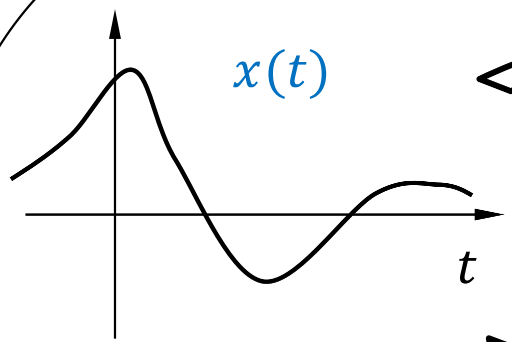
$$x[n] = \frac{1}{\sqrt{N}} \cdot \sum_{n=0}^{N-1} X[k] \cdot e^{i \cdot k \cdot n \cdot \frac{2\pi}{N}}, n = 0, \dots, N-1$$

Non-periodic functions, Fourier transform

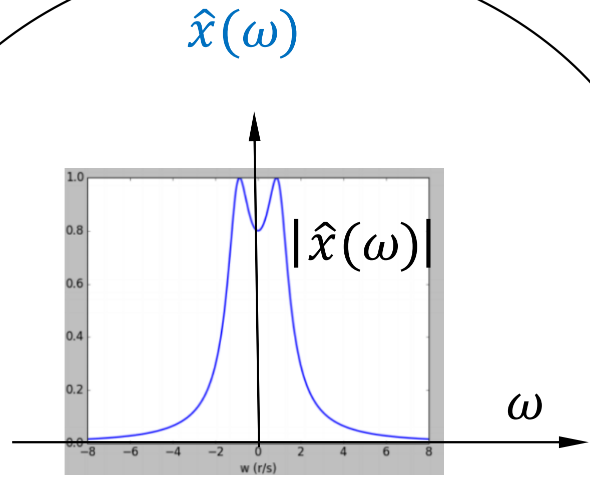
integrable signals

continuous complex spectrum

$$\hat{x}(\omega) = \frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty} x(t) \cdot e^{-i\cdot\omega\cdot t}\, dt, \omega \in \mathbb{R}$$



$$\left\{e^{i\cdot\omega\cdot t}\right\}, \omega \in \mathbb{R}$$



$$x(t) = \frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\cdot\omega\cdot t}\, d\omega, t \in \mathbb{R}$$

Orthogonality and decomposition in $\mathbb{R}^n$ (n-dimensional Hilbert space)

The inner product of elements $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

The norm of element $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Elements $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are orthogonal if

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = 0.$$

A basis $\{b_1, \dots, b_n\} \subset \mathbb{R}^n$ is orthogonal if its elements are pairwise orthogonal, and it is orthonormal if, additionally, the elements are unit vectors (with unit length).

The coefficients in the decomposition of $x \in \mathbb{R}^n$ with respect to a given orthonormal basis $\{b_1, \dots, b_n\}$ can be calculated as the inner products of x and the basis vectors b_i as follows

$$\langle x, b_i \rangle, i = 1, .., n,$$

The decomposition of x is

$$x = \sum_{i=1}^{n} \langle x, b_i \rangle \cdot b_i$$

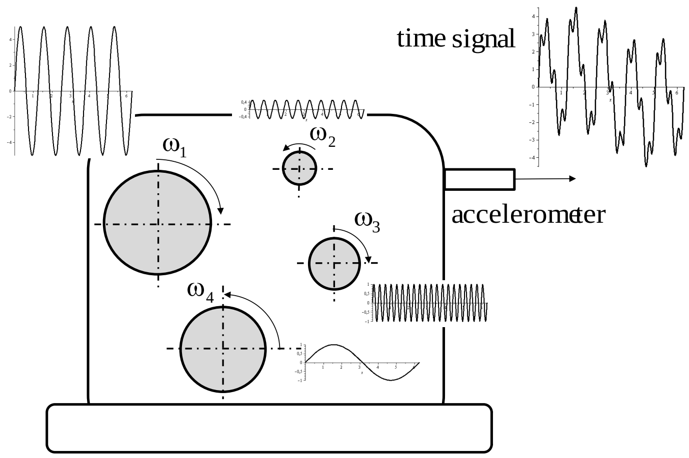
E.g. $\{\bar{i}, \bar{j}, \bar{k}\}$ is an orthonormal basis of $\mathbb{R}^3$ and

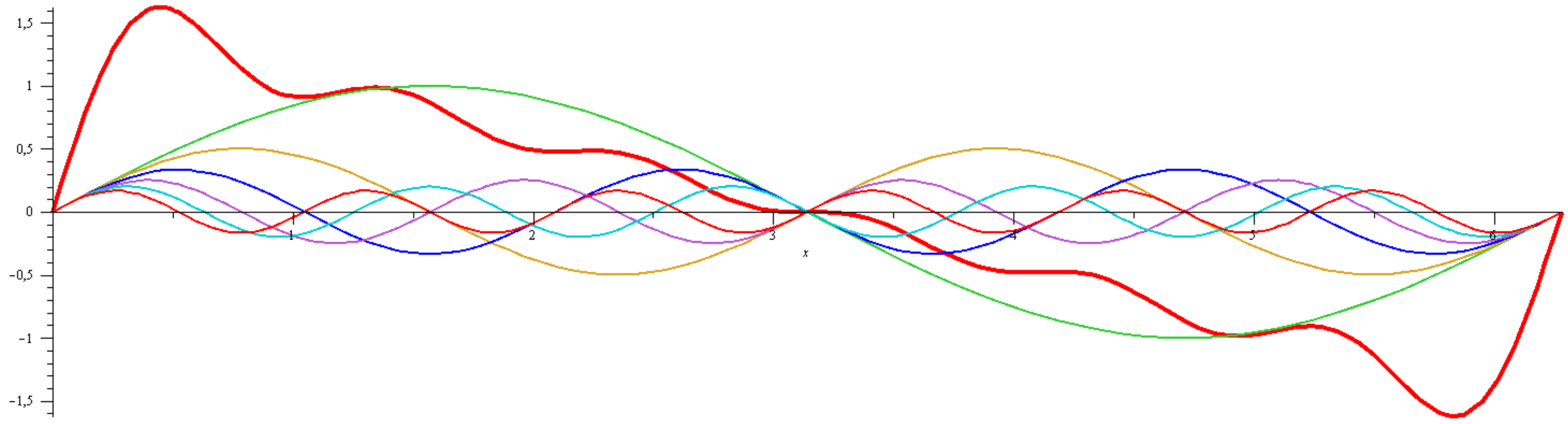
$$\bar{x} = \langle \bar{x}, \bar{i} \rangle \cdot \bar{i} + \langle \bar{x}, \bar{j} \rangle \cdot \bar{j} + \langle \bar{x}, \bar{k} \rangle \cdot \bar{k}$$

is the decomposition of $\bar{x}$ with respect to $\{\bar{i}, \bar{j}, \bar{k}\}$.

Orthogonality and decomposition in function spaces (infinite dimensional Hilbert spaces)

A goal in vibration diagnostics is to identify the frequency, amplitude and phase of harmonic vibrations (signals) characteristic to the machine elements with certain failures from the superposition of vibrations (time signal).





This problem can be solved with the decomposition of the vibration signal with respect to an appropriate orthonormal system.

Commonly used systems for the decomposition of T-periodic functions

Orthonormal trigonometric system	Orthonormal exponential system
$\left\{\frac{1}{\sqrt{T}}, \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$	$\left\{\frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k\in\mathbb{Z}}$
Trigonometric system	Exponential system
$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$	$\left\{e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k\in\mathbb{Z}}$
$\{1, \cos(\boldsymbol{k} \cdot 2\pi \cdot \boldsymbol{f_0} \cdot t), \sin(\boldsymbol{k} \cdot 2\pi \cdot \boldsymbol{f_0} \cdot t)\}_{\mathrm{k}\in\mathbb{N}}$	$\left\{e^{i \cdot k \cdot 2\pi \cdot \boldsymbol{f_0} \cdot t}\right\}_{k\in\mathbb{Z}}$

Remark: As we will see later, the trigonometric system and the exponential system are orthogonal but not orthonormal.

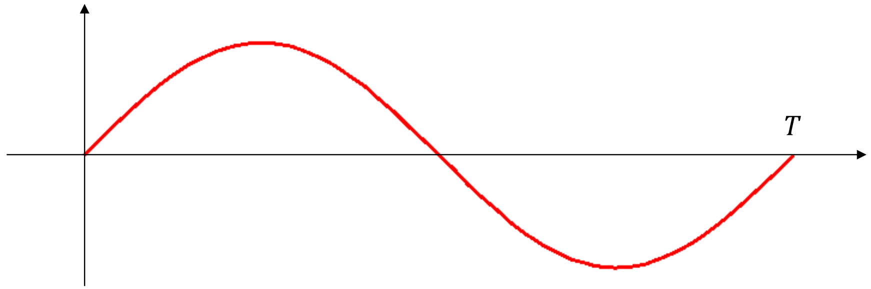
Some elements of the trigonometric system with T-periodic basic functions

A trigonometric system contains T-periodic sin and cos basic functions of frequency $f_0 = \frac{1}{T}$, and harmonics of frequencies $k \cdot f_0, k = 1,2, \ldots$:

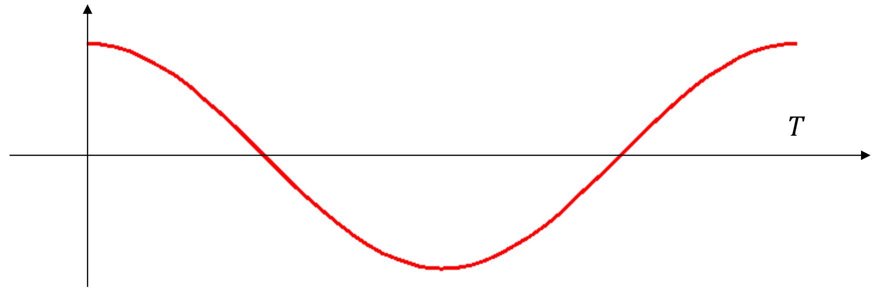
	period	frequency
$\cos\left(\frac{2\pi}{T} \cdot t\right) = \cos(2\pi f_0 \cdot t)$	$T = \frac{1}{f_0}$	$f_0 = \frac{1}{T}$
$\sin\left(\frac{2\pi}{T} \cdot t\right) = \sin(2\pi f_0 \cdot t)$	$T = \frac{1}{f_0}$	$f_0 = \frac{1}{T}$
$\cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(2 \cdot 2\pi f_0 \cdot t)$	$T/2$	$2f_0$
$\sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(2 \cdot 2\pi f_0 \cdot t)$	$T/2$	$2f_0$
$\cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(3 \cdot 2\pi f_0 \cdot t)$	$T/3$	$3f_0$
$\sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(3 \cdot 2\pi f_0 \cdot t)$	$T/3$	$3f_0$
$\cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(4 \cdot 2\pi f_0 \cdot t)$	$T/4$	$4f_0$
$\sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(4 \cdot 2\pi f_0 \cdot t)$	$T/4$	$4f_0$

Every element of the trigonometric system is related to a frequency which is a physical quantity.

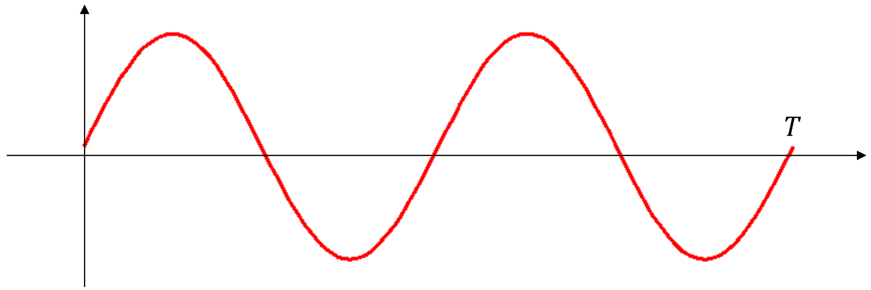
$$t \to \sin\left(\frac{2\pi}{T} \cdot t\right) = \sin(2\pi f_0 \cdot t)$$



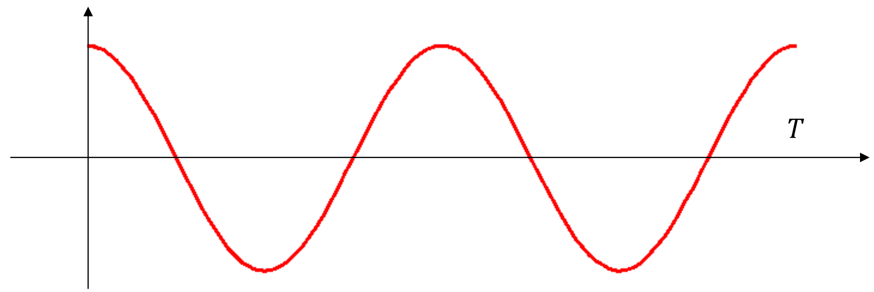
$$t \to \cos\left(\frac{2\pi}{T} \cdot t\right) = \cos(2\pi f_0 \cdot t)$$



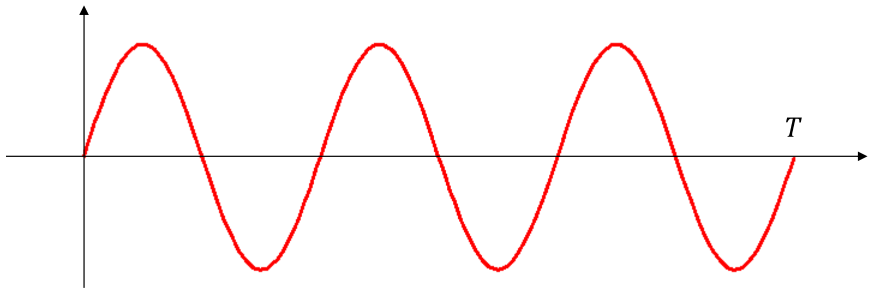
$$t \to \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(2 \cdot 2\pi f_0 \cdot t)$$



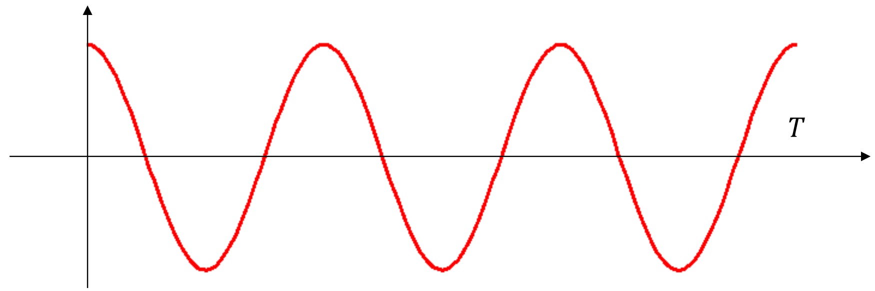
$$t \to \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



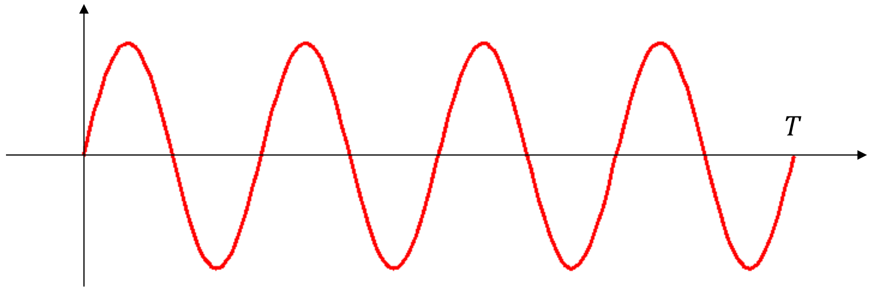
$$t \to \sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(3 \cdot 2\pi f_0 \cdot t)$$



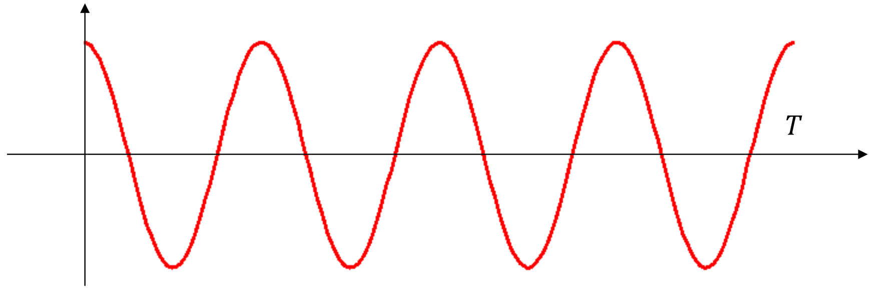
$$t \to \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(3 \cdot 2\pi f_0 \cdot t)$$



$$t \to \sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(4 \cdot 2\pi f_0 \cdot t)$$



$$t \to \cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(4 \cdot 2\pi f_0 \cdot t)$$

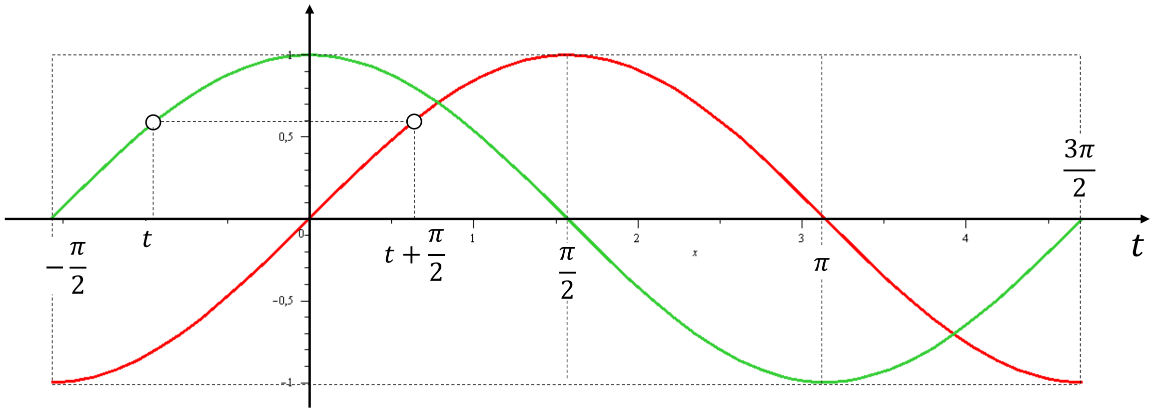


About sin and cos functions

Remark: Since

$$\cos t = \sin\left(t + \frac{\pi}{2}\right)$$

the trigonometric system can be written only with sin functions.



Three equivalent formulas are used to describe harmonic vibrations

$$A \cdot \sin(\omega \cdot t + \varphi) = A \cdot \sin(2\pi f \cdot t + \varphi) = A \cdot \sin\left(\frac{2\pi}{T} \cdot t + \varphi\right),$$

where the physical quantities are

– ω is the angular frequency (angular velocity in physics) $\omega = \left[\frac{rad}{s}\right]$

– f is the frequency $f = \left[\frac{1}{s}\right] = \left[\frac{rad}{s}\right] = [Hz]$

– T is the period $T = [s]$

– φ is the phase $\varphi = [rad]$.

A linear combination of sin and cos functions of the same frequency can be written as a "shifted" sin function of the common frequency as follows

$$A \cdot \sin(\omega \cdot t) + B \cdot \cos(\omega \cdot t) = \sqrt{A^2 + B^2} \cdot \sin(\omega \cdot t + \varphi),$$

where

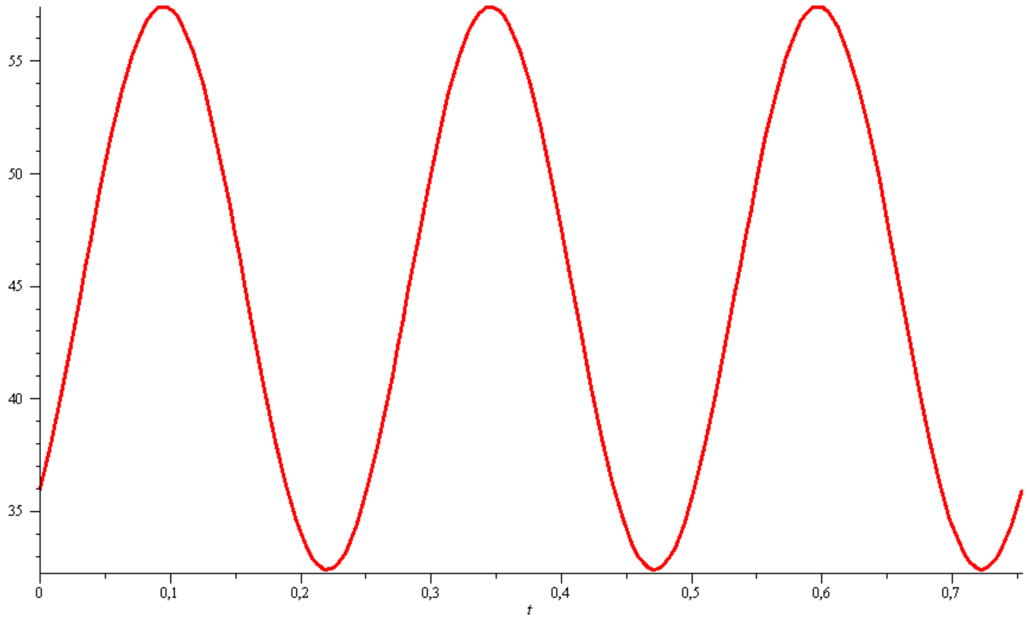
$$\varphi = \begin{cases} \operatorname{arctg}\dfrac{B}{A}, & \text{ha } A \geq 0 \\ \operatorname{arctg}\dfrac{B}{A} \pm \pi, & \text{ha } A < 0 \end{cases}$$

Consequently, a decomposition with respect to the trigonometric can be written with help of shifted sin functions, thus one frequency cannot appear twice.

Example

$$x(t) = 12.5 \cdot \sin(25t - 0.8) + 44.9, \quad t \in [0,3T]$$

where T is the period.

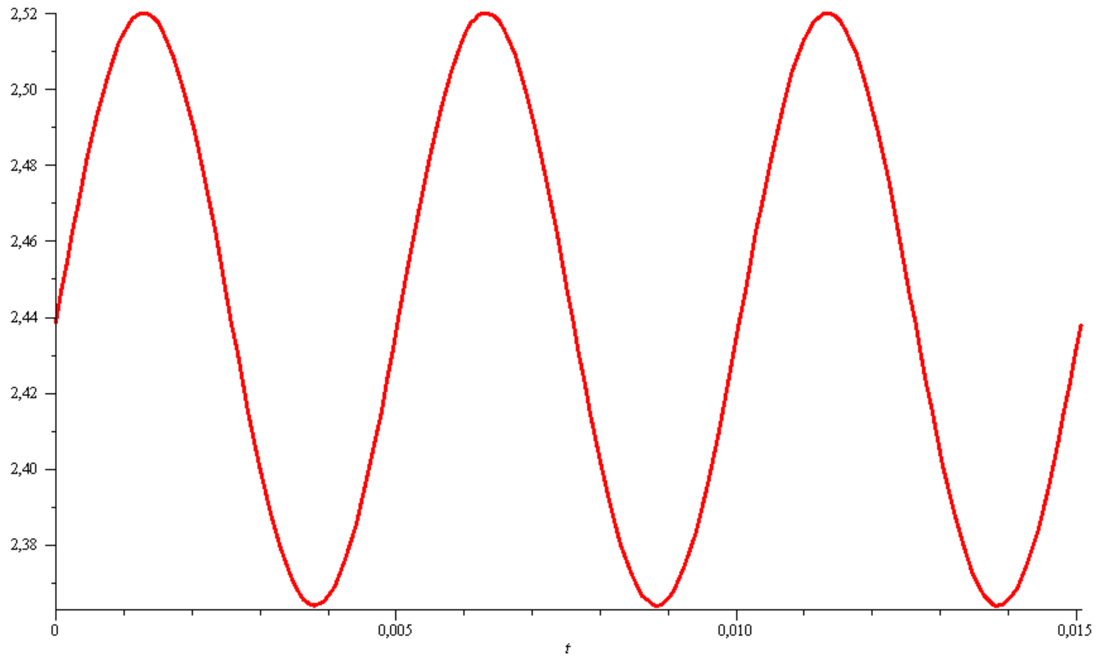


amplitude	12.5
maximum	57.4
minimum	32.4
period	$\frac{2\pi}{25} \approx 0.251$
frequency	$\frac{25}{2\pi} \approx 3.979$
angular frequency	25
phase	-0.8

Example

$$x(t) = 0.078 \cdot \sin(1250t - 0.05) + 2.442, \quad t \in [0,3T]$$

where T is the period.



amplitude	0.078
maximum	2.520
minimum	2.364
period	$\frac{2\pi}{1250} \approx 0.005$
frequency	$\frac{1250}{2\pi} \approx 200$
angular frequency	1250
phase	−0.05

The complex exponential function

Complex sin, cos and exponential functions are defined as power series:

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k+1)!} \cdot z^{2k+1}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k)!} \cdot z^{2k}$$

$$EXP(z) = e^z = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^k, \quad z \in \mathbb{C}$$

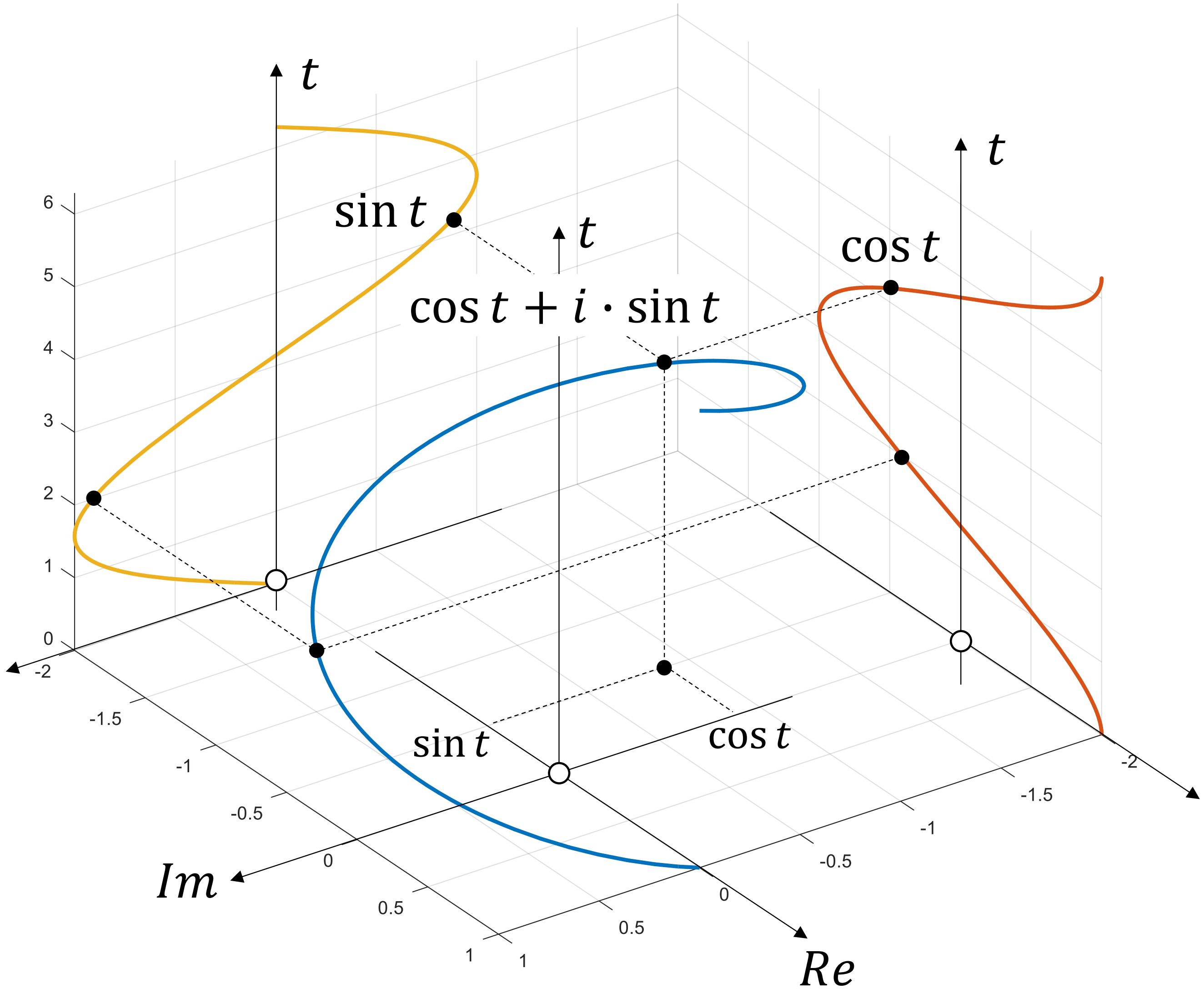
Remark: The real sin, cos and exponential functions are obtained as restrictions to $\mathbb{R}$.

The Euler formula, which comes directly from the definitions, show the connection of the three functions

$$e^{i \cdot t} = \cos t + i \cdot \sin t, \quad t \in \mathbb{R}$$

Seeing the Euler formula, it is not surprising that somehow the complex exponential functions can also be used for decomposition.

Representation of the complex valued function $t \to e^{i \cdot t} = \cos t + i \cdot \sin t$



Values of the complex exponential function can be calculated from values of real trigonometric and exponential functions: for an arbitrary complex number $z = \sigma + s \cdot i, (\sigma, s \in \mathbb{R})$

$$e^z = e^{\sigma + s \cdot i} = e^{\sigma} \cdot e^{s \cdot i} = e^{\sigma} \cdot (\cos s + i \cdot \sin s)$$

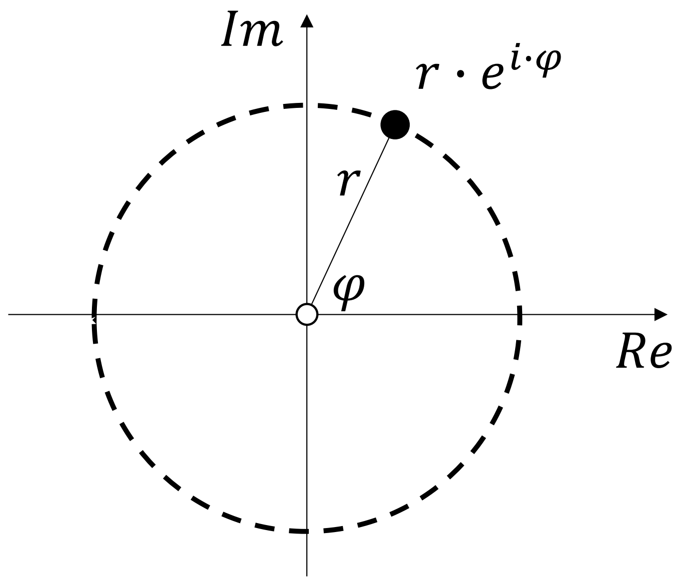
Since e^{σ} is a positive real number and

$$\left|e^{s \cdot i}\right| = |\cos s + i \cdot \sin s| = \sqrt{\cos^2 s + \sin^2 s} = 1,$$

in formula

$$e^z = e^{\sigma} \cdot e^{s \cdot i}$$

$r = e^{\sigma}$ is the norm and $\varphi = s$ is the argument ('angle') of e^z.



‘Frequency’ of the complex exponential function

Functions

$$t \to e^{i \cdot 2\pi \cdot f \cdot t}, \quad t, f \in \mathbb{R}$$

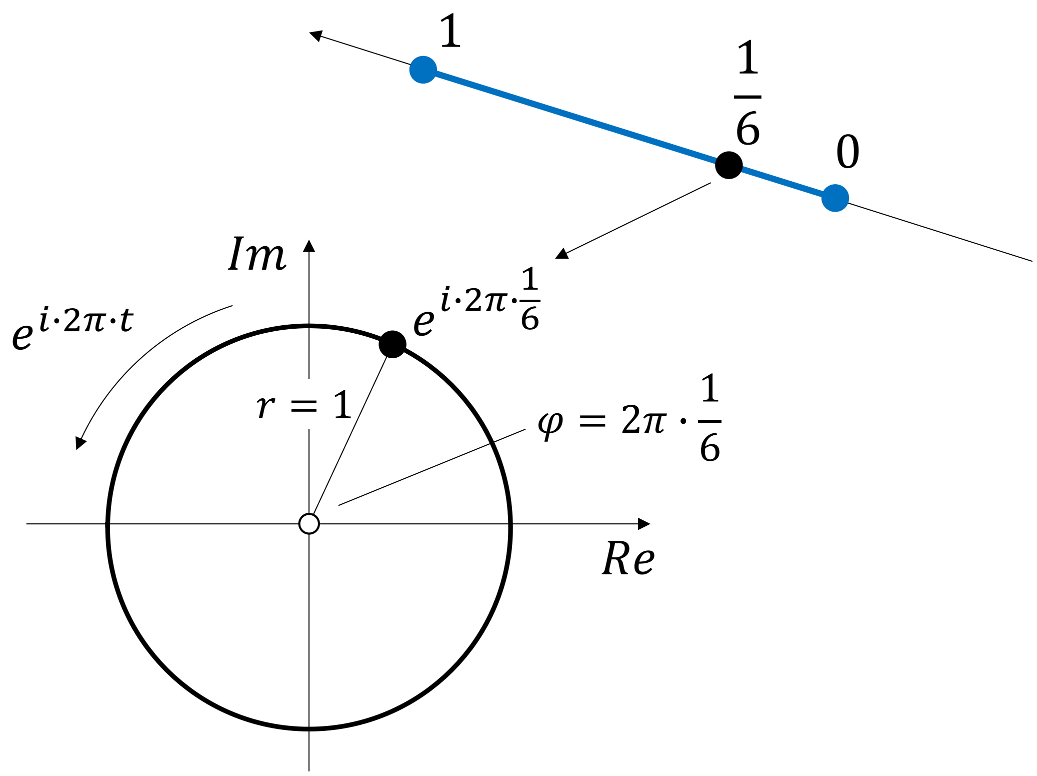
have an important role in the topic of Fourier series and Fourier transforms.

Function $t \to e^{i \cdot 2\pi \cdot t}, t \in \mathbb{R}$ is 1-periodic.

The range of function

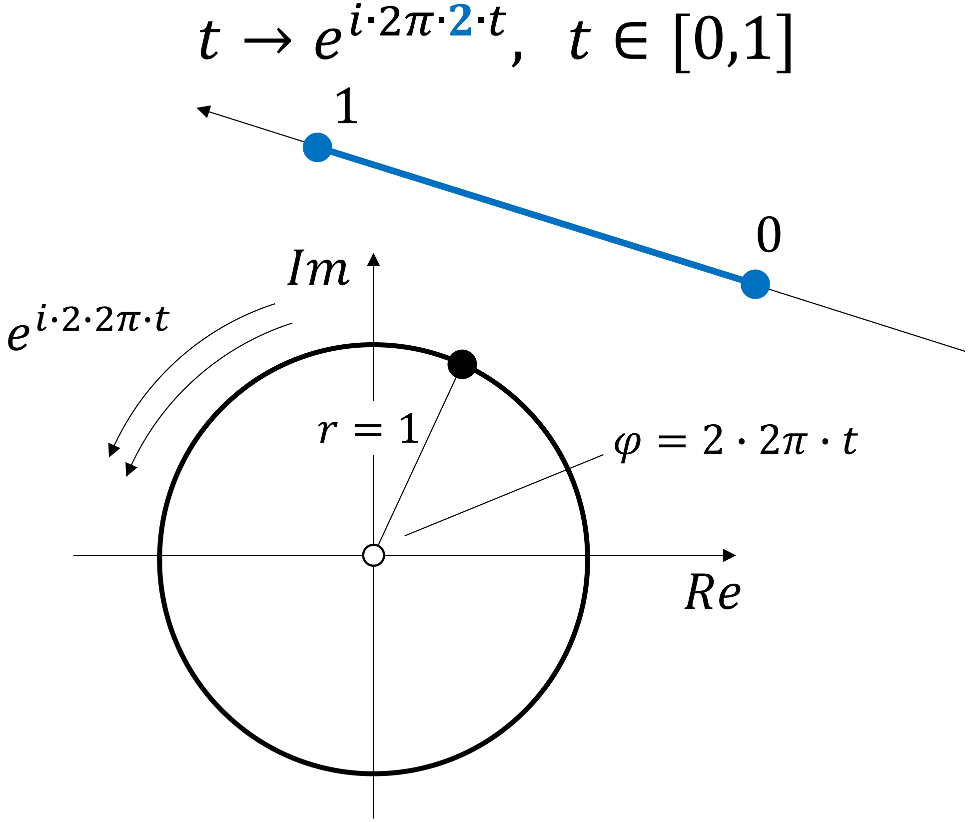
$$t \to e^{i \cdot 2\pi \cdot t}, t \in [0,1]$$

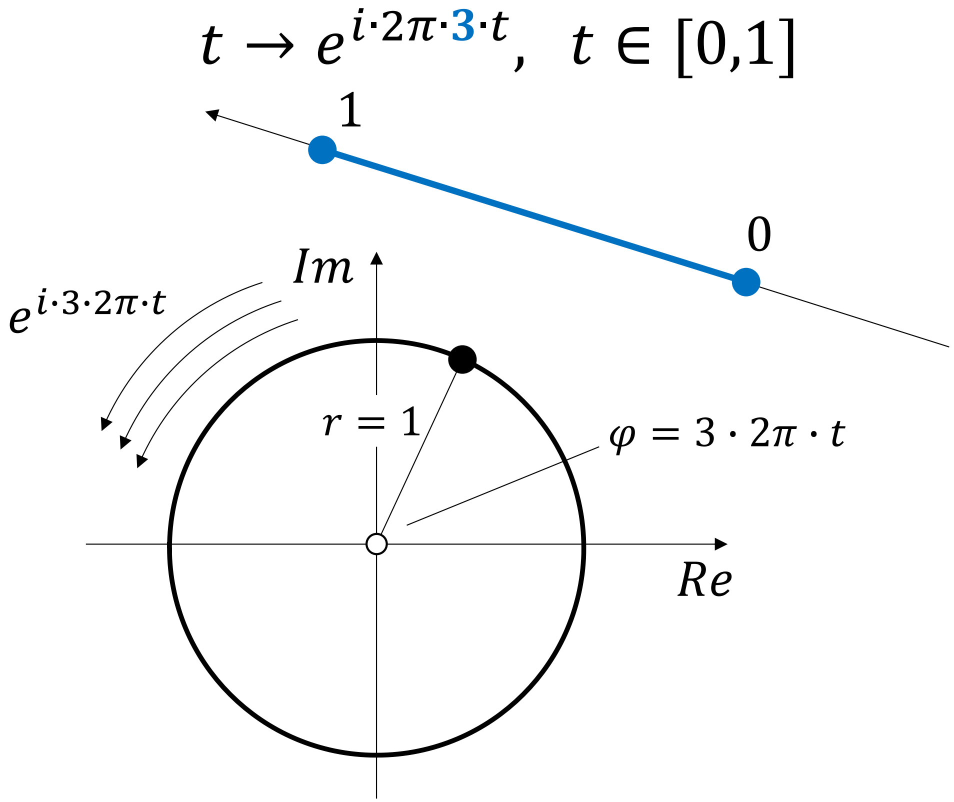
is the unit circle of the complex plane.



The period of function $t \to e^{i \cdot \boldsymbol{f} \cdot 2\pi \cdot t}$ is $T = 1/f$.

Considering $t \to e^{i \cdot f \cdot 2\pi \cdot t}$ as a 'position-time function' in the complex plane $f = 1/T$ can be called 'rotational frequency' which gives the number of rotations per second.





Remark: Since $\omega = 2\pi \cdot f$, we can write $e^{i \cdot \boldsymbol{f} \cdot \mathbf{2\pi} \cdot t} = e^{i \cdot \boldsymbol{\omega} \cdot t}$ as well.

For fixed N, the following values of the complex exponential function are used when calculating the discrete Fourier transform:

$$e^{i\cdot 2\pi\cdot k\cdot\frac{n}{N}},\quad n=0,1,\dots,N-1$$

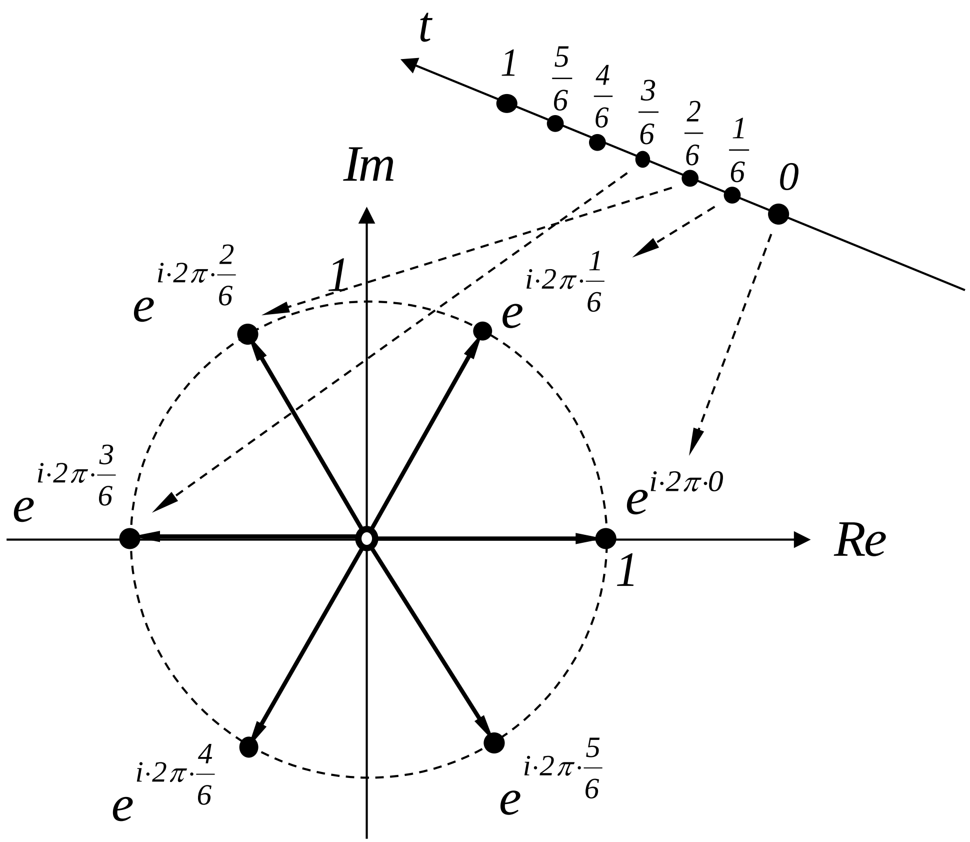
k values give the different ‘rotational frequencies’

$\frac{n}{N}\in[0,1]$ values are ‘discrete’ time values

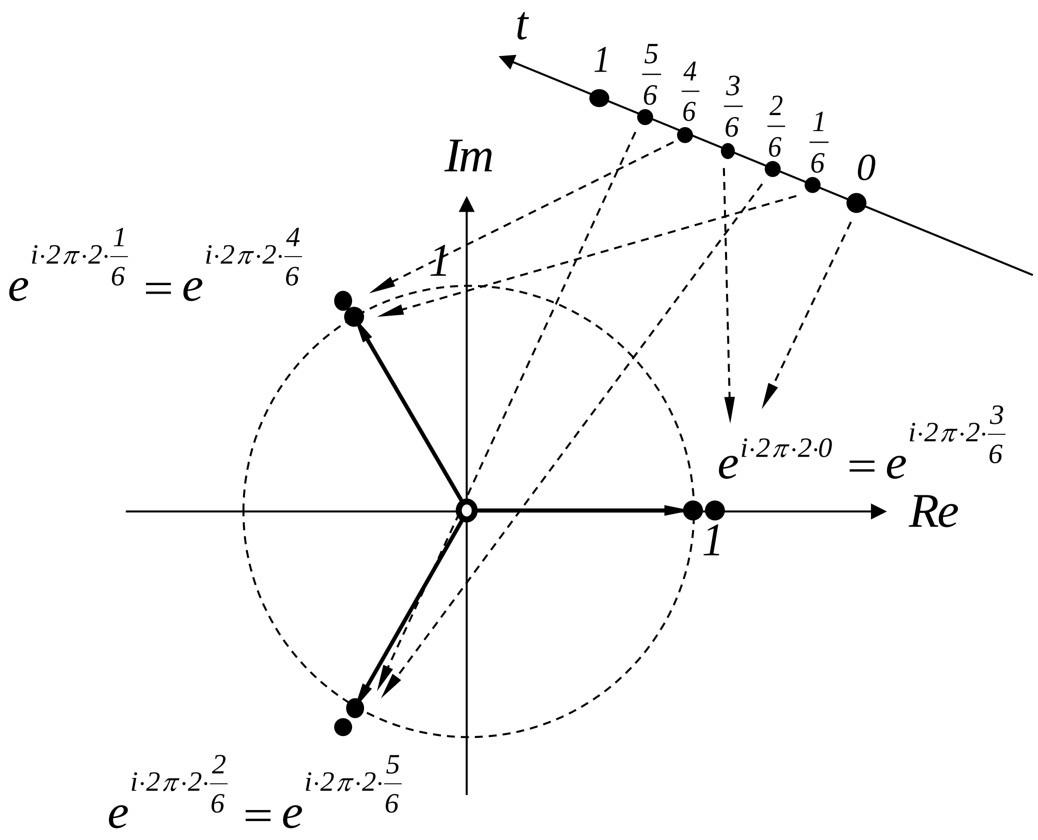
Example

Plot values $e^{i\cdot 2\pi\cdot k\cdot\frac{n}{6}}, n = 0, \dots, 5$ for $k = 1, \dots, 3$ on the complex plane.

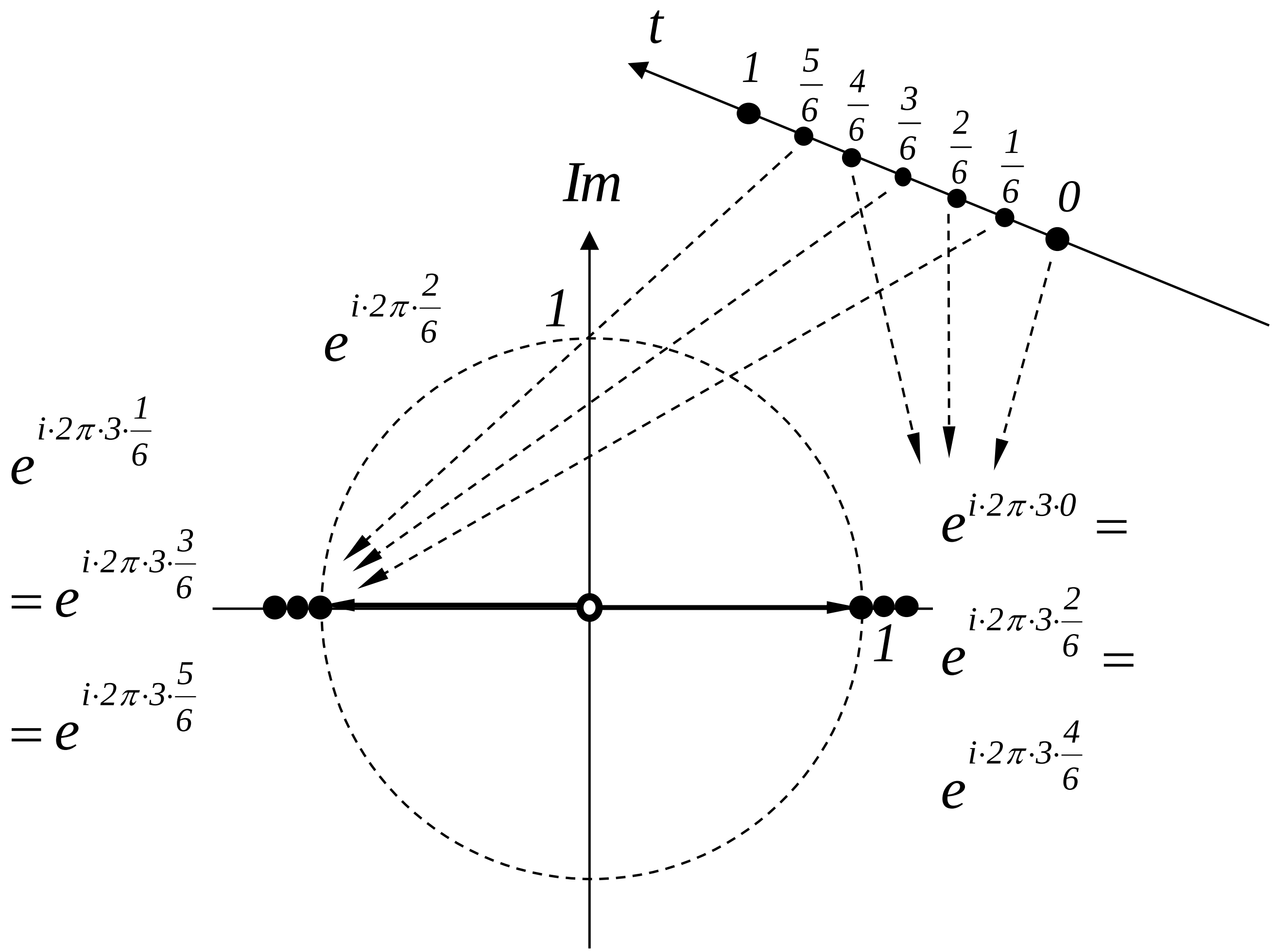
Case $k = 1, n = 0, \dots, 5$



Case $k = 2, n = 0, \dots, 5$



Case $k = 3$, $n = 0, \dots, 5$



Example: Show that

$$\sum_{n=0}^{5} e^{i\cdot 2\pi\cdot\frac{n}{6}} = 0$$

$$\sum_{n=0}^{5} e^{i\cdot 2\pi\cdot\frac{n}{6}} = e^0 + e^{i\cdot\frac{2\pi}{6}} + e^{i\cdot\frac{4\pi}{6}} + e^{i\cdot\frac{6\pi}{6}} + e^{i\cdot\frac{8\pi}{6}} + e^{i\cdot\frac{10\pi}{6}} =$$

$$= e^0 + e^{i\cdot\frac{\pi}{3}} + e^{i\cdot\frac{2\pi}{3}} + e^{i\cdot\pi} + e^{i\cdot\frac{4\pi}{3}} + e^{i\cdot\frac{5\pi}{3}} =$$

$$= 1 + \left(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3}\right) + \left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right) +$$

$$+(\cos\pi + i\cdot\sin\pi) + \left(\cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3}\right) + \left(\cos\frac{5\pi}{3} + i\cdot\sin\frac{5\pi}{3}\right) =$$

$$= 1 + \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} - \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} - 1 - \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2} + \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2} = 0$$

Remark: It can be proven that $\sum_{n=0}^{N-1} e^{i\cdot 2\pi\cdot\frac{n}{N}} = 0$ holds for all positive integers N.

The Concept of Hilbert Spaces

Let X be a real or complex linear space. A function $\langle\quad\rangle: X \times X \to \mathbb{C}$ is called inner product if

$$\mathbb{R} \ni \langle x, x\rangle \geq 0, \ \ \langle x, x\rangle = 0 \iff x = 0$$

$$\langle x, y\rangle = \langle y, x\rangle^*$$ *conjugate symmetry*

$$\langle \lambda \cdot x, y\rangle = \lambda \cdot \langle x, y\rangle$$ *homogeneity in the first argument*

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$ *additivity in the first argument*

hold for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$.

x^* denotes the complex conjugate of $x \in \mathbb{C}$.

Remark

Further properties that follow from the definition

$$\langle x, \lambda \cdot y\rangle = \langle \lambda \cdot y, x\rangle^* = (\lambda \cdot \langle y, x\rangle)^* = \lambda^* \cdot \langle y, x\rangle^* = \lambda^* \cdot \langle x, y\rangle$$

conjugate homogeneity in the second argument

$$\langle x, y + z\rangle = \langle y + z, x\rangle^* = \langle y, x\rangle^* + \langle z, x\rangle^* = \langle x, y\rangle + \langle x, z\rangle$$

additivity in the second argument

Remark: If the inner product is defined as a real-valued function $\langle\quad\rangle: X \times X \to \mathbb{R}$ then symmetry $\langle x, y\rangle = \langle y, x\rangle$ holds.

The pair $(X, \langle\quad\rangle)$ is called inner product space.

An inner product space $(X, \langle\quad\rangle)$ is called Hilbert space if X is a complete metric space with respect to the distance function induced by the inner product.

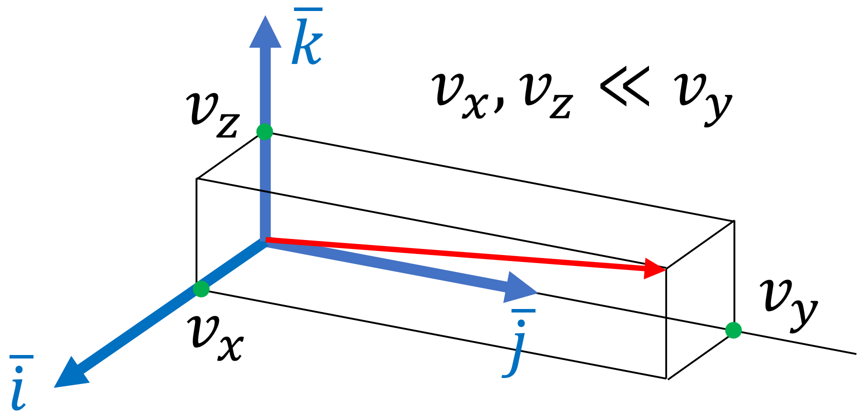
Remark

Each Hilbert space $(X, \langle\quad\rangle)$ is a normed space with the norm

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad x \in X.$$

The value of inner product characterizes the 'similarity' of elements in a Hilbert space.

The higher the value of the inner product is, the more 'similar' the two elements are.



Finite Dimensional Hilbert spaces

A Hilbert space $(X, \langle \quad \rangle)$ is finite dimensional if X is a finite dimensional linear space.

Let n be a positive integer and suppose that X is an n-dimensional Hilbert space.

A system of vectors $\{b_1, \dots, b_k\} \subset X, k \in \mathbb{N}$ is orthogonal if its elements are pairwise orthogonal.

The system is orthonormal if orthogonal and normed, that is, the elements are unit vectors.

If an orthogonal (orthonormal) system $\{b_1, \dots, b_n\} \subset X$ is a basis of X, it is called orthogonal (orthonormal) basis of X.

Orthonormal bases have important role in Hilbert spaces:

if $\{b_1, \dots, b_n\} \subset X$ is an orthonormal basis of X and $x \in X$ then

$$x = \sum_{i=1}^{n} \langle x, b_i \rangle \cdot b_i.$$

This sum is also called the decomposition $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$.

The coefficients

$$\langle x, b_i \rangle, \quad i = 1, \dots, n$$

are the coordinates of $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$.

Remark: the statement above is not true for arbitrary bases.

The space of real n-tuples

$\mathbb{R}^n$ is a n-dimensional Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i\,, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, y = (y_1, \dots, y_n) \in \mathbb{R}^n\,.$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are called orthogonal if

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i = 0.$$

The 'natural' orthonormal basis is in $\mathbb{R}^n$ is $\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\right\}$

which is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$ and $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^2$.

The space of complex n-tuples

$\mathbb{C}^n$ (which is an n-dimensional linear space over $\mathbb{C}$) is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i^* \,, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n, y = (y_1, \dots, y_n) \in \mathbb{C}^n \,.$$

$x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are called orthogonal if

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i^* = 0.$$

The Hilbert space of square integrable functions

Orthogonality and Similarity of Functions

Let I be an interval. A function $x: I \to \mathbb{C}$ is square integrable if

$$\int_I |x(t)|^2 < \infty.$$

$| \quad |$ denotes the magnitude (norm) of a complex number. The space of the square integrable functions defined on I is denoted by $L_2(I)$.

Remark

A real valued function $x: I \to \mathbb{R}$ is square integrable if $\int_I \; x^2 < \infty$.

Remark

Square integrable functions are mathematical representations of finite energy signals.

The inner product of functions $x \in L_2(I)$ and $\psi \in L_2(I)$ is

$$\langle x, \psi \rangle = \int_I x \cdot \psi^* = \int_I x(t) \cdot \psi^*(t)\, dt.$$

Remark

If $x \in L_2(I)$ and $\psi \in L_2(I)$ are real-valued functions then $\psi^* = \psi$, and we can write

$$\langle x, \psi \rangle = \int_I x \cdot \psi = \int_I x(t) \cdot \psi(t)\, dt.$$

The norm of function $x \in L_2(I)$ is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_I x \cdot x^*} = \sqrt{\int_I |x|^2}.$$

Remark

If $x \in L_2(I)$ is a real-valued function we can write $\|x\| = \sqrt{\int_I x^2}$.

Functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are orthogonal if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = 0.$$

Remark

The real-valued functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are orthogonal if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi) = 0.$$

Remark

The value of the inner product characterizes the 'similarity' of the functions.

Example

Functions in $L_2([0,2\pi])$

$$x_1(t) = \sin t, \qquad x_2(t) = \cos t,$$

$$x_3(t) = \sin 2t, \qquad x_4(t) = \cos 2t, \qquad t \in [0,2\pi]$$

are pairwise orthogonal, that is, $\langle x_i, x_j \rangle = 0$ if $i \neq j$.

Furthermore, the norm of all the four functions is $\sqrt{\pi}$.

$$\langle x, \psi \rangle = \int_I x \cdot \psi$$

Calculation of $\langle x_1, x_4 \rangle$ and $\langle x_3, x_4 \rangle$ is as follows

$$\langle x_1, x_4 \rangle = \int_0^{2\pi} \sin t \cdot \cos 2t \, dt = \left[-\frac{2}{3} \cdot \cos^3 t - \cos t \right]_0^{2\pi} = 0$$

Details of the integration:

$$\int \sin t \cdot \cos 2t \, dt = \int \sin t \cdot (\cos^2 t - \sin^2 t) \, dt = \int \sin t \cdot (2\cos^2 t - 1) \, dt =$$

$$= -2 \int -\sin t \cdot \cos^2 t \, dt - \int \sin t \, dt = -\frac{2}{3} \cdot \cos^3 t - \cos t$$

$$\langle x_3, x_4\rangle = \int_0^{2\pi} \sin 2t \cdot \cos 2t \, dt = \left[-\frac{1}{8}\cdot \cos 4t\right]_0^{2\pi} = 0$$

Details of the integration:

$$\int \sin 2t \cdot \cos 2t \, dt = \frac{1}{2}\int \sin 4t \, dt = -\frac{1}{8}\cdot \cos 4t$$

$$\|x\| = \sqrt{\int_I x^2}$$

Calculation of squared norms:

$$\|x_1\|^2 = \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2}\cdot \int_0^{2\pi} (1 - \cos 2t)\, dt = \frac{1}{2}\cdot \left[t - \frac{1}{2}\cdot \sin 2t\right]_0^{2\pi} = \pi$$

$$\|x_2\|^2 = \int_0^{2\pi} \cos^2 t \, dt = \frac{1}{2}\cdot \int_0^{2\pi} (1 + \cos 2t)\, dt = \frac{1}{2}\cdot \left[t + \frac{1}{2}\cdot \sin 2t\right]_0^{2\pi} = \pi$$

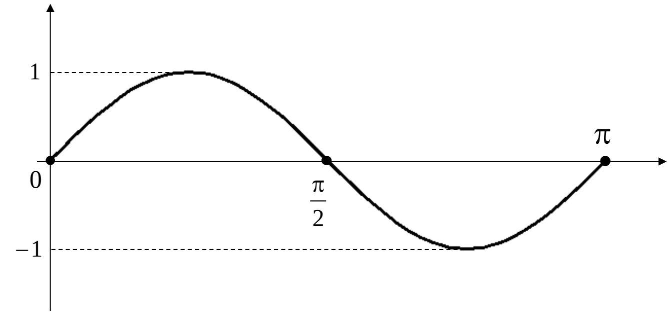
$$\|x_3\|^2 = \int_0^{2\pi} \sin^2 2t\, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 - \cos 4t)\, dt = \frac{1}{2} \cdot \left[t - \frac{1}{4} \cdot \sin 4t \right]_0^{2\pi} = \pi$$

$$\|x_4\|^2 = \int_0^{2\pi} \cos^2 2t\, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 + \cos 4t)\, dt = \frac{1}{2} \cdot \left[t + \frac{1}{4} \cdot \sin 4t \right]_0^{2\pi} = \pi$$

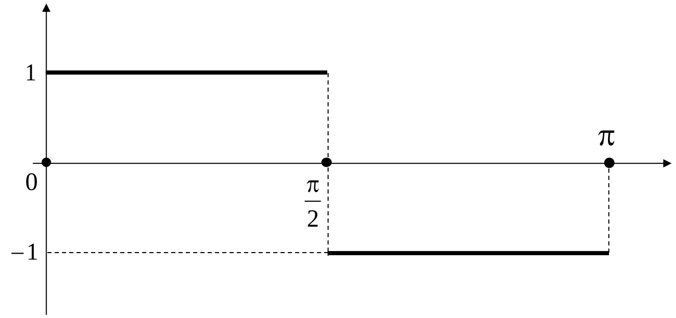
Thus $\|x_i\| = \sqrt{\pi}, i = 1,2,3,4.$

Example: Consider the following functions in $L_2([0,\pi])$:

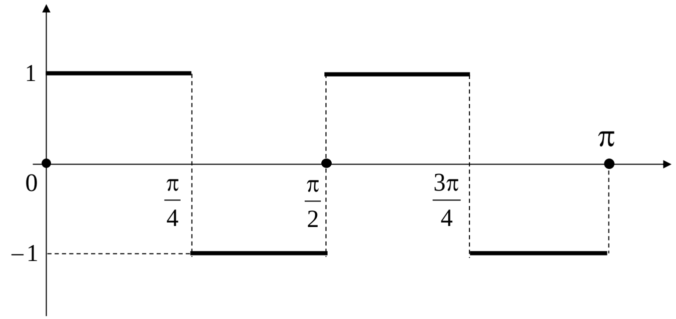
$$\psi(t) = \sin 2t\,,\, t \in [0,\pi]$$



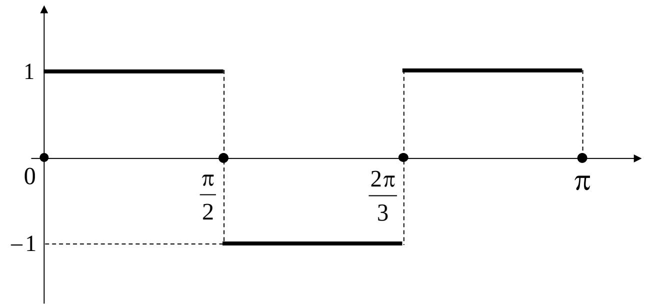
$$x_1(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{\pi}{2}\right[\\ -1 & \text{if } t \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$



$$x_2(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{\pi}{4}\right[\text{ or } t \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right[\\ -1 & \text{if } t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right[\text{ or } t \in \left[\frac{3\pi}{4}, \pi\right] \end{cases}$$



$$x_3(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{\pi}{3}\right[\text{ or } t \in \left[\frac{2\pi}{3}, \pi\right] \\ -1 & \text{if } t \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right[\end{cases}$$



Calculate the inner product of ψ with x_1, x_2 and x_3, respectively, and compare the similarity of ψ with the three functions.

The inner products:

$$\langle x_1, \psi \rangle = \int_0^{\pi} x_1(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{2}} \sin 2t\, dt - \int_{\frac{\pi}{2}}^{\pi} \sin 2t\, dt = -\frac{1}{2} \cdot [\cos 2t]_0^{\frac{\pi}{2}} + \frac{1}{2} \cdot [\cos 2t]_{\frac{\pi}{2}}^{\pi} = 2$$

$$\langle x_2, \psi \rangle = \int_0^{\pi} x_2(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{4}} \sin 2t\, dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2t\, dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2t\, dt - \int_{\frac{3\pi}{4}}^{\pi} \sin 2t\, dt = 0$$

$$\langle x_3, \psi \rangle = \int_0^{\pi} x_3(t) \cdot \psi(t)\, dt = \int_0^{\frac{\pi}{3}} \sin 2t\, dt - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin 2t\, dt + \int_{\frac{2\pi}{3}}^{\pi} \sin 2t\, dt = 1$$

That is, $0 = \langle x_2, \psi \rangle < \langle x_3, \psi \rangle < \langle x_1, \psi \rangle$.

This result implies that the similarity is the highest between x_1 and ψ, while x_2 and ψ are not similar (actually, they are orthogonal).

Exercise

Show that functions

$$x_1(t) = \sin\left(\frac{6\pi}{T} \cdot t\right) \quad \text{and} \quad x_2(t) = \cos\left(\frac{6\pi}{T} \cdot t\right)$$

are orthogonal in $L_2([0,T])$ space.

Give the norm of x_2.

$$\langle x_1, x_2 \rangle = \int_0^T \left(\sin\left(\frac{6\pi}{T} \cdot t\right) \cdot \cos\left(\frac{6\pi}{T} \cdot t\right)\right) dt = \frac{1}{2} \cdot \int_0^T \sin\left(\frac{12\pi}{T} \cdot t\right) dt =$$

$$= -\frac{1}{2} \cdot \frac{T}{12\pi} \cdot \left[\cos\left(\frac{12\pi}{T} \cdot t\right)\right]_0^T = -\frac{T}{24\pi} \cdot (1 - 1) = 0$$

$$\|x_2\|^2 = \int_0^T \cos^2\left(\frac{6\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \int_0^T 1 + \cos\left(\frac{12\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \left[t + \frac{T}{12\pi} \cdot \sin\left(\frac{12\pi}{T} \cdot t\right)\right]_0^T = \frac{T}{2}$$

$$\|x_2\| = \sqrt{T/2}$$

Example

Show that functions

$$x_1(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{6\pi}{T} \cdot t} \quad \text{and} \quad x_2(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{10\pi}{T} \cdot t}$$

are orthogonal in $L_2([0,T])$ space. Give the norm of x_2.

$$\langle x, \psi \rangle = \int_I x \cdot \psi^*$$

$$\langle x_1, x_2 \rangle = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{6\pi}{T} \cdot t} \cdot \frac{1}{\sqrt{T}} \cdot \boldsymbol{e^{-i \cdot \frac{10\pi}{T} \cdot t}} \right) dt = \frac{1}{T} \cdot \int_0^T e^{i \cdot \frac{-4\pi}{T} \cdot t}\, dt =$$

$$= \frac{1}{T} \cdot \frac{1}{i \cdot \frac{-4\pi}{T}} \cdot \left[e^{i \cdot \frac{-4\pi}{T} \cdot t} \right]_0^T = \frac{-1}{4\pi \cdot i} \cdot \left(e^{-4\pi \cdot i} - 1 \right) = 0$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_I x \cdot x^*}$$

$$\|x_2\|^2 = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i \cdot \frac{10\pi}{T} \cdot t} \cdot \frac{1}{\sqrt{T}} \cdot \boldsymbol{e^{-i \cdot \frac{10\pi}{T} \cdot t}} \right) dt = \int_0^T \frac{1}{T} dt = 1$$

Orthonormal Systems

Let I be an interval. A sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}} \subset L_2(I)$ is orthonormal if its elements are pairwise orthogonal and the norm of each element is 1. An orthonormal sequence is also called orthonormal system.

The Fourier coefficients of a function $x \in L_2(I)$ with respect to the orthonormal system $\{\varphi_j\}_{j\in\mathbb{N}} \subset L_2(I)$ are

$$\hat{x}_k = \langle x, \varphi_k \rangle = \int_I x \cdot \varphi_k^*, \quad k \in \mathbb{N}$$

The series of functions

$$\mathcal{FS}(x) = \sum_{k=1}^{\infty} \hat{x}_k \cdot \varphi_k = \sum_{k=1}^{\infty} \langle x, \varphi_k \rangle \cdot \varphi_k$$

is called the Fourier series of x with respect to the orthonormal system $\{\varphi_j\}_{j\in\mathbb{N}}$.

Connection between $x \in L_2(I)$ and $\mathcal{FS}(x)$ is important question in the Fourier theory.

From the point of view of engineering practice, it is generally enough to know that the Fourier series of a piecewise continuous function

– converges to the value of the function at every point t where the function is continuous and
– converges to the midpoint of the discontinuity (the average of the left- and right-hand limits) wherever the function is discontinuous.

The Parseval equality

$$\|x\|^2 = \sum_{k=-\infty}^{\infty} |\hat{x}_k|^2$$

states that the square norm of a function (energy content of a signal) can be calculated directly from its Fourier coefficients.

The Trigonometric System

The Orthonormal Trigonometric System

Let $T > 0$. System of functions

$$\left\{\mathrm{CONST}(\mathrm{t}) = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k} \in \mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

T-periodic functions

$$t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(\frac{2\pi}{T} \cdot t\right) \quad \text{and} \quad t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(\frac{2\pi}{T} \cdot t\right)$$

of 'frequency' $f_0 = \frac{1}{T}$ are called the basic functions of the system, while T/k-periodic functions

$$t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad t \to \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad k = 2,3, \ldots$$

of frequency $k \cdot f_0 = k/T$ are the harmonics.

The Fourier series of a function $x \in L_2([0,T])$ with respect to the orthonormal trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{A}_0 \cdot \mathrm{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \mathrm{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \mathrm{SIN}_k(t),$$

where

$$\hat{A}_0 = \langle x, \mathrm{CONST} \rangle = \int_0^T x(t) \cdot \mathrm{CONST}(\mathrm{t})\, dt = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} dt$$

$$\hat{A}_k = \langle x, \mathrm{COS}_k \rangle = \int_0^T x(t) \cdot \mathrm{COS}_k(t)\, dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \ \ k = 1,2,\dots$$

$$\hat{B}_k = \langle x, \mathrm{SIN}_k \rangle = \int_0^T x(t) \cdot \mathrm{SIN}_k(t)\, dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) dt, \ \ k = 1,2,\dots$$

$\hat{A}_0, \hat{A}_k$ and $\hat{B}_k, k = 1,2,\dots$ are the Fourier coefficients of x with respect to the orthonormal trigonometric system.

Remark

When calculating the Fourier coefficients of a T-periodic function we can take the integrals on any interval of length T.

E.g. we often do the calculations on interval $\left[-\frac{T}{2},\frac{T}{2}\right]$.

Remark

If function x is odd, then $\hat{A}_k = 0$, $k = 0,1,2,\ldots$

(no constant or cos *function in the decomposition = in the Fourier series)*

If x is even, then $\hat{B}_k = 0$, $k = 1,2,\ldots$

(no sin *function in the decomposition = in the Fourier series)*

The Parseval's equality in the case of the orthonormal trigonometric system is

$$\|x\|^2 = \int_0^T x^2 = \hat{A}_0^2 + \sum_{k=1}^{\infty} \hat{A}_k^2 + \sum_{k=1}^{\infty} \hat{B}_k^2$$

In the special case $T = 2\pi$ the orthonormal trigonometric system is

$$\left\{\mathrm{CONST}(t) = \frac{1}{\sqrt{2\pi}}, \mathrm{COS}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t), \mathrm{SIN}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right\}_{k \in \mathbb{N}}$$

and the Fourier coefficients of x are

$$\hat{A}_0 = \int_0^{2\pi} x(t) \cdot \frac{1}{\sqrt{2\pi}} dt$$

$$\hat{A}_k = \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t)\right) dt, \quad k = 1,2, \ldots$$

$$\hat{B}_k = \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right) dt, \quad k = 1,2, \ldots$$

Example

Let $T > 0$. Show that the system of functions

$$\left\{\mathrm{CONST}(\mathrm{t}) = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

Solution

$$\int_0^T \mathrm{CONST}^2(t)\, dt = \int_0^T \frac{1}{T} dt = 1$$

$$\int_0^T \mathrm{COS}_k^2(t)\, dt = \int_0^T \frac{2}{T} \cdot \cos^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{T} \cdot \int_0^T \left(1 + \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) dt =$$

$$= \frac{1}{T} \cdot \left[t + \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right]_0^T = 1$$

$$\int_0^T \mathrm{SINC}_k^2(t)\, dt = \int_0^T \frac{2}{T} \cdot \sin^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{T} \cdot \int_0^T \left(1 - \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) \mathrm{dt} =$$

$$= \frac{1}{T} \cdot \left[t - \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right) \right]_0^T = 1$$

$$\int_0^T \mathrm{COS}_k(t) \cdot \mathrm{SIN}_n(t)\, dt = \frac{2}{T} \cdot \int_0^T \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \left[\sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) \right]_0^T +$$

$$+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \left[\cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \right]_0^T =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot (\sin(k \cdot 2\pi) \cdot \sin(n \cdot 2\pi) - \sin 0 \cdot \sin 0) +$$

$$+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot (\cos(k \cdot 2\pi) \cdot \cos(n \cdot 2\pi) - \cos 0 \cdot \cos 0) = 0$$

Details of the integration:

$$\int \cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$\begin{bmatrix} g(t)=\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & g'(t)=n\cdot\frac{2\pi}{T}\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right) \\ f'(t)=\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & f(t)=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{bmatrix}$$

$$=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)-\frac{n}{k}\cdot\int\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$\begin{bmatrix} g(t)=\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & g'(t)=-n\cdot\frac{2\pi}{T}\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right) \\ f'(t)=\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right) & \Longrightarrow & f(t)=-\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right) \end{bmatrix}$$

$$=\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)-$$

$$-\frac{n}{k}\cdot\left(-\frac{1}{k\cdot\frac{2\pi}{T}}\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\cos\left(n\cdot\frac{2\pi}{T}\cdot t\right)-\frac{n}{k}\cdot\int\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)\cdot\sin\left(n\cdot\frac{2\pi}{T}\cdot t\right)dt\right)=$$

$$= \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) +$$

$$+ \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n^2}{k^2} \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt$$

$$\left(1 - \frac{n^2}{k^2}\right) \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)$$

$$\int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)$$

We can show similarly that if $k \neq n$ then

$$\int_0^T \mathrm{SIN}_k(t) \cdot \mathrm{SIN}_n(t)\, dt = 0 \quad \text{and} \quad \int_0^T \mathrm{COS}_k(t) \cdot \mathrm{COS}_n(t)\, dt = 0$$

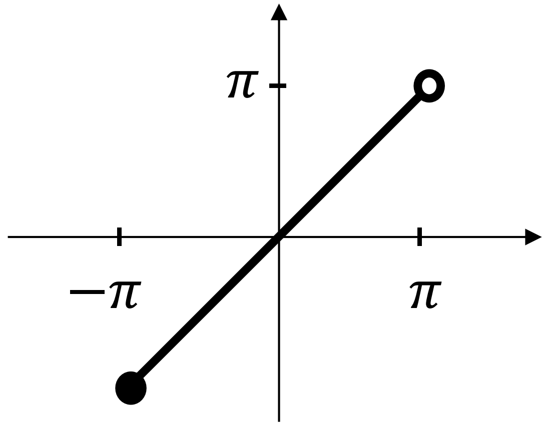
Example

Calculate the Fourier coefficients of the 2π-periodic function x defined as

$$x(t) = t, \quad -\pi \leq t < \pi$$

with respect to the orthonormal trigonometric system.

Use the Parseval's equality to give the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$.



Solution

Since function x is odd, $\hat{A}_k = 0, k = 0,1,2, \dots$

$$\hat{B}_k = \int_{-\pi}^{\pi} t \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t)\right) dt = \frac{1}{\sqrt{\pi}} \cdot \left[-\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t)\right]_{-\pi}^{\pi} =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(\left(-\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) + \frac{1}{k^2} \cdot \sin(k \cdot \pi)\right) - \left(\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) - \frac{1}{k^2} \cdot \sin(k \cdot \pi)\right)\right) =$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(-\frac{2}{k} \cdot \pi \cdot \cos(k \cdot \pi)\right) = 2 \cdot \sqrt{\pi} \cdot (-1)^{k+1} \cdot \frac{1}{k}$$

Details of the calculation (integration by parts):

$$\int t \cdot \sin(k \cdot t)\, dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k} \cdot \int \cos(k \cdot t)\, dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t)$$

$$\begin{bmatrix} g(t) = t & \Longrightarrow & g'(t) = 1 \\ f'(t) = \sin(k \cdot t) & \Longrightarrow & f(t) = -\frac{1}{k} \cdot \cos(k \cdot t) \end{bmatrix}$$

According to the Parseval's equality

$$\|x\|^2 = \int_{-\pi}^{\pi} t^2\, dt = \sum_{k=1}^{\infty} \hat{B}_k^2 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Since $\int_{-\pi}^{\pi} t^2\, dt = \frac{1}{3} \cdot [t^3]_{-\pi}^{\pi} = \frac{2}{3} \cdot \pi^3$ we have

$$\frac{2}{3} \cdot \pi^3 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

that is

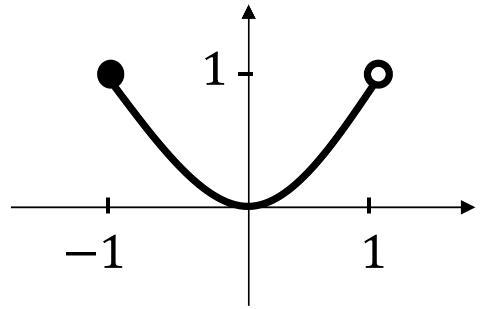
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Example

Calculate the Fourier coefficients of the 2-periodic function x defined as

$$x(t) = t^2, \quad -1 \leq t < 1$$

with respect to the orthonormal trigonometric system.



Solution

Since function x is even, $\hat{B}_k = 0, k = 1,2, \ldots$

$$\hat{A}_0 = \langle x, \mathrm{CONST} \rangle = \int_{-1}^{1} t^2 \cdot \frac{1}{\sqrt{2}} dt = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot [t^3]_{-1}^{1} = \frac{\sqrt{2}}{3}$$

$$\hat{A}_k = \langle x, \mathrm{COS}_k \rangle = \int_{-1}^{1} t^2 \cdot (\cos(k \cdot \pi \cdot t))\, dt =$$

$$= \left[\frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) + \frac{2}{k^2 \cdot \pi^2} \cdot t \cdot \cos(k \cdot \pi \cdot t) - \frac{2}{k^3 \pi^3} \cdot \sin(k \cdot \pi \cdot t) \right]_{-1}^{1} =$$

$$= \left(\frac{1}{k \cdot \pi} \cdot \sin(k \cdot \pi) + \frac{2}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) \right) -$$

$$- \left(- \frac{1}{k \cdot \pi} \cdot \sin(k \cdot \pi) - \frac{2}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) + \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) \right) =$$

$$= \frac{2}{k \cdot \pi} \cdot \sin(k \cdot \pi) + \frac{4}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) - \frac{4}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) = \frac{4}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi)$$

$$\hat{A}_k = \begin{cases} \dfrac{4}{k^2 \cdot \pi^2} & \text{if } k \text{ is even} \\ -\dfrac{4}{k^2 \cdot \pi^2} & \text{if } k \text{ is odd} \end{cases}$$

Details of the calculation (integration by parts):

$$\int t^2 \cdot (\cos(k \cdot \pi \cdot t))\, dt = \frac{1}{k\pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) - \frac{2}{k\pi} \cdot \int t \cdot \sin(k \cdot \pi \cdot t)\, dt =$$

$$\begin{bmatrix} g(t) = t^2 & \Longrightarrow & g'(t) = 2t \\ f'(t) = \cos(k \cdot \pi \cdot t) & \Longrightarrow & f(t) = \dfrac{1}{k\pi} \cdot \sin(k \cdot \pi \cdot t) \end{bmatrix}$$

$$\begin{bmatrix} g(t) = t & \Longrightarrow & g'(t) = 1 \\ f'(t) = \sin(k \cdot \pi \cdot t) & \Longrightarrow & f(t) = -\dfrac{1}{k\pi} \cdot \cos(k \cdot \pi \cdot t) \end{bmatrix}$$

$$= \frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) - \frac{2}{k \cdot \pi} \cdot \left(-\frac{1}{k \cdot \pi} \cdot t \cdot \cos(k \cdot \pi \cdot t) + \frac{1}{k \cdot \pi} \cdot \int \cos(k \cdot \pi \cdot t)\, dt\right) =$$

$$= \frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) + \frac{2}{k^2 \cdot \pi^2} \cdot t \cdot \cos(k \cdot \pi \cdot t) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi \cdot t)$$

The Trigonometric System

Let $T > 0$. System of functions

$$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{k \in \mathbb{N}}$$

is orthogonal (but not orthonormal) in $L_2([0,T])$.

It is called the trigonometric system.

The Fourier coefficients of a function $x \in L_2([0,T])$ with respect to the trigonometric system are

$$\hat{a}_0 = \frac{1}{T} \cdot \int_0^T x(t)\, dt$$

$$\hat{a}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1,2, \dots$$

$$\hat{b}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1,2, \dots$$

The Fourier series of x with respect to the trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{a}_0 + \sum_{k=1}^{\infty} \hat{a}_k \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + \sum_{k=1}^{\infty} \hat{b}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)$$

Remark

If function x is odd, then $\hat{a}_k = 0$, $k = 0,1,2, \ldots$

(no constant or cos *function in the decomposition = in the Fourier series)*

If x is even, then $\hat{b}_k = 0$, $k = 1,2, \ldots$

(no sin *function in the decomposition = in the Fourier series)*

Using the trigonometric equality

$$A \cdot \sin x + B \cdot \cos x = \sqrt{A^2 + B^2} \cdot \sin(x + \varphi)\text{, where } \varphi = \begin{cases} \operatorname{arctg}\dfrac{B}{A}, & \text{if A} \geq 0 \\ \operatorname{arctg}\dfrac{B}{A} + \pi, & \text{if A} < 0 \end{cases}$$

an alternative form of the Fourier series

$$\mathcal{FS}(x)(t) = \hat{c}_0 + \sum_{k=1}^{\infty} \hat{c}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t + \varphi_k\right),$$

is obtained, where

$$\hat{c}_0 = \hat{a}_0,\;\; \hat{c}_k = \sqrt{\hat{a}_k^2 + \hat{b}_k^2}, k = 1,2, \ldots$$

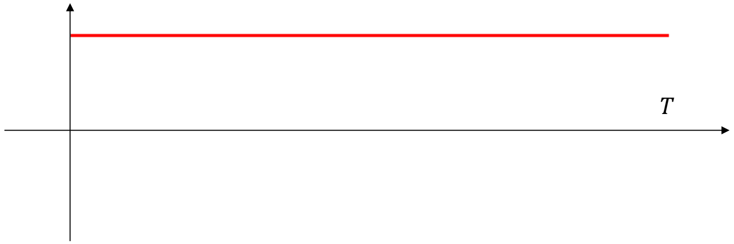
and

$$\varphi_k = \begin{cases} \operatorname{arctg}\dfrac{\hat{b}_k}{\hat{a}_k}, & \text{if } \hat{a}_k \geq 0 \\ \operatorname{arctg}\dfrac{\hat{b}_k}{\hat{a}_k} + \pi, & \text{if } \hat{a}_k < 0 \end{cases}$$

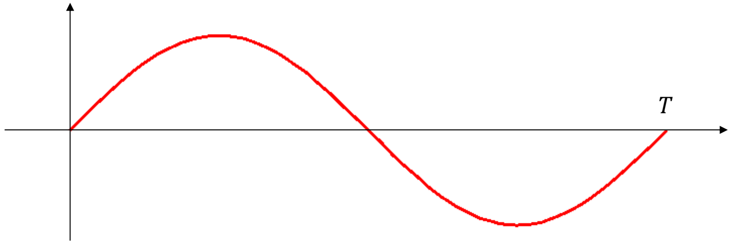
is the phase of the harmonic belonging to index k.

The graph of the constant function and the first four cosine and the first four sine functions of the trigonometric system belonging to the period T on the interval $[0, T]$ are

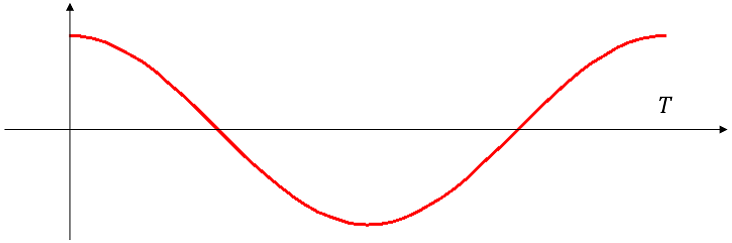
$$t \to 1$$



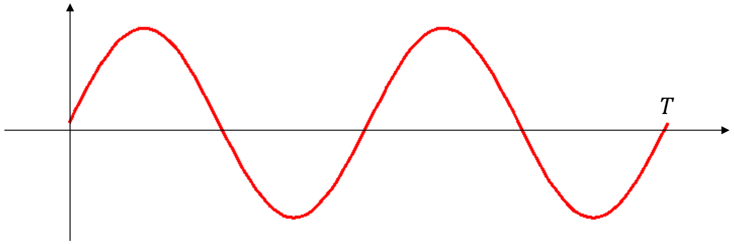
$$t \to \sin\left(\frac{2\pi}{T} \cdot t\right)$$



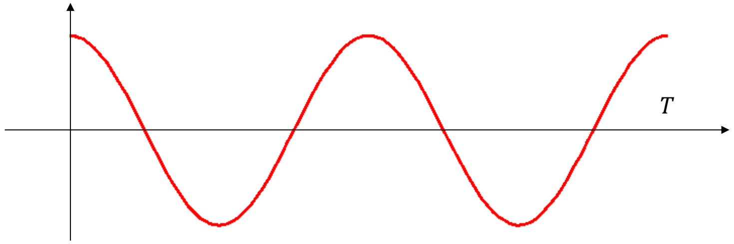
$$t \to \cos\left(\frac{2\pi}{T} \cdot t\right)$$



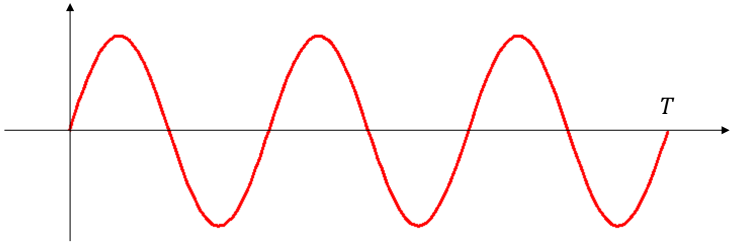
$$t \to \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



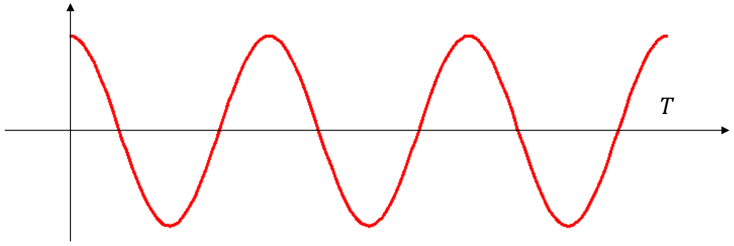
$$t \to \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$$



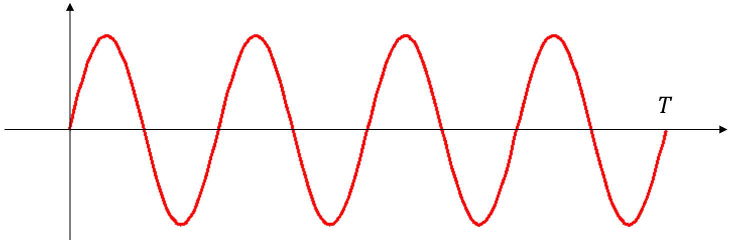
$$t \to \sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$



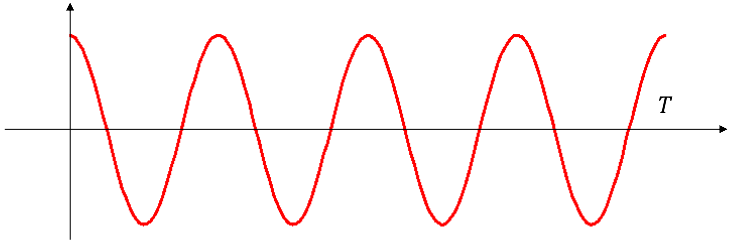
$$t \to \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$



$$t \to \sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



$$t \to \cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



In the special case $T = 2\pi$ the trigonometric system is

$$\{1, \cos(k \cdot t), \sin(k \cdot t)\}_{k \in \mathbb{N}}$$

and the Fourier coefficients are

$$\hat{a}_0 = \frac{1}{2\pi} \cdot \int_0^{2\pi} x(t)\, dt$$

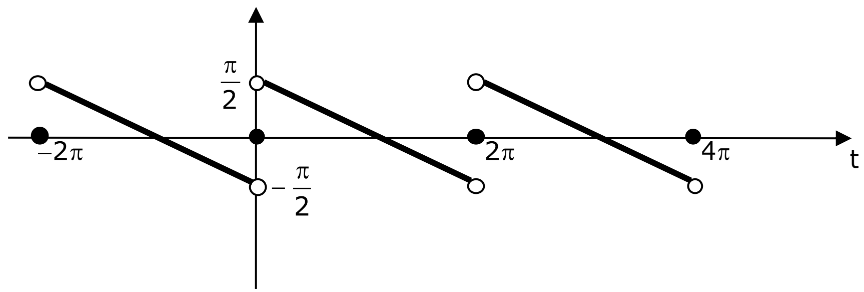
$$\hat{a}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \cos(k \cdot t)\, dt, \quad k = 1,2, \dots$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \sin(k \cdot t)\, dt, \quad k = 1,2, \dots$$

Example

Determine the Fourier coefficients of the 2π-periodic function x defined as

$$x(t) = \begin{cases} 0, & \text{if} \quad t = 0 \\ -\frac{1}{2}t + \frac{\pi}{2}, & \text{if} \quad 0 < t < 2\pi \end{cases}$$



with respect to the trigonometric system.

Solution

Function x is odd, so $\hat{a}_k = 0, k = 0,1,2, \ldots$

We can get the coefficients $\hat{b}_k$ by integration by parts

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \sin(k \cdot t)\, dt =$$

$$= \frac{1}{\pi} \cdot \left[\left(\frac{1}{2k}t - \frac{\pi}{2k}\right) \cdot \cos(k \cdot t) - \frac{1}{2k^2} \cdot \sin(k \cdot t)\right]_0^{2\pi} =$$

$$= \frac{1}{\pi} \cdot \left(\left(\left(\frac{\pi}{k} - \frac{\pi}{2k}\right) \cdot \cos(k \cdot 2\pi) - \frac{1}{2k^2} \cdot \sin(k \cdot 2\pi)\right) - \left(-\frac{\pi}{2k} \cdot \cos 0 - \frac{1}{2k^2} \cdot \sin 0\right)\right) = \frac{1}{k}$$

Details of the calculation (integration by parts):

$$\int\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\sin(k\cdot t)\,dt=-\frac{1}{k}\cdot\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\cos(k\cdot t)-\frac{1}{2k}\cdot\int\cos(k\cdot t)\,dt=$$

$$\begin{bmatrix} g(t)=-\frac{1}{2}t+\frac{\pi}{2} & \Rightarrow & g'(t)=-\frac{1}{2} \\ f'(t)=\sin(k\cdot t) & \Rightarrow & f(t)=-\frac{1}{k}\cdot\cos(k\cdot t) \end{bmatrix}$$

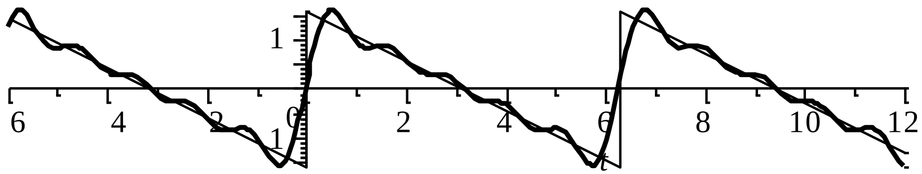
$$=-\frac{1}{k}\cdot\left(-\frac{1}{2}t+\frac{\pi}{2}\right)\cdot\cos(k\cdot t)-\frac{1}{2k^2}\cdot\sin(k\cdot t)=\left(\frac{1}{2k}t-\frac{\pi}{2k}\right)\cdot\cos(k\cdot t)-\frac{1}{2k^2}\cdot\sin(k\cdot t)$$

Since $\hat{a}_k=0, k=0,1,2,\dots$ and $\hat{b}_k=\frac{1}{k}, k=1,2,\dots$ the Fourier series of x is

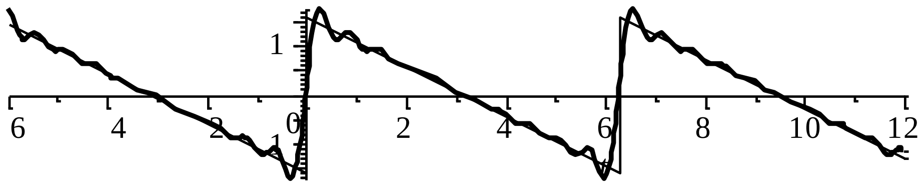
$$\mathcal{FS}x(t)=\sum_{k=1}^{\infty}\frac{\sin(k\cdot t)}{k}.$$

The sum of the first 5 terms and the sum of the first 10 terms in the Fourier series.

$$t\to\sum_{k=1}^{5}\frac{\sin(k\cdot t)}{k}$$



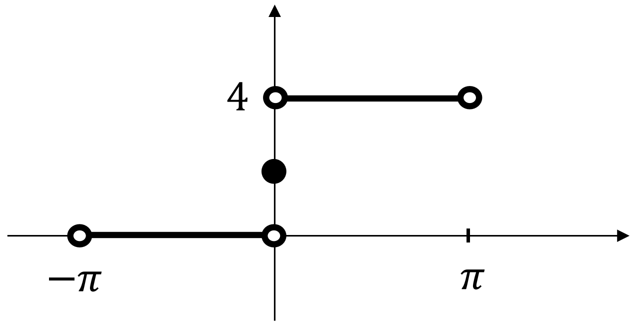
$$t\to\sum_{k=1}^{10}\frac{\sin(k\cdot t)}{k}$$



Example

Determine the Fourier coefficients of the 2π-periodic function x defined as

$$x(t)=\begin{cases}0, & \text{if} \quad -\pi<t<0\\ 2, & \text{if} \quad t=0\\ 4, & \text{if} \quad 0<t<\pi\\ 2, & \text{if} \quad t=\pi\end{cases}$$



with respect to the trigonometric system.

Give the sum of the first 4 terms and the sum of the first 8 terms in the Fourier series.

Solution

$$\hat{a}_0=\frac{1}{2\pi}\cdot\int_0^{\pi}4\,dt=2$$

$$\hat{a}_k=\frac{1}{\pi}\cdot\int_0^{\pi}4\cdot\cos(k\cdot t)\,dt=\frac{4}{k\cdot\pi}\cdot[\sin(k\cdot t)]_0^{\pi}=0$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{\pi} 4 \cdot \sin(k \cdot t)\, dt = \frac{-4}{k \cdot \pi} \cdot [\cos(k \cdot t)]_0^{\pi} = \frac{4}{k \cdot \pi} \cdot (1 - \cos(k \cdot \pi))$$

We have that

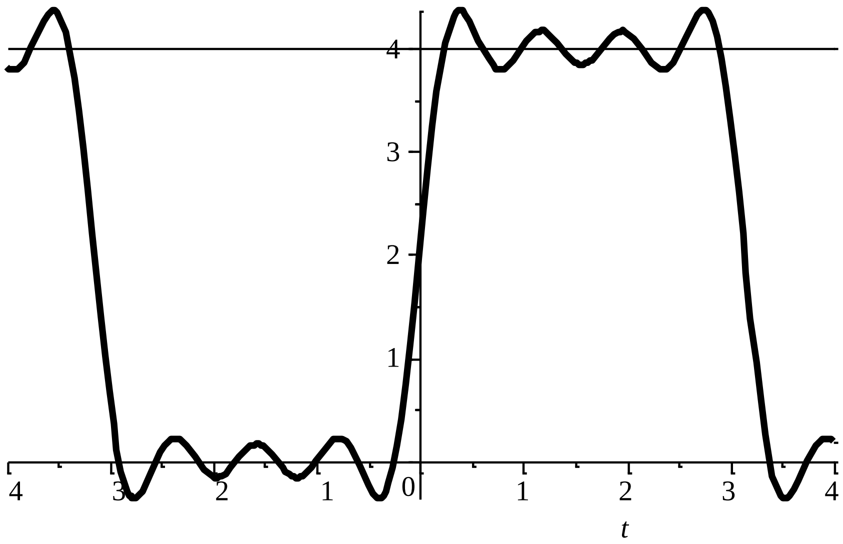
$$\hat{b}_k = \begin{cases} \dfrac{8}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

writing the odd numbers k in the form $k = 2n - 1$ the Fourier series of x is

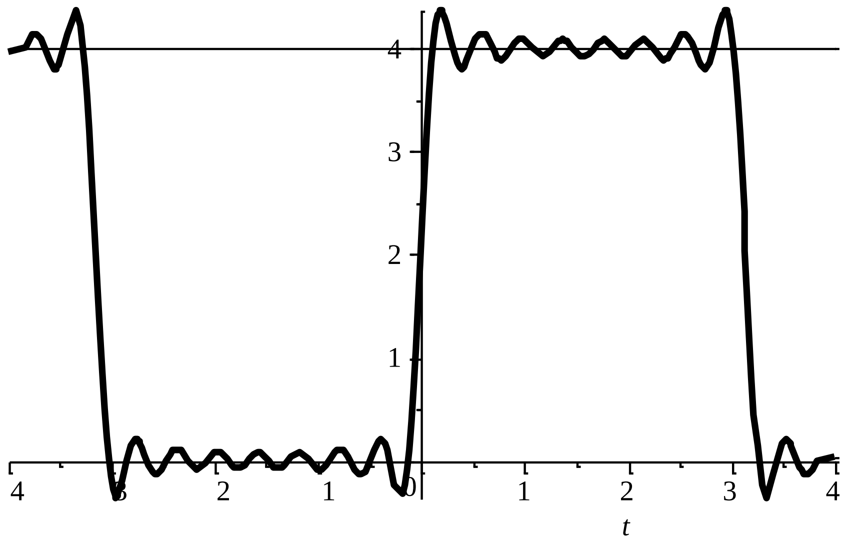
$$x(t) = 2 + \frac{8}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin((2n-1) \cdot t)}{2n - 1}$$

The two partial sums are

$$t \to 2 + \sum_{k=1}^{4} \frac{8}{(2k-1)\cdot\pi} \cdot \sin((2k-1)\cdot t)$$



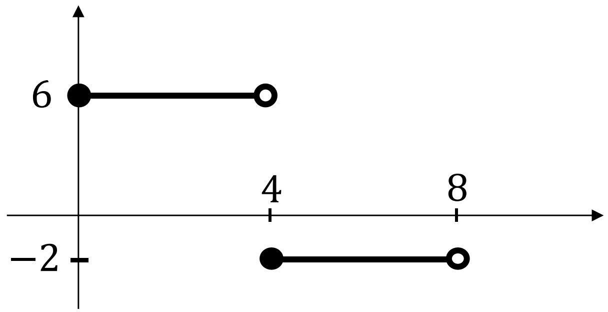
$$t \to 2 + \sum_{k=1}^{8} \frac{8}{(2k-1)\cdot\pi} \cdot \sin((2k-1)\cdot t)$$



Example

Determine the Fourier coefficients of the 8-periodic function x defined as

$$x(t) = \begin{cases} 6 & \text{if } 0 \leq t < 4 \\ -2 & \text{if } 4 \leq t < 8 \end{cases}$$



Solution

$$\hat{a}_0 = \frac{1}{8} \cdot \int_0^4 6\, dt + \frac{1}{8} \cdot \int_4^8 -2\, dt = 2$$

$$\hat{a}_k = \frac{1}{4} \cdot \int_0^4 6 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt =$$

$$= \frac{6}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 - \frac{2}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 = 0$$

$$\hat{b}_k = \frac{1}{4} \cdot \int_0^4 6 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt =$$

$$= \frac{-6}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 + \frac{2}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 =$$

$$= \frac{-6}{k \cdot \pi} \cdot (\cos(k \cdot \pi) - 1) + \frac{2}{k \cdot \pi} \cdot (\cos(2k \cdot \pi) - \cos(k \cdot \pi))$$

$$\hat{b}_k = \begin{cases} \dfrac{16}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

Writing the odd numbers k in the form $k = 2n - 1$ the Fourier series of x is

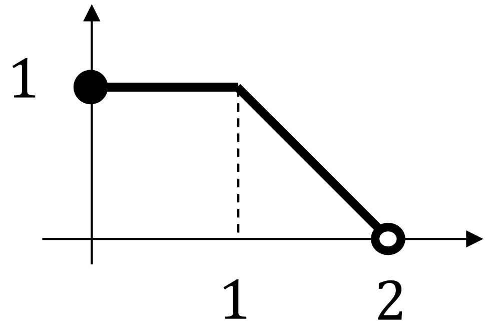
$$x(t) = 2 + \frac{16}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin\left((2n-1) \cdot \frac{\pi}{4} \cdot t\right)}{2n-1}$$

Example

Calculate the Fourier coefficient $\hat{b}_{10}$ of the 2-periodic function x defined as

$$x(t) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ 2 - t, & \text{if } 1 < t < 2 \end{cases}$$

with respect to the trigonometric system.



Solution

$$\hat{b}_{10} = \int_0^2 x(t) \cdot \sin(10 \cdot \pi \cdot t)\, dt = \int_0^1 \sin(10\pi \cdot t)\, dt + \int_1^2 (2 - t) \cdot \sin(10\pi \cdot t)\, dt =$$

$$= -\frac{1}{10\pi} \cdot [\cos(10\pi \cdot t)]_0^1 + \left[\frac{t-2}{10\pi} \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t)\right]_1^2 = \frac{1}{10\pi}$$

Details of the calculation (integration by parts):

$$\int (2 - t) \cdot \sin(10\pi \cdot t)\, dt = -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{10\pi} \cdot \int \cos(10\pi \cdot t)\, dt =$$

$$= -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t)$$

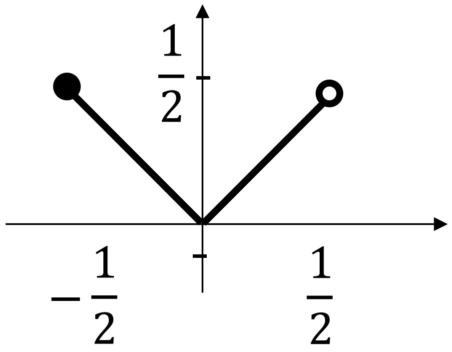
$$\begin{bmatrix} g(t) = 2 - t & \Longrightarrow & g'(t) = -1 \\ f'(t) = \sin(10\pi \cdot t) & \Longrightarrow & f(t) = -\dfrac{1}{10\pi} \cdot \cos(10\pi \cdot t) \end{bmatrix}$$

Example

Calculate the Fourier coefficient $\hat{a}_2$ of the 1-periodic function x defined as

$$x(t) = |t|, \quad -\frac{1}{2} \leq t < \frac{1}{2}$$

with respect to the trigonometric system.



Solution

$$\hat{a}_2 = 2 \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| \cdot \cos(2 \cdot 2\pi \cdot t)\, dt = 4 \cdot \int_0^{\frac{1}{2}} t \cdot \cos(4\pi \cdot t)\, dt =$$

$$= 4 \cdot \left[\frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t)\right]_0^{1/2} =$$

$$= 4 \cdot \left(\frac{1}{4\pi} \cdot \frac{1}{2} \cdot \sin(2\pi) + \frac{1}{16\pi^2} \cdot \cos(2\pi) - \frac{1}{16\pi^2}\right) = 0$$

Details of the calculation (integration by parts):

$$\int t \cdot \cos(4\pi \cdot t)\, dt = \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) - \frac{1}{4\pi} \cdot \int \sin(4\pi \cdot t)\, dt =$$

$$= \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t)$$

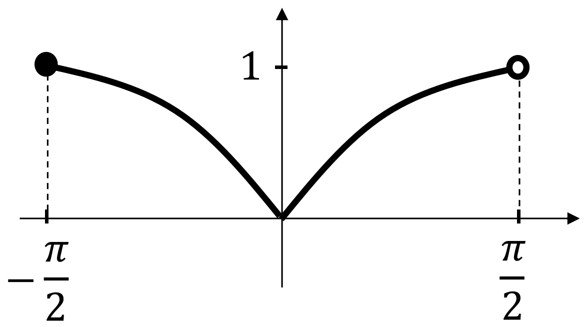
$$\begin{bmatrix} g(t) = t & \Longrightarrow & g'(t) = 1 \\ f'(t) = \cos(4\pi \cdot t) & \Longrightarrow & f(t) = \frac{1}{4\pi} \cdot \sin(4\pi \cdot t) \end{bmatrix}$$

Example

Calculate the Fourier coefficient $\hat{a}_9$ of the π-periodic function x defined as

$$x(t) = |\sin t|, \quad -\frac{\pi}{2} \le t < \frac{\pi}{2}$$

with respect to the trigonometric system.



Solution

$$\hat{a}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt,$$

$$\hat{a}_9 = \frac{2}{\pi} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \cdot \cos(9 \cdot 2 \cdot t)\, dt = \frac{4}{\pi} \cdot \int_0^{\frac{\pi}{2}} \sin t \cdot \cos(18 \cdot t)\, dt =$$

$$= \frac{4}{\pi} \cdot \left[\frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t)\right]_0^{\pi/2} =$$

$$= \frac{4}{\pi} \cdot \left(\frac{18}{323} \cdot \sin\frac{\pi}{2} \cdot \sin(9\pi) + \frac{1}{323} \cdot \cos\frac{\pi}{2} \cdot \cos(9\pi) - \frac{1}{323}\right) = \frac{-4}{323 \cdot \pi}$$

Details of the calculation (integration by parts):

$$\int \sin t \cdot \cos(18 \cdot t)\, dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \int \cos t \cdot \sin(18 \cdot t)\, dt =$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \left(-\frac{1}{18} \cdot \cos t \cdot \cos(18 \cdot t) - \frac{1}{18} \cdot \int \sin t \cdot \cos(18 \cdot t)\, dt\right) =$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t)\, dt$$

$$\begin{bmatrix} g(t) = \sin t & \Longrightarrow & g'(t) = \cos t \\ f'(t) = \cos(18 \cdot t) & \Longrightarrow & f(t) = \frac{1}{18} \cdot \sin(18 \cdot t) \end{bmatrix}$$

$$\begin{bmatrix} g(t) = \cos t & \Longrightarrow & g'(t) = -\sin t \\ f'(t) = \sin(18 \cdot t) & \Longrightarrow & f(t) = -\frac{1}{18} \cdot \cos(18 \cdot t) \end{bmatrix}$$

$$\int \sin t \cdot \cos(18 \cdot t)\, dt$$

$$= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t)\, dt$$

$$\left(1 - \frac{1}{324}\right) \cdot \int \sin t \cdot \cos(18 \cdot t)\, dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t)$$

$$\int \sin t \cdot \cos(18 \cdot t)\, dt = \frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t)$$

Example

Determine the period of the signal

$$x(t) = 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$$

and give the Fourier coefficients $\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3$.

Solution

Period of function $t \to 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)$ is 20, period of function $t \to 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$ is 30.

It is easy to see, that period of their sum is equal to the smallest common multiple of 20 and 30, that is $T = 60$.

Now it is evident that signal x contains two harmonic components, namely

$$t \to 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) = 6 \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right)$$

and

$$t \to 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) = 12 \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right)$$

thus $\hat{b}_3 = 6$ and $\hat{a}_2 = 12$. All other Fourier coefficients are equal to zero.

We can calculate the Fourier coefficients according to the formulas. $T = 60$ thus

$$\hat{b}_3 = \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)\right) \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \int_0^{60} \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \left[\frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right)\right)\right]_0^{60} +$$

$$+ \frac{24}{60} \cdot \left[-\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)\right]_0^{60} = 6$$

Details of the calculation

$$\int \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt = \frac{1}{2} \cdot \int 1 - \cos\left(\frac{2\pi}{10} \cdot t\right) dt = \frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right)\right)$$

$$\int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$= -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \int \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) dt =$$

$$\begin{bmatrix} g(t) = \cos\left(\frac{2\pi}{30}\cdot t\right) & \Longrightarrow & g'(t) = -\frac{2\pi}{30}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right) \\ f'(t) = \sin\left(\frac{2\pi}{20}\cdot t\right) & \Longrightarrow & f(t) = -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) \end{bmatrix}$$

$$\begin{bmatrix} g(t) = \sin\left(\frac{2\pi}{30}\cdot t\right) & \Longrightarrow & g'(t) = \frac{2\pi}{30}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right) \\ f'(t) = \cos\left(\frac{2\pi}{20}\cdot t\right) & \Longrightarrow & f(t) = \frac{20}{2\pi}\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) \end{bmatrix}$$

$$= -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) -$$

$$-\frac{2}{3}\cdot\left(\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt\right) =$$

$$= -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right) + \frac{4}{9}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt$$

$$\frac{5}{9}\cdot\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt = -\frac{20}{2\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{2}{3}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)$$

$$\int\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)dt = -\frac{18}{\pi}\cdot\cos\left(\frac{2\pi}{30}\cdot t\right)\cdot\cos\left(\frac{2\pi}{20}\cdot t\right) - \frac{6}{5}\cdot\sin\left(\frac{2\pi}{30}\cdot t\right)\cdot\sin\left(\frac{2\pi}{20}\cdot t\right)$$

$$\hat{a}_2 = \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)\right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \int_0^{60} \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt =$$

$$= \frac{12}{60} \cdot \left[\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)\right]_0^{60} +$$

$$+ \frac{24}{60} \cdot \left[\frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right)\right)\right]_0^{60} = 12$$

Details of the calculation

$$\int \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{1}{2} \cdot \int 1 + \cos\left(\frac{2\pi}{15} \cdot t\right) dt = \frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right)\right)$$

$$\int \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)$$

(for further details see the calculations above)

Example

Give the spectrum of the following signal

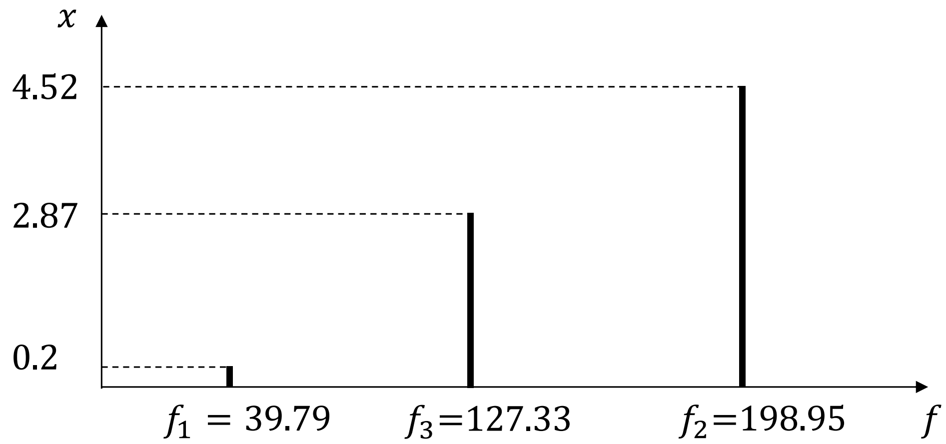
$$x(t) = 0.2 \cdot \sin(250 \cdot t - 5.6) - 4.52 \cdot \sin(1250 \cdot t - 3.2) + 2.87 \cdot \sin(800 \cdot t)$$

Solution

$$\omega_1 = 250 \left[\frac{rad}{s}\right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 39.79[Hz]$$

$$\omega_2 = 1250 \left[\frac{rad}{s}\right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 198.95[Hz]$$

$$\omega_3 = 800 \left[\frac{rad}{s}\right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 127.33[Hz]$$



Example

Give the spectrum of the following signal

$$x(t) = 100 \cdot \sin(5.48 \cdot t - 0.6) + 55 \cdot \sin(6.28 \cdot t - 3) + $$
$$+21 \cdot \sin(7.27 \cdot t + 1) + 66 \cdot \sin(t - 1.9)$$

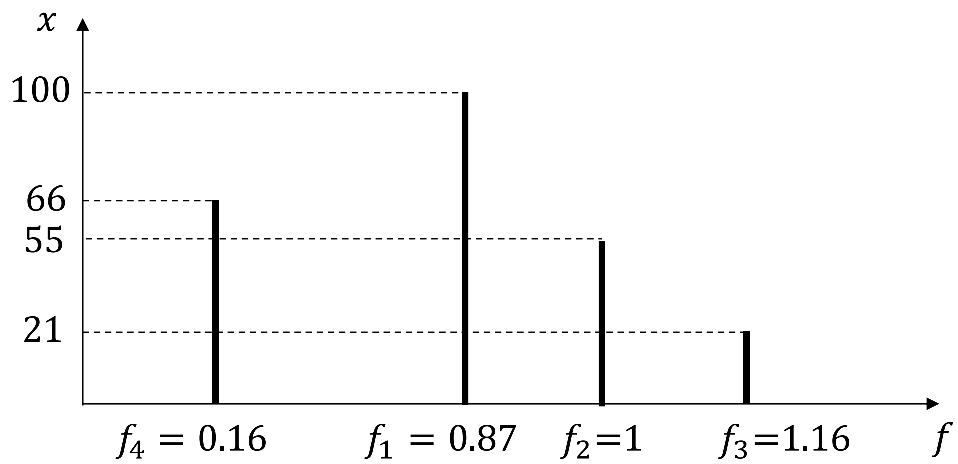
Solution

$$\omega_1 = 5.48 \left[\frac{rad}{s}\right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 0.87[Hz]$$

$$\omega_2 = 6.28 \left[\frac{rad}{s}\right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 1[Hz]$$

$$\omega_3 = 7.27 \left[\frac{rad}{s}\right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 1.16[Hz]$$

$$\omega_4 = 1 \left[\frac{rad}{s}\right] \Rightarrow f_4 = \frac{\omega_4}{2\pi} = 0.16[Hz]$$



The Exponential System

The Orthonormal Exponential System

Let $T > 0$. System of functions

$$\left\{ \mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right\}_{k \in \mathbb{Z}}$$

is orthonormal in $L_2([0, T])$. This system is called orthonormal exponential system.

Remark

In the exponential system index k is from $\mathbb{Z}$, that is, there are negative indices as well. But negative (physical) frequencies do not exist.

The Fourier coefficients of a function $x \in L_2([0, T])$ with respect to the orthonormal exponential system are

$$\hat{X}_k = \langle x, \mathrm{EXP}_k \rangle = \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) dt, \quad k \in \mathbb{Z},$$

The Fourier series (decomposition) of x is

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} \hat{X}_k \cdot \mathrm{EXP}_k = \sum_{k=-\infty}^{\infty} \langle x, \mathrm{EXP}_k \rangle \cdot \mathrm{EXP}_k$$

Remark

When calculating the Fourier coefficients of the T-periodic functions we can take the integrals on any interval of length T.

E.g. we often do the calculations on interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

Example

Show that system of functions

$$\left\{\mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

is orthonormal in $L_2([0,T])$.

Solution

For arbitrary $k \in \mathbb{Z}$ we have

$$\|\mathrm{EXP}_k\|^2 = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right) dt = \int_0^T \frac{1}{T} dt = 1$$

(we used that $\left(e^{i\cdot\alpha}\right)^* = e^{-i\cdot\alpha},\ \alpha \in \mathbb{R}$)

For arbitrary $k, l \in \mathbb{Z}, k \neq l$ we have

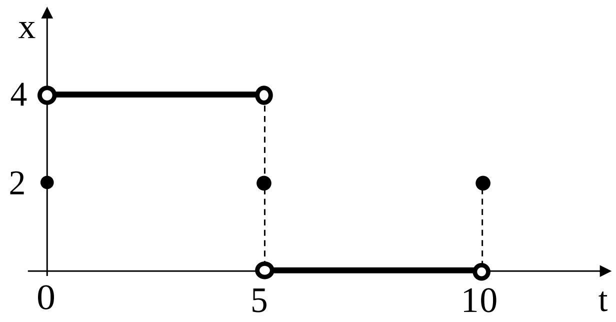
$$\langle \mathrm{EXP}_k, \mathrm{EXP}_l \rangle = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i\cdot l\cdot\frac{2\pi}{T}\cdot t}\right) dt = \frac{1}{T} \cdot \int_0^T e^{i\cdot(k-l)\cdot\frac{2\pi}{T}\cdot t}\, dt =$$

$$= \frac{1}{T} \cdot \frac{1}{i\cdot(k-l)\cdot\frac{2\pi}{T}} \cdot \left[e^{i\cdot(k-l)\cdot\frac{2\pi}{T}\cdot t}\right]_0^T = \frac{1}{2\pi\cdot i\cdot(k-l)} \cdot \left(e^{2\pi\cdot i\cdot(k-l)} - 1\right) = 0$$

Example

Give the Fourier series of 10-periodic function x defined as

$$x(t) = \begin{cases} 4, & \text{if} \quad 0 < t < 5 \\ 0, & \text{if} \quad 5 < t < 10 \\ 2, & \text{if} \quad x \in \{0,5,10\} \end{cases}$$



with respect to the orthonormal exponential system.

Solution

$$\hat{X}_0 = \langle x, \mathrm{EXP}_0 \rangle = \int_0^5 4 \cdot \frac{1}{\sqrt{10}} dt = \frac{20}{\sqrt{10}}$$

If $k \neq 0$

$$\hat{X}_k = \langle x, \mathrm{EXP}_k \rangle = \int_0^5 4 \cdot \left(\frac{1}{\sqrt{10}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{10} \cdot t} \right) dt = \frac{4}{\sqrt{10}} \cdot \frac{-10}{2\pi \cdot i \cdot k} \cdot \left[e^{-i \cdot k \cdot \frac{2\pi}{10} \cdot t} \right]_0^5 =$$

$$= \frac{20 \cdot i}{\sqrt{10} \cdot \pi \cdot k} \cdot \left(e^{-i \cdot k \cdot \pi} - 1 \right) = \begin{cases} \frac{-40 \cdot i}{\sqrt{10} \cdot \pi \cdot k}, & \text{if} \quad k \text{ is odd} \\ 0, & \text{if} \quad k \text{ is even}, k \neq 0 \end{cases}$$

Using the notation $k = 2l - 1, l \in \mathbb{Z}$ the Fourier series of x is

$$\mathcal{FS}f(x)(t) = 2 + \sum_{l=-\infty}^{\infty} \left(\frac{-40 \cdot i}{\sqrt{10} \cdot \pi \cdot (2l - 1)} \cdot \frac{1}{\sqrt{10}} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} \cdot t} \right) =$$

$$= 2 + \sum_{l=-\infty}^{\infty} \left(\frac{-4 \cdot i}{\pi \cdot (2l - 1)} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} \cdot t} \right)$$

The Exponential System

System of functions

$$\left\{e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

is orthogonal (but not orthonormal) in $L_2([0,T])$.

It is called exponential system.

The (complex) Fourier coefficients of function $x \in L_2([0,T])$ with respect to the exponential system are

$$\hat{x}_k = \frac{1}{T}\cdot\int_0^T x(t)\cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\,dt,\quad k\in\mathbb{Z},$$

the Fourier series of x is

$$\mathcal{FS}(x)(t) = \sum_{k=-\infty}^{\infty}\left(\hat{x}_k\cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right).$$

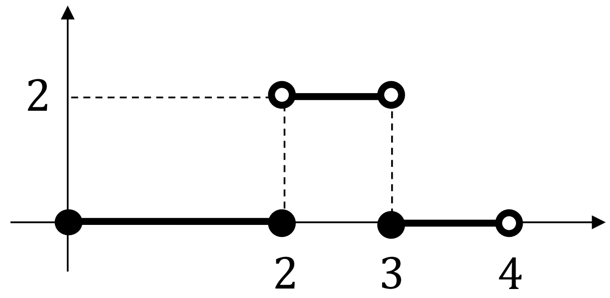
Functions

- $k \to |\hat{x}_k|$,
- $k \to |\hat{x}_k|^2$, and
- $k \to \arg\hat{x}_k$

are called amplitude spectrum, energy spectrum and phase spectrum, respectively.

Example

Calculate the Fourier coefficient $\hat{x}_5$ of the 4-periodic function x defined as



$$x(t) = \begin{cases} 2, & \text{if } 2 < t < 3 \\ 0, & \text{otherwise on } [0,4[\end{cases}$$

with respect to the exponential system.

Solution

$$\hat{x}_5 = \frac{1}{4} \cdot \int_2^3 2 \cdot e^{-i \cdot 5 \cdot \frac{2\pi}{4} \cdot t}\, dt = \frac{1}{4} \cdot \frac{-2}{5\pi \cdot i} \cdot \left[e^{-i \cdot \frac{5\pi}{2} \cdot t} \right]_0^4 =$$

$$= \frac{-1}{10\pi \cdot i} \cdot \left(e^{-i \cdot 10\pi} - 1 \right) = \frac{-1}{10\pi \cdot i} \cdot (\cos(-10\pi) + i \cdot \sin(-10\pi) - 1) = 0$$

Real and Complex Fourier Coefficients

If $x \in L_2([0,T])$ is a real-valued function, we have

$$\hat{x}_{-k} = \hat{x}_k^* \quad k \in \mathbb{Z}$$

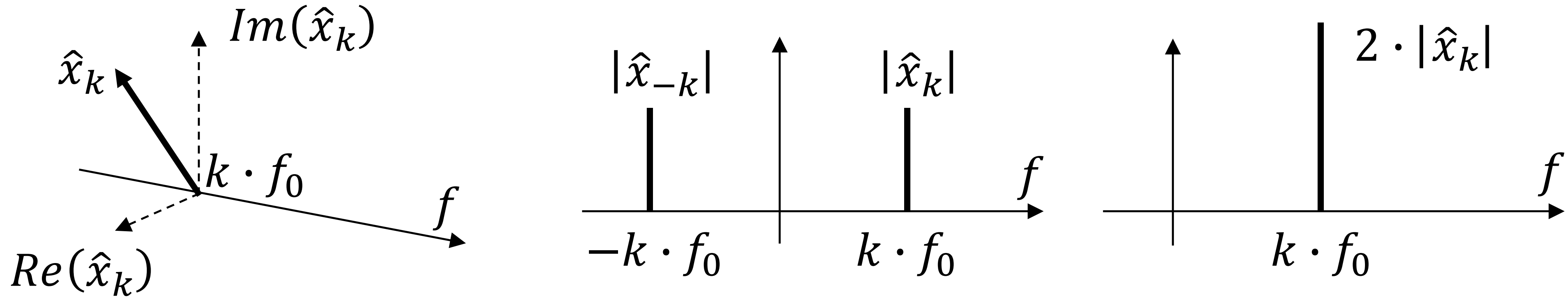
and, consequently

$$|\hat{x}_{-k}| = |\hat{x}_k|, \quad k \in \mathbb{Z}$$

showing that the complex spectrum has symmetric nature and the fact that the Fourier coefficients of a real-valued function belonging to 'negative frequencies' have not independent meaning.

The complex spectrum can be displayed in different ways.

We can draw a “3D” diagram showing the complex values (the real and the imaginary part of the coefficients), or we can plot only the values $|\hat{x}_k|$, and finally we can plot values $2 \cdot |\hat{x}_k|$ on the non-negative frequency axis.



Consider the orthonormal trigonometric system

$$\left\{\mathrm{CONST}(\mathrm{t}) = \frac{1}{\sqrt{T}}, \mathrm{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \mathrm{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$$

and the orthonormal exponential system

$$\left\{\mathrm{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

in $L_2([0,T])$.

Since both the real and the complex Fourier coefficients $(\hat{A}_k, \hat{B}_k, \hat{X}_k)$ belong to frequency $k \cdot \frac{2\pi}{T}$, they are expected to be connected. In fact

$$\hat{X}_0 = \hat{A}_0,$$

furthermore the properties of sine, cosine and exponential functions imply that for $k \in \mathbb{Z}, k > 0$ we have

$$\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k - \hat{B}_k \cdot i\right), \quad \hat{X}_{-k} = \frac{1}{\sqrt{2}} \cdot \left(\hat{A}_k + \hat{B}_k \cdot i\right), \text{ and } \left|\hat{X}_k\right| = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}.$$

Considering the trigonometric system

$$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{\mathrm{k}\in\mathbb{N}}$$

and the exponential system

$$\left\{e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right\}_{k\in\mathbb{Z}}$$

the connection between the real and complex Fourier coefficients $(\hat{a}_k, \hat{b}_k, \hat{x}_k)$ is as follows:

$$\hat{x}_0 = \hat{a}_0,$$

furthermore, for $k \in \mathbb{Z}, k > 0$, we have

$$\hat{x}_k = \frac{1}{2} \cdot \left(\hat{a}_k - \hat{b}_k \cdot i\right), \quad \hat{x}_{-k} = \frac{1}{2} \cdot \left(\hat{a}_k + \hat{b}_k \cdot i\right), \quad |\hat{x}_k| = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

Example

Using the Euler formula $e^{i\cdot t} = \cos t + i\cdot\sin t\,, t\in\mathbb{R}$ show that

$$\hat{X}_k = \frac{1}{\sqrt{2}}\cdot\left(\hat{A}_k - \hat{B}_k\cdot i\right)$$

$$\hat{X}_k = \frac{1}{\sqrt{2}}\cdot\left(\hat{A}_k + \hat{B}_k\cdot i\right)$$

and

$$\left|\hat{X}_k\right| = \frac{1}{2}\cdot\sqrt{\hat{A}_k^2 + \hat{B}_k^2},\ \ k\in\mathbb{Z}, k>0$$

Express $\hat{X}_k + \hat{X}_{-k}$ and $i\cdot\left(\hat{X}_k - \hat{X}_{-k}\right), k\in\mathbb{Z}, k>0$.

Solution

Let $k \in \mathbb{Z}, k > 0$.

$$\hat{X}_k = \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) dt =$$

$$= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$$

The sine function is odd while the cosine function is even thus

$$\hat{X}_{-k} = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{\sqrt{2}} \cdot \hat{A}_k + \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$$

Thus

$\hat{X}_k + \hat{X}_{-k} = \sqrt{2} \cdot \hat{A}_k, \quad k \in \mathbb{Z}, k > 0$

$i \cdot \left(\hat{X}_k - \hat{X}_{-k}\right) = \sqrt{2} \cdot \hat{B}_k, \quad k \in \mathbb{Z}, k > 0$

From formula $\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$ we have that

$$\mathrm{Re}\left(\hat{X}_k\right) = \frac{1}{\sqrt{2}} \cdot \hat{A}_k \quad \text{and} \quad \mathrm{Im}\left(\hat{X}_k\right) = -\frac{1}{\sqrt{2}} \cdot \hat{B}_k,$$

thus

$$\left|\hat{X}_k\right| = \sqrt{\frac{1}{2} \cdot \hat{A}_k^2 + \frac{1}{2} \cdot \hat{B}_k^2} = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}$$

Example

Using the formulas obtained in the previous exercise, manipulate the Fourier series of a function $x \in L_2([0,T])$ with respect to the orthonormal exponential system to get the Fourier series of x with respect to the orthonormal trigonometric system.

Solution

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} \hat{X}_k \cdot \mathrm{EXP}_k = \sum_{k=-\infty}^{\infty} \hat{X}_k \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{i\cdot k\cdot\frac{2\pi}{T}\cdot t}\right) =$$

$$= \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + i \cdot \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{X}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty} \left(\left(\hat{X}_k + \hat{X}_{-k}\right) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + \sum_{k=1}^{\infty} \left(i \cdot \left(\hat{X}_k - \hat{X}_{-k}\right) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{A}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty} \left(\hat{A}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) + \sum_{k=1}^{\infty} \left(\hat{B}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right) =$$

$$= \hat{A}_0 \cdot \mathrm{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \mathrm{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \mathrm{SIN}_k(t)$$

Example

Using the Euler formula $e^{i\cdot t} = \cos t + i \cdot \sin t\,, t \in \mathbb{R}$ show that

$$\hat{x}_k = \frac{1}{2}\cdot\left(\hat{a}_k - \hat{b}_k \cdot i\right), \quad \hat{X}_{-k} = \frac{1}{2}\cdot\left(\hat{a}_k + \hat{b}_k \cdot i\right), \quad \left|\hat{X}_k\right| = \frac{1}{2}\cdot\sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

Solution

Let $k \in \mathbb{Z}, k > 0$.

$$\hat{x}_k = \frac{1}{T}\cdot\int_0^T x(t)\cdot e^{-i\cdot k\cdot\frac{2\pi}{T}\cdot t}\,dt$$

$$= \frac{1}{T}\cdot\int_0^T x(t)\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt - i\cdot\frac{1}{T}\cdot\int_0^T x(t)\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$= \frac{1}{2}\cdot\frac{2}{T}\cdot\int_0^T x(t)\cdot\cos\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt - \frac{1}{2}\cdot i\cdot\frac{2}{T}\cdot\int_0^T x(t)\cdot\sin\left(k\cdot\frac{2\pi}{T}\cdot t\right)dt = \frac{1}{2}\cdot\hat{a}_k - \frac{1}{2}\cdot i\cdot\hat{b}_k.$$

The sine function is odd while the cosine function is even thus

$$\hat{x}_{-k} = \frac{1}{T}\cdot\int_0^T x(t)\cdot\cos\left(-k\cdot\frac{2\pi}{T}\cdot t\right)dt - i\cdot\frac{1}{T}\cdot\int_0^T x(t)\cdot\sin\left(-k\cdot\frac{2\pi}{T}\cdot t\right)dt =$$

$$= \frac{1}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \frac{1}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt =$$

$$= \frac{1}{2} \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{2} \cdot i \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \hat{a}_k + \frac{1}{2} \cdot i \cdot \hat{b}_k.$$

In the formula $\hat{x}_k = \frac{1}{2} \cdot \hat{a}_k - \frac{1}{2} \cdot i \cdot \hat{b}_k$ we can see that $\mathrm{Re}(\hat{x}_k) = \frac{1}{2} \cdot \hat{a}_k$ and $\mathrm{Im}(\hat{x}_k) = -\frac{1}{2} \cdot \hat{b}_k$ thus

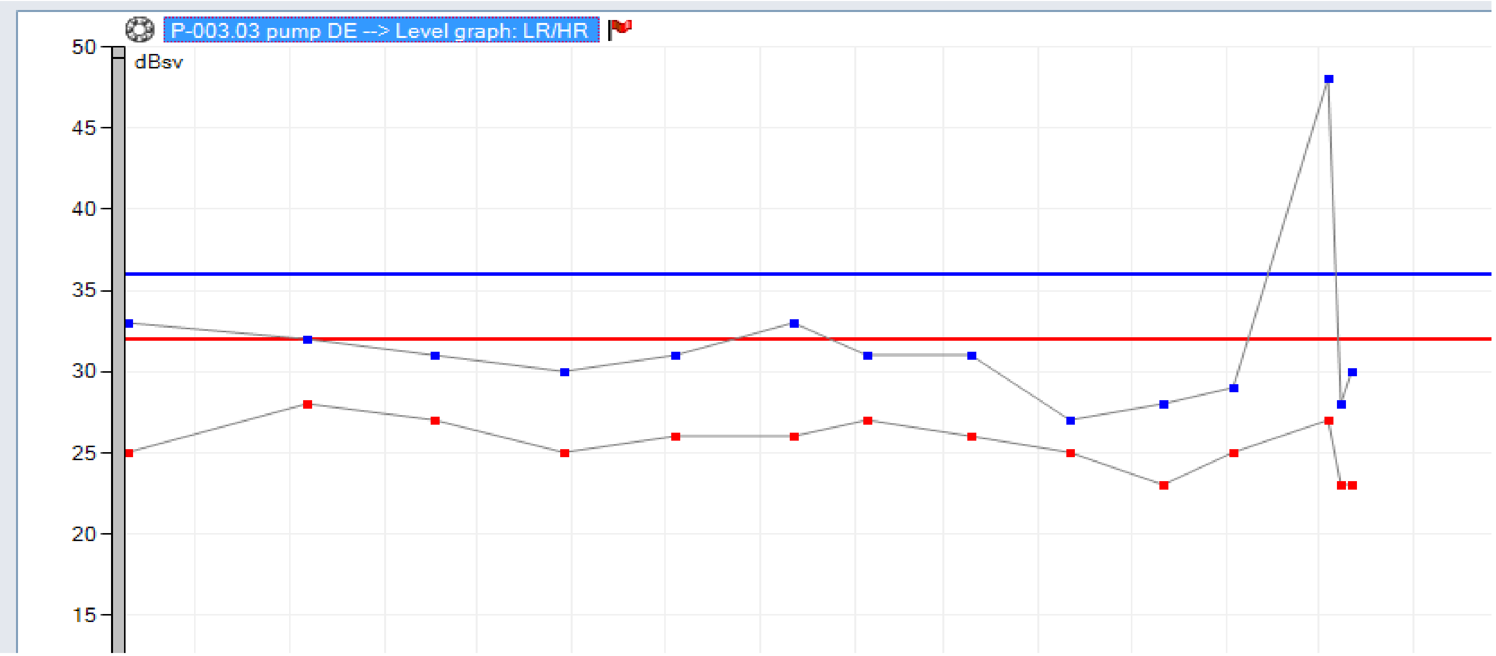
$$|\hat{x}_k| = \sqrt{\frac{1}{4} \cdot \hat{a}_k^2 + \frac{1}{4} \cdot \hat{b}_k^2} = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

Special diagrams related to the frequency spectrum in the SPM condition monitoring system

In predictive maintenance of machinery the control of the propagation of failures in time is even more important than the determination of the current condition.

Several graphical tools are available in SPM Condmaster software providing information about changes in time.

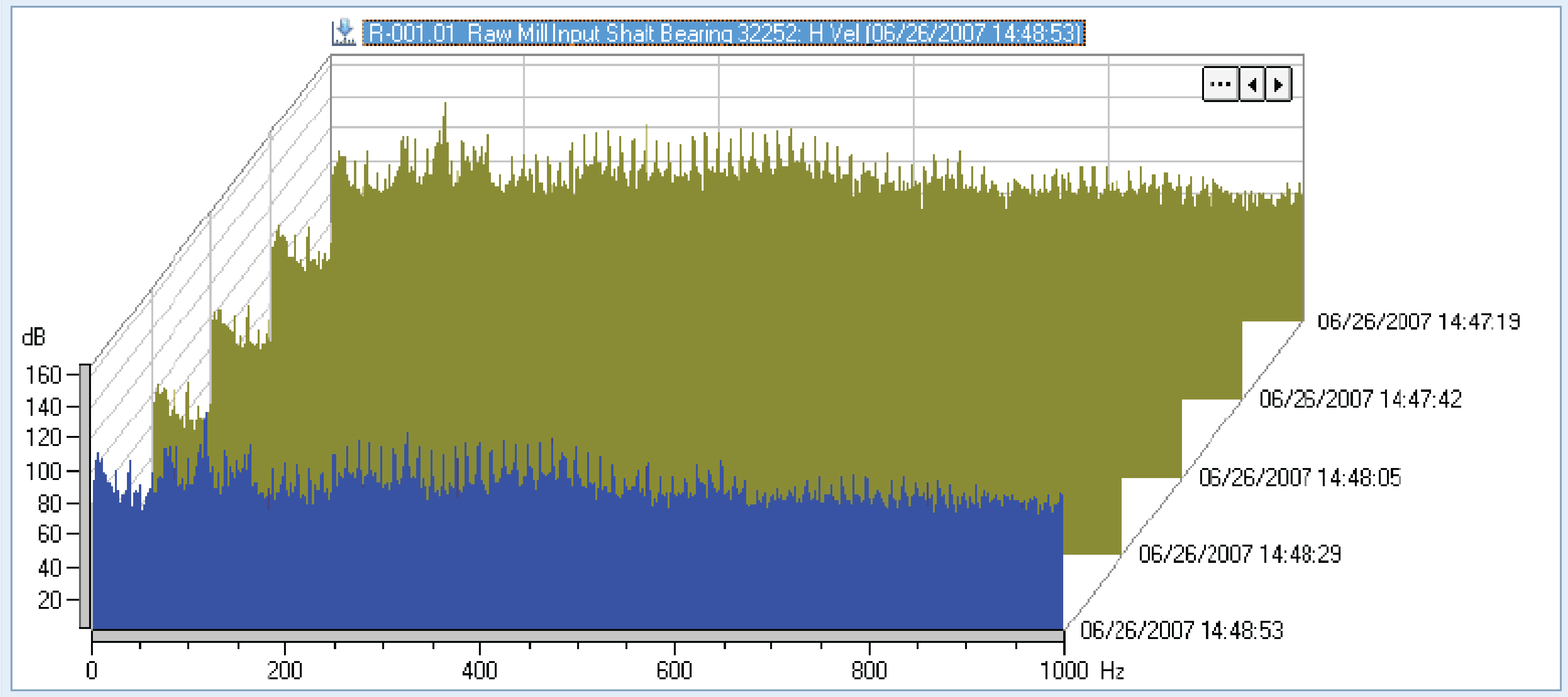
A kind of these diagrams shows some important numerical values as a function of time and also the related control limits.



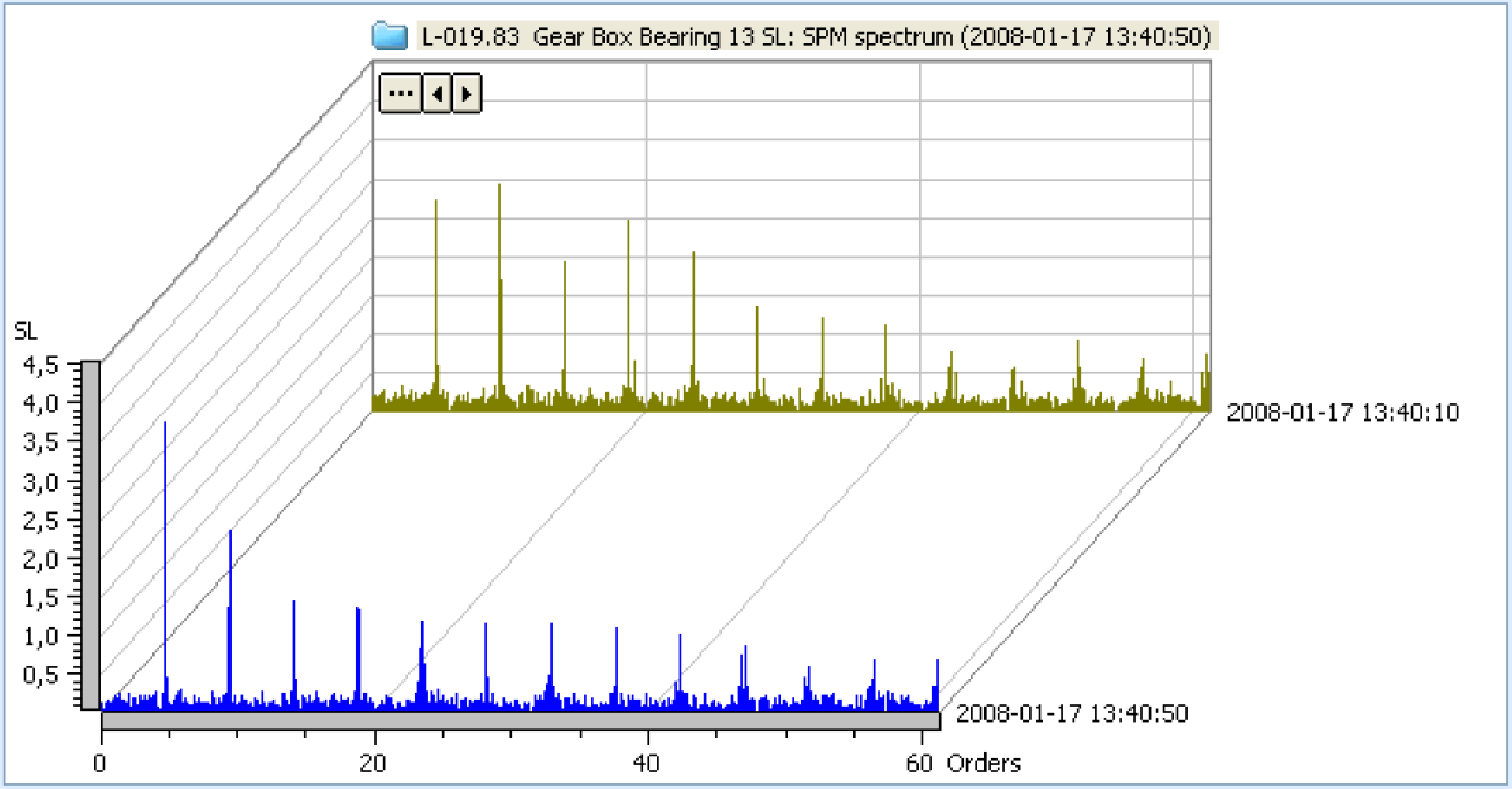
Another type of diagrams shows the change of graphs, for instance the change of the spectrum.

When the amplitudes belonging to critical frequencies increase or new frequencies appear in the spectrum the machine or process must be checked and the root cause of the change must be identified to avoid the further propagation of the failure.

A useful tool is the so-called Waterfall diagram which is a three-dimensional display of up to 50 vibration spectra. The different readings are displayed along an axis, with the latest being the nearest the viewer.

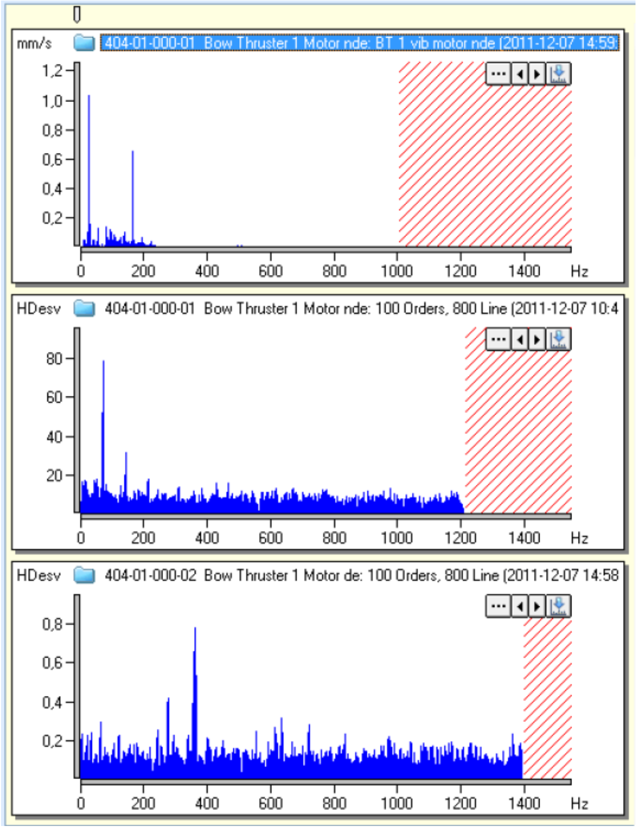


Change of the spectrum at a measuring point (gearbox bearing)



With the Compare spectrum function we can view more than one frequency range and/or resolution at a time.

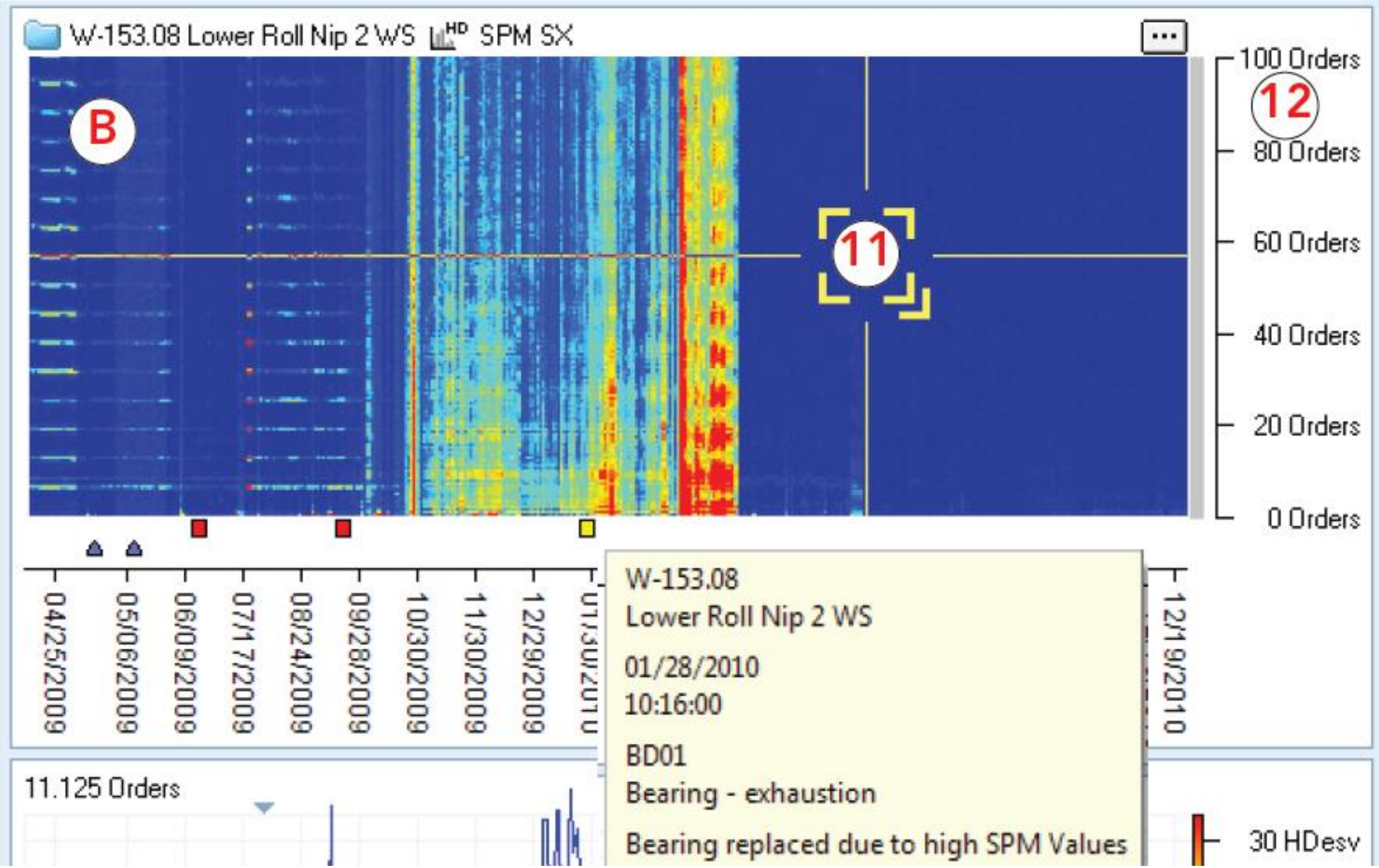
This means that we can implement a variable frequency range from one measuring assignment to another and also between measuring points.



The Coloured Spectrum Overview is a three-dimensional view of all spectra under a particular measuring assignment.

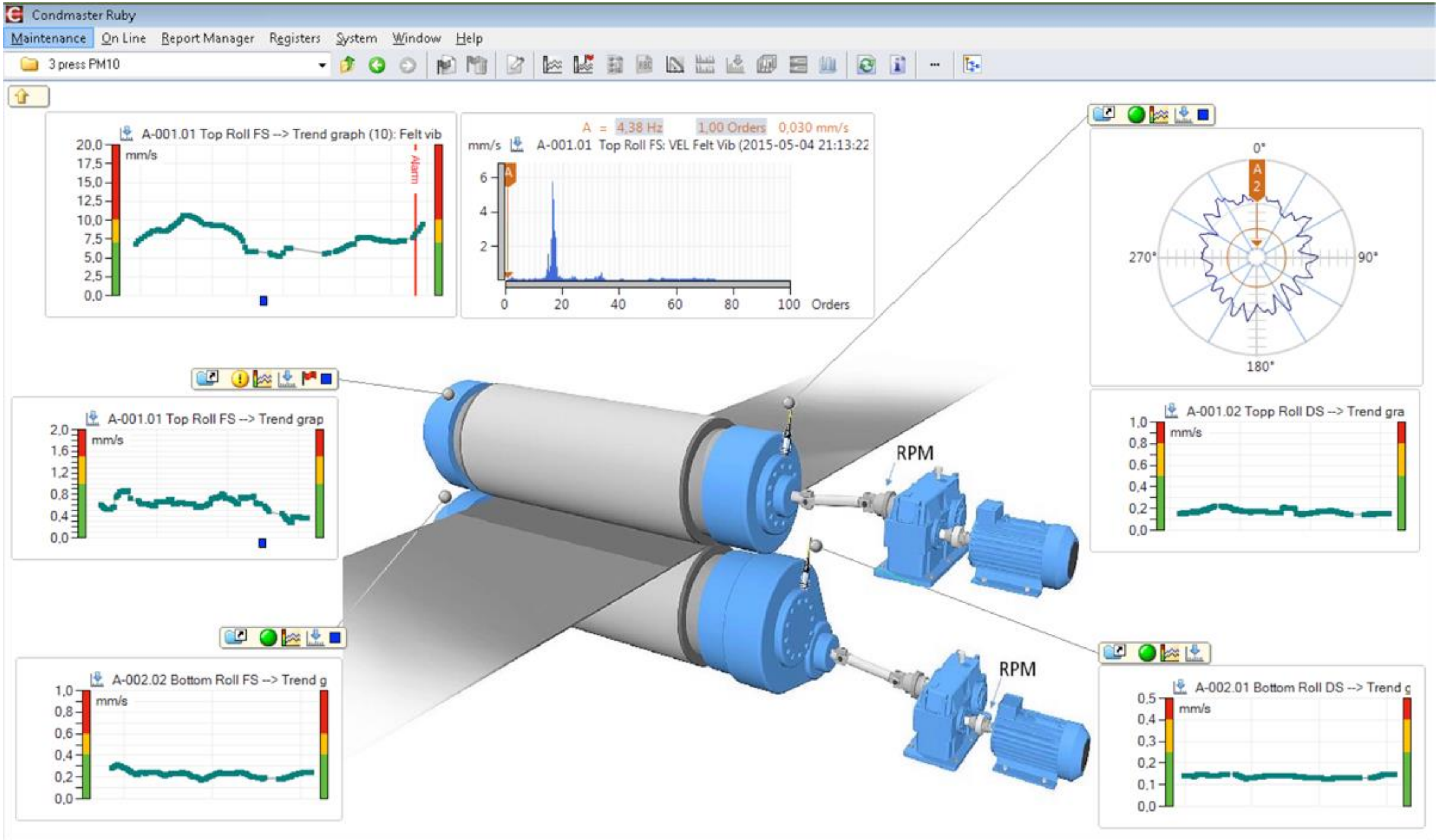
Its purpose is to simplify the process of identifying in spectra the patterns and trends which indicate damages.

Signals which are always present in the machine are clearly distinguished from signals caused by developing damages. It provides a very good overall picture of machine condition development.


W-153.08 Lower Roll Nip 2 WS
SPM SX
100 Orders
80 Orders
60 Orders
40 Orders
20 Orders
0 Orders
B
11
12
04/25/2009
05/06/2009
06/09/2009
07/17/2009
08/24/2009
09/28/2009
10/30/2009
11/30/2009
12/29/2009
12/19/2010
W-153.08
Lower Roll Nip 2 WS
01/28/2010
10:16:00
BD01
Bearing - exhaustion
Bearing replaced due to high SPM Values
11.125 Orders
30 HDesv

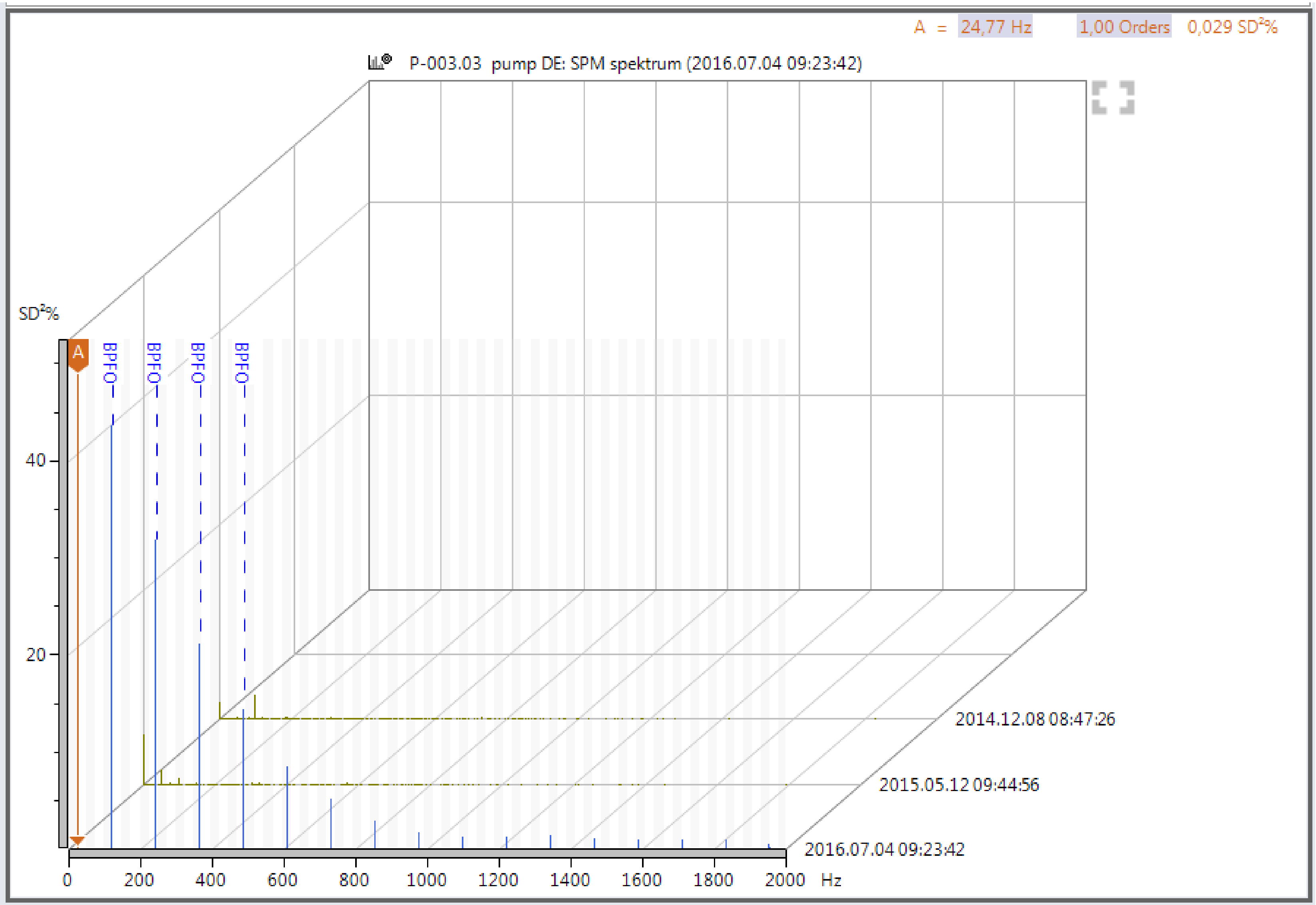
A typical online condition monitoring system with fixed transducers (paper mill).

The current condition of bearings can be checked anytime through internet.



Case study: condition monitoring of a pump bearing

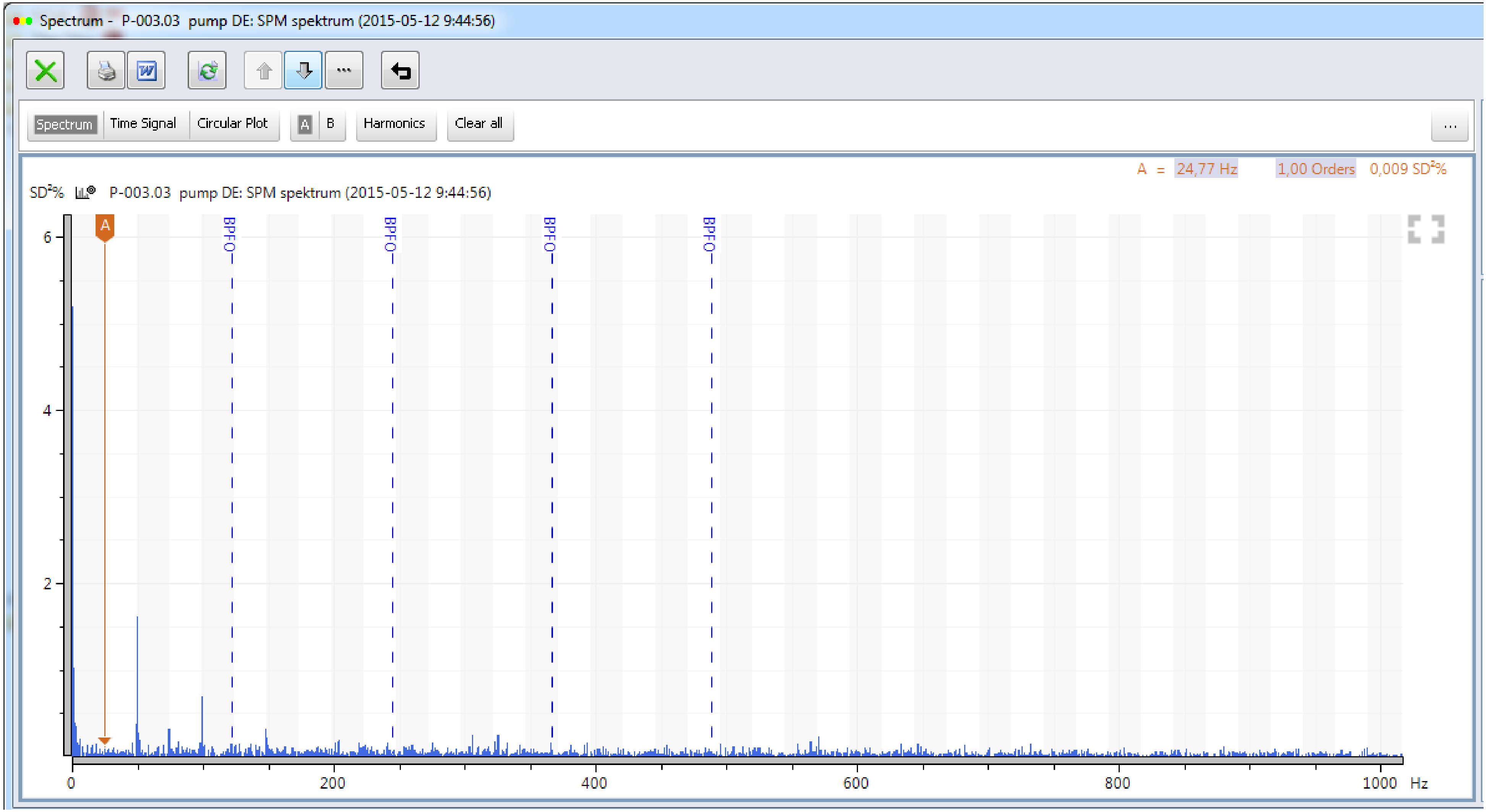
The waterfall diagram below shows clearly, that the measure of amplitude enhancement was significant at certain frequencies.

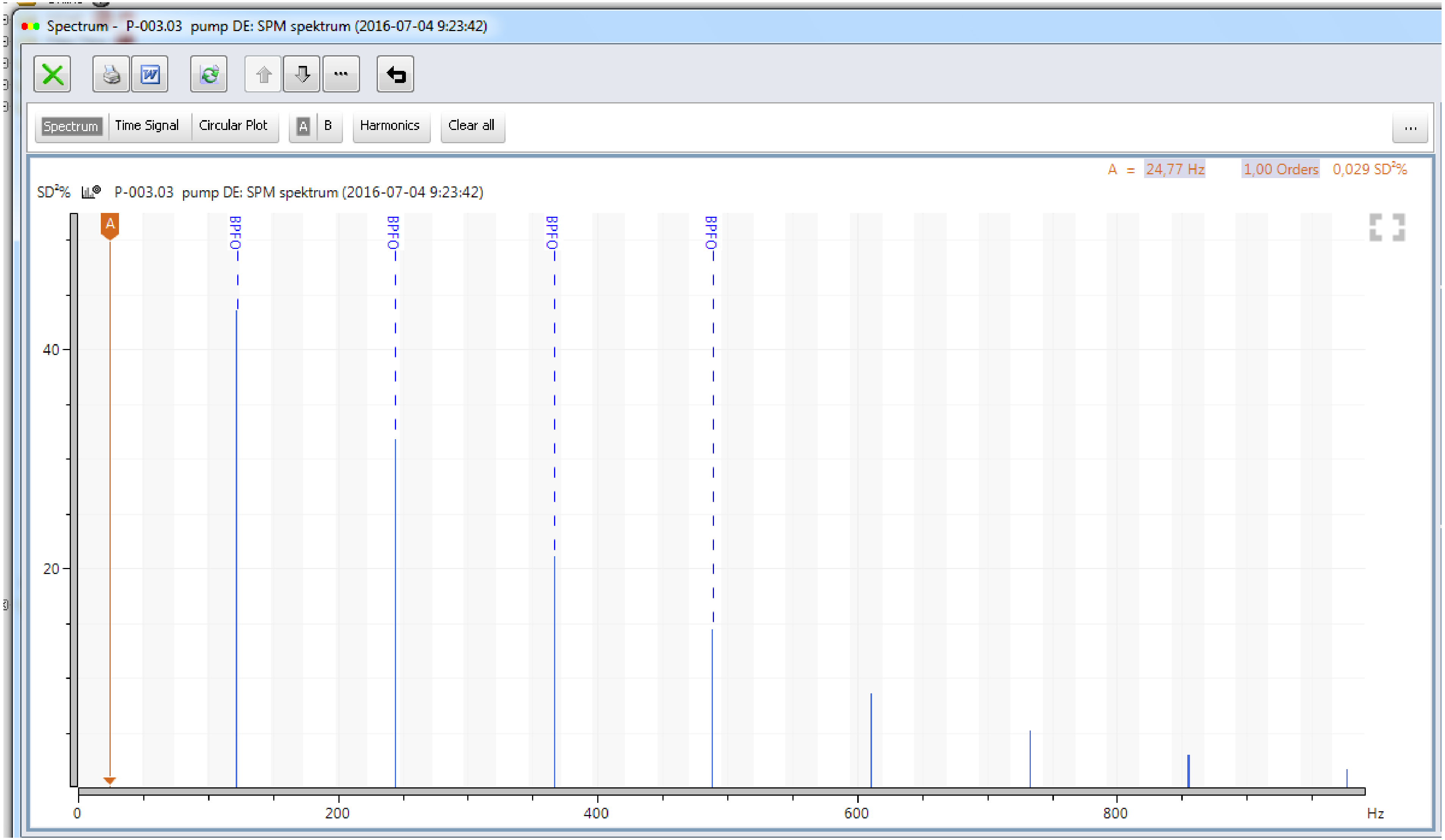


Further investigations showed that the high lines matched the symptom lines belonging to the outer ring fault (BPFO), that is, a failure of the outer ring was detected.

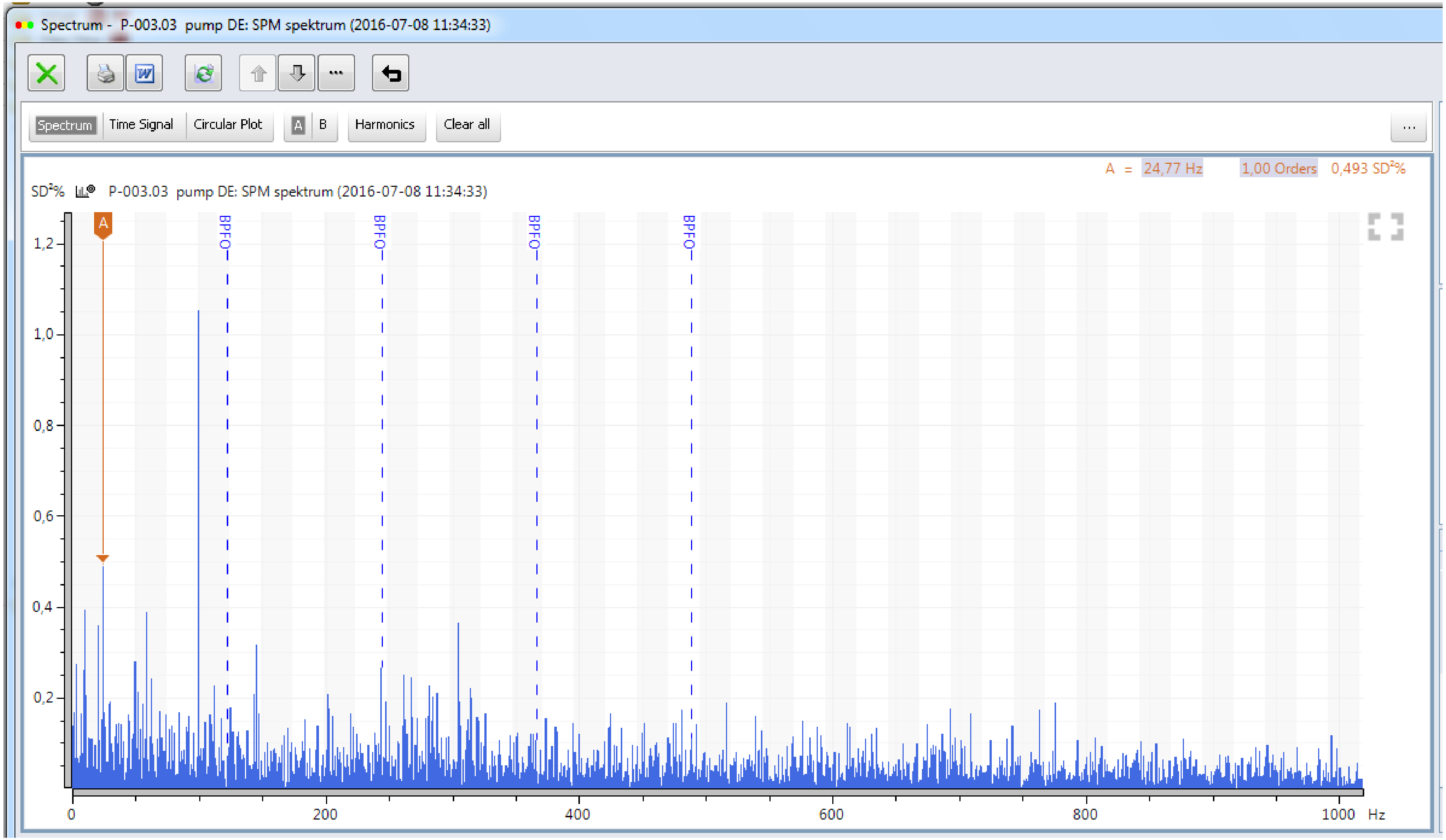
The following figures show the spectrum measured

– before outer ring fault appeared (good condition),
– when the problem developed (defective outer ring), and
– after installing a new bearing (good condition again).





Spectrum - P-003.03 pump DE: SPM spektrum (2016-07-04 9:23:42)
Spectrum
Time Signal
Circular Plot
A
B
Harmonics
Clear all
A = 24,77 Hz
1,00 Orders
0,029 SD²%
SD²%
P-003.03 pump DE: SPM spektrum (2016-07-04 9:23:42)
BPFO
BPFO
BPFO
BPFO
40
20
0
200
400
600
800
Hz



Spectrum - P-003.03 pump DE: SPM spektrum (2016-07-08 11:34:33)
Spectrum
Time Signal
Circular Plot
A
B
Harmonics
Clear all
A = 24,77 Hz
1,00 Orders
0,493 SD²%
SD²%
P-003.03 pump DE: SPM spektrum (2016-07-08 11:34:33)
BPFO
BPFO
BPFO
BPFO
1,2
1,0
0,8
0,6
0,4
0,2
0
200
400
600
800
1000
Hz

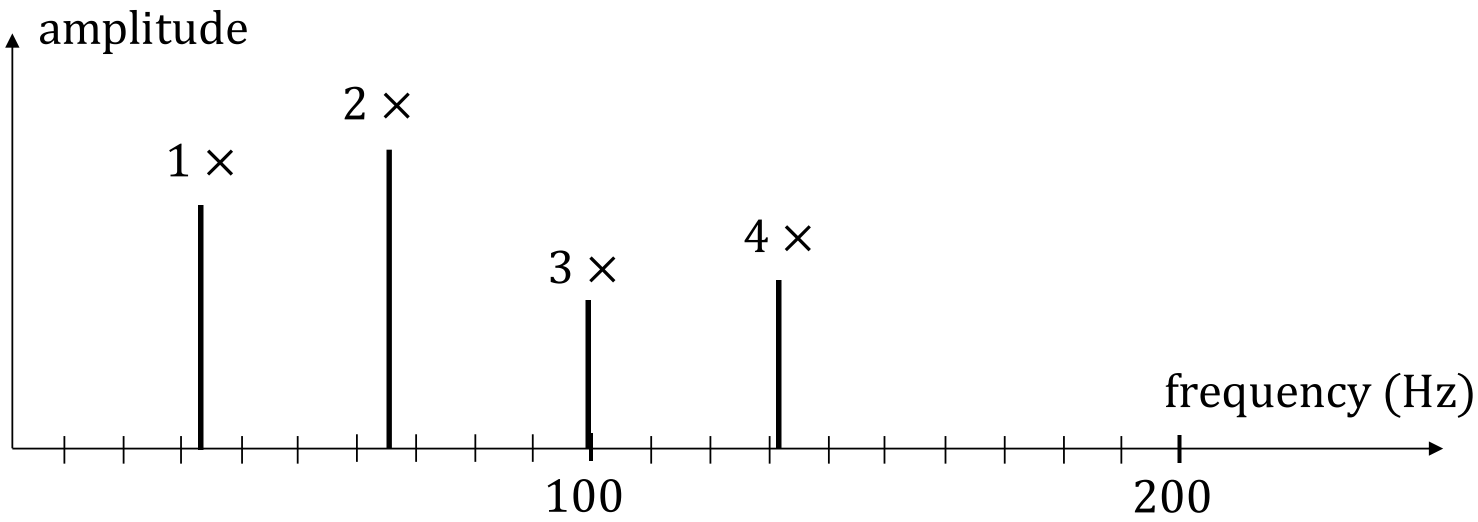
Example

The bearing fault coefficients for the ISO6302 bearing can be seen in the picture.

The rotational speed of the shaft during the measurement is 1140 rpm.

Bearing number	6302			
Manufacturer	NSK			
TYPE no.	1			
Inner diameter	15	mm	BPO	2,558
Outer diameter	42	mm	BPI	4,442
Mean diameter	28,5	mm	BS	1,724
Width	13	mm	FT	0,365

The sketch of the spectrum provided by SPM Condmaster Ruby is



Which element of the bearing is damaged: outer ring, inner ring, ball, cage, or none of them?

Solution

The bearing fault frequencies belonging to the rotational speed of $1140\ rpm = 19\ rps$ are

fault type	coefficient	rotational speed (rps)	fault frequency (Hz)
outer ring	2.558	19	48.60
inner ring	4.442	19	84.40
ball	1.724	19	32.76
cage	0.365	19	6.94

The spectrum contains a frequency near to $32.765\ Hz$ (ball spin frequency) and its harmonics. It suggests that there is a fault on a ball.

Example

It is known that the specific symptom of a coupling problem is a high line in the frequency spectrum at 2nd order.

Determine the specific frequency belonging to the coupling problem if the rotational speed of the shaft is 1800 rpm.

Solution

The rotational speed is 1800 $rpm = 50\ rps$.

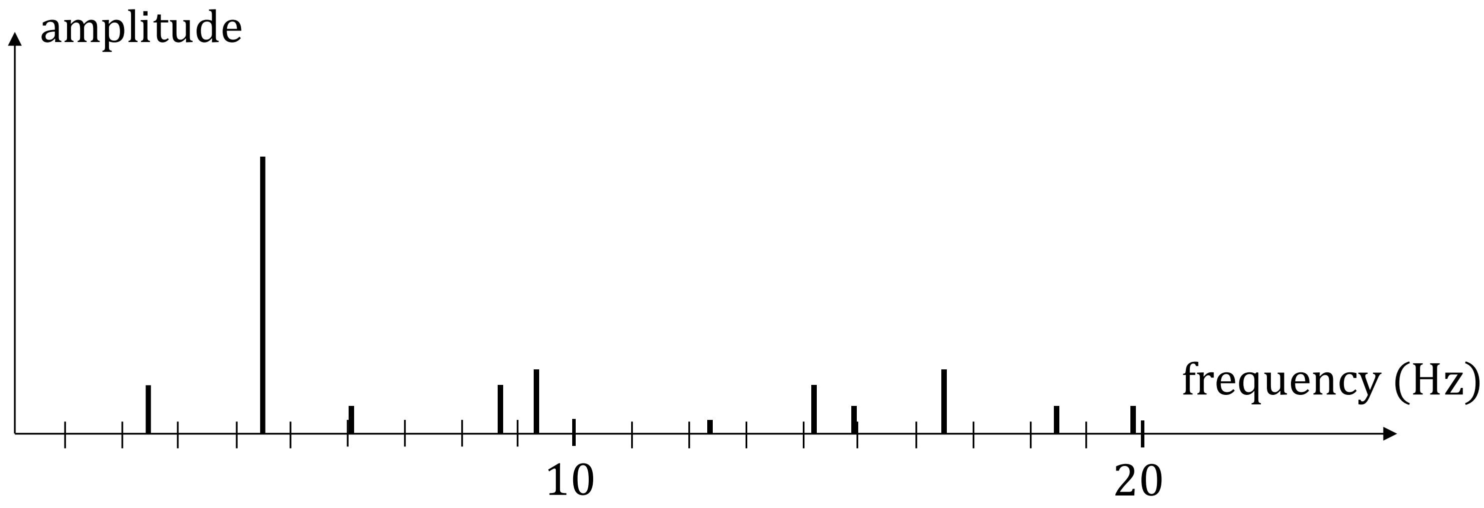
The line belonging to the coupling problem is at $2 \times 50 = 100\ Hz$ in the frequency spectrum.

Example

In the majority of cases the highest spectrum line is at the 1st order which belongs to speed of the shaft (characteristics frequency of unbalance).

Finding the frequency of the highest energy harmonic component in the signal, the shaft speed can be identified.

Give the likely value of the rotational speed of the shaft on the basis of the following spectrum.



Solution

The highest line in the frequency spectrum is near to 4.5 Hz. It suggests that the rotational speed of the shaft is 4.5 $rps = 4.5 \times 60 = 225\ rpm.$

Integral Transforms

Let $K: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a given integrable function. Function

$$F(s) = \int_a^b f(t) \cdot K(s,t)\, dt, \quad s \in \mathbb{C}$$

is called the integral transform of function $f: [a, b] \to \mathbb{C}$ if the integral is convergent.

Function K is called kernel function.

The formula provides different transforms for different kernel functions.

The Fourier transform and the Laplace transform are two well-known integral transforms, which are frequently used in different fields of engineering and sciences.

Some special transformations appear in special applications, e.g. the wavelet transform is important tool, for example in technical diagnostics.

Some transformations (e.g. Fourier and wavelet) have continuous and discrete forms. Discrete transformations are used in discrete signal processing where only a sampled signal is available rather than the formula of the function (signal).

The Continuous Fourier Transform

Function

$$\mathcal{FT}(x)(\omega) = \hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t} \, dt, \quad \omega \in \mathbb{R}$$

is the Fourier transform of function $x: \mathbb{R} \to \mathbb{R}$ if the integral is convergent.

The Fourier integral of $x: \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{FI}(x)(t) = \mathcal{FT}^{-1}(\hat{x})(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i \cdot \omega \cdot t} \, d\omega, \quad t \in \mathbb{R}.$$

Functions

$$\omega \to |\hat{x}(\omega)|, \quad \omega \to |\hat{x}(\omega)|^2, \quad \text{and} \quad \omega \to \angle\, \hat{x}(\omega)$$

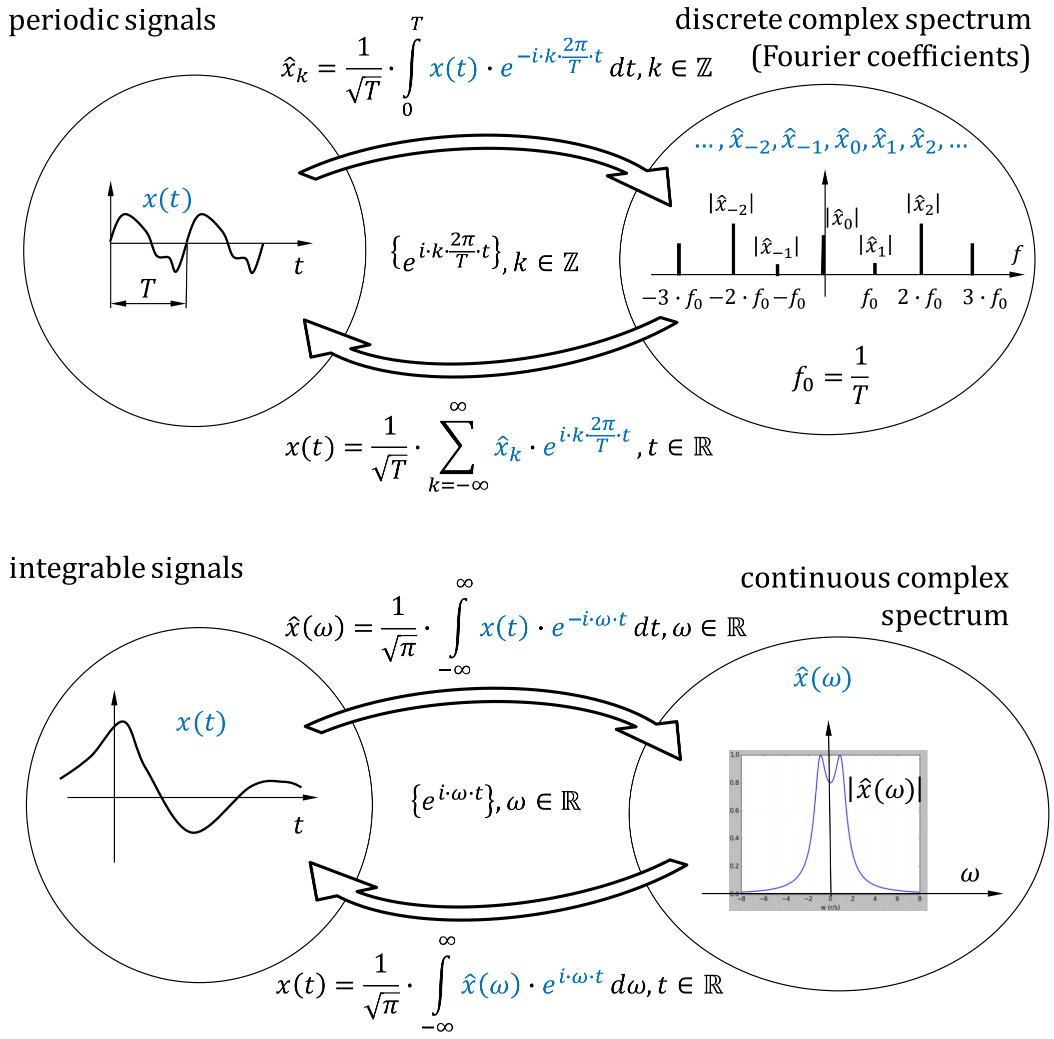
are called amplitude spectrum, energy spectrum and phase spectrum, respectively, in engineering literature.

Remark

Instead of the angular frequency ω frequency f can also be used as the variable in the formula of the Fourier transform, but some correction factors must be used in this case.

Remark

The Fourier coefficients of periodic functions have discrete nature, while the Fourier transform gives a 'continuous' spectrum



Using the Euler's formula $e^{i\cdot t} = \cos t + i \cdot \sin t\,, t \in \mathbb{R}$ we can write the Fourier transform of $x\colon \mathbb{R} \to \mathbb{R}$ as

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\cdot\omega\cdot t}\, dt =$$

$$= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(-\omega \cdot t)\, dt + i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(-\omega \cdot t)\, dt =$$

$$= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt - i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt = \hat{\boldsymbol{a}}(\boldsymbol{\omega}) - i \cdot \hat{\boldsymbol{b}}(\boldsymbol{\omega}), \quad \omega \in \mathbb{R}.$$

When x is even, then $\hat{b}_x = 0$ and we have

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}.$$

When x is odd, then $\hat{a}_x = 0$ and we have

$$\hat{x}(\omega) = -i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt = -i \cdot 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}.$$

Integrals

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

and

$$\mathcal{FT}_{\sin}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t)\, dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

are called the cosine Fourier transform and the sine Fourier transform of function $x\colon [0, \infty[\to \mathbb{R}$.

The Fourier cosine integral of x is

$$\mathcal{FI}_{\cos}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\cos}(x)(\omega) \cdot \cos(\omega \cdot t)\, d\omega, \quad t \in \mathbb{R}, t \geq 0,$$

while the Fourier sine integral of x is

$$\mathcal{FI}_{sin}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\sin}(x)(\omega) \cdot \sin(\omega \cdot t)\, d\omega, \quad t \in \mathbb{R}, t \geq 0.$$

Remark

Each real function $x: \mathbb{R} \to \mathbb{R}$ (having Fourier transform) can be analysed with its cosine and sine Fourier transform since x can be written as

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = g(t) + h(t), \quad t \in \mathbb{R}.$$

where g is even and h is odd.

Thus

$$\mathcal{FT}(x) = \mathcal{FT}(g) + \mathcal{FT}(h) = \mathcal{FT}_{\cos}(g) - i \cdot \mathcal{FT}_{\sin}(h)$$

If function x is piecewise continuous then $\mathcal{FI}(x)$ is equal to x wherever x is continuous, and $\mathcal{FI}(x)$ is the average the left- and right-hand limits wherever x is discontinuous.

Remark

Since a piecewise continuous function (signal) can be reconstructed from its Fourier transform (through its Fourier integral) we can say that the Fourier transform contains all information about the function, and can be considered as an alternative representation.

For instance, a vibration process can be described in the ‘time domain’ (e.g. vibration velocity vs. time function) and also in ‘frequency domain’ (e.g. vibration frequency spectrum).

Parseval’s equality (energy of a signal):

$$\int_{t=-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} |\hat{x}(\omega)|^2\,d\omega.$$

The following table shows the Fourier transform of some functions.

We can find some 'dual' properties of the Fourier transform which show how the Fourier transform changes (in the frequency domain) when the function is changed in the time domain, and vice versa.

For $\alpha, \beta, T, \omega_0 \in \mathbb{R}$

	time domain	**frequency domain**
	$t \to \boldsymbol{x}(\boldsymbol{t}) = \mathcal{FT}^{-1}(\hat{x})(t)$	$\omega \to \hat{\boldsymbol{x}}(\boldsymbol{\omega}) = \mathcal{FT}(x)(\omega)$
linearity	$t \to \alpha \cdot x(t) + \beta \cdot y(t)$	$\omega \to \alpha \cdot \hat{x}(\omega) + \beta \cdot \hat{y}(\omega)$
shift in the time domain	$t \to x(t-T)$	$\omega \to \hat{x}(\omega) \cdot e^{-i \cdot T \cdot \omega}$
shift in the frequency domain (modulation)	$t \to x(t) \cdot e^{i \cdot \omega_0 \cdot t}$	$\omega \to \hat{x}(\omega - \omega_0)$
scaling	$t \to x(\alpha \cdot t)$	$\omega \to \frac{1}{\lvert\alpha\rvert} \cdot \hat{x}\left(\frac{\omega}{\alpha}\right)$
convolution	$t \to (x * y)(t)$	$\omega \to \hat{x}(\omega) \cdot \hat{y}(\omega)$

Example

Determine the complex Fourier transform and the Fourier integral of the rectangular pulse function

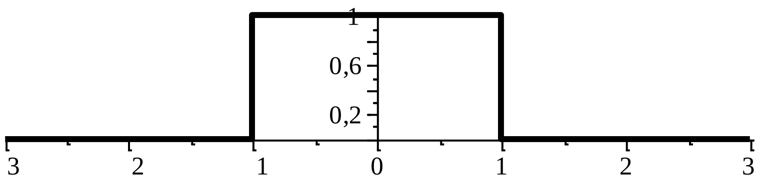
$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if} \quad |t| \le 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

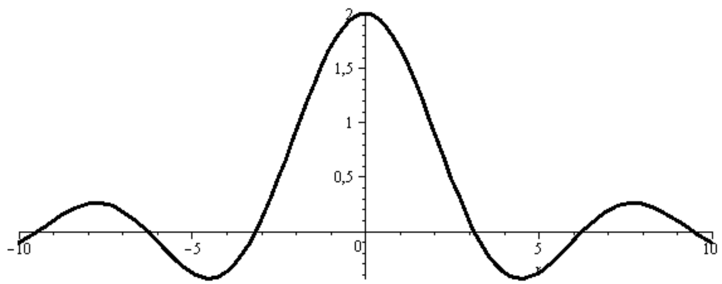
Solution

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t} \, dt = \int_{t=-1}^{1} e^{-i \cdot \omega \cdot t} \, dt = \left[\frac{e^{-i \cdot \omega \cdot t}}{-i \cdot \omega} \right]_{t=-1}^{1} =$$

$$= \frac{1}{-i \cdot \omega} \cdot \left(e^{-i \cdot \omega} - e^{i \cdot \omega} \right) = \frac{2}{\omega} \cdot \frac{e^{i \cdot \omega} - e^{-i \cdot \omega}}{2i} = 2 \cdot \frac{\sin \omega}{\omega} = 2 \cdot \operatorname{sinc} \omega$$

$$t \to x(t) = \Pi(t) \qquad\qquad \omega \to \hat{x}(\omega) = 2 \cdot \frac{\sin \omega}{\omega}$$





The Fourier integral of x is

$$\mathcal{FI}(x)(t) = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\cdot\omega\cdot t}\, d\omega = \frac{1}{\pi} \cdot \int_{\omega=-\infty}^{\infty} \frac{\sin(\omega)}{\omega} \cdot e^{i\cdot\omega\cdot t}\, d\omega$$

Example

Determine the sine and cosine Fourier transform of the of the rectangular pulse function

$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if} \quad |t| \le 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

Solution

Since x is even $\mathcal{FT}_{\sin}(x) = 0$.

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t)\, dt = 2 \cdot \int_{t=0}^{1} \cos(\omega \cdot t)\, dt = 2 \cdot \frac{\sin \omega}{\omega}$$

The Fourier cosine integral of x is

$$\mathcal{FI}(x)(t) = \frac{2}{\pi} \cdot \int_{\omega=0}^{\infty} \frac{\sin \omega}{\omega} \cdot \cos(\omega \cdot t)\, d\omega$$

Example

Determine the Fourier transform of the shifted rectangular pulse function

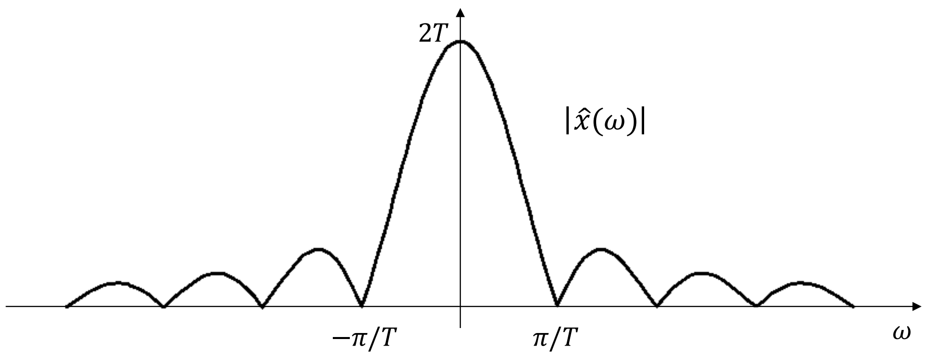
$$x(t) = \begin{cases} 1 & \text{if} \quad 1 - T \le t \le 1 + T \\ 0 & \text{if} \quad t < 1 - T \text{ or } t > 1 + T \end{cases}$$

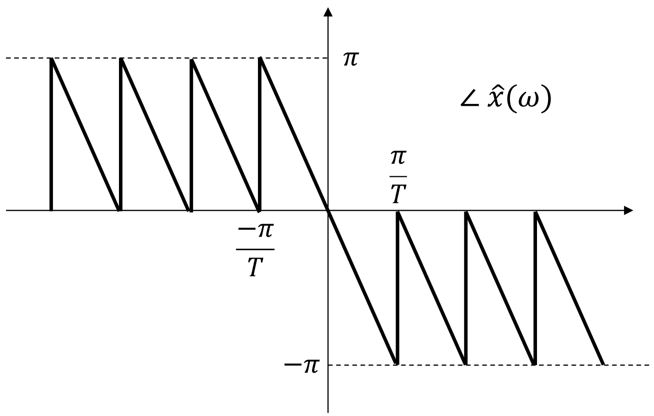
Solution

$$\hat{x}(\omega) = \int_{t=1-T}^{1+T} e^{-i\cdot\omega\cdot t}\, dt = \frac{-1}{i \cdot \omega} \cdot \left[e^{-i\cdot\omega\cdot t}\right]_{t=1-T}^{1+T} = \frac{-1}{i \cdot \omega} \cdot \left(e^{-i\cdot\omega\cdot(1+T)} - e^{i\cdot\omega\cdot(1-T)}\right) =$$

$$= \frac{-1}{i \cdot \omega} \cdot e^{-i\cdot\omega} \cdot \left(e^{-i\cdot\omega\cdot T} - e^{i\cdot\omega\cdot T}\right) = 2 \cdot e^{-i\cdot\omega} \cdot \frac{1}{\omega} \cdot \frac{e^{i\cdot\omega\cdot T} - e^{-i\cdot\omega\cdot T}}{2i}$$

$$= 2 \cdot \frac{\sin(\omega \cdot T)}{\omega} \cdot e^{-i\cdot\omega}$$





Remark

The unit step function $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ has not Fourier transform since the integral

$$\int_{t=0}^{\infty} e^{-i\cdot\omega\cdot t}\, dt = \int_{t=0}^{\infty} \cos(\omega\cdot t)\, dt - i\cdot \int_{t=0}^{\infty} \sin(\omega\cdot t)\, dt$$

is not convergent.

The unit step function can be considered as the limit of function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a\cdot t} & \text{if } t \geq 0 \end{cases}, a > 0$$

as $a \to 0 + 0$, thus we can define, symbolically, the 'Fourier transform' of unit step function as $\pi \cdot \delta(\omega) + \frac{1}{i\cdot\omega}$.

This definition yields Fourier transform of further important functions.

Example

Determine the Fourier transform of triangle function

$$x(t) = \mathrm{tri}(t) = \begin{cases} t+2 & if \quad -2 \le t \le 0 \\ -t+2 & if \quad 0 \le t \le 2 \\ 0 & if \quad |t| > 2 \end{cases}$$

using the convolution theorem $\mathcal{FT}(x * h) = \mathcal{FT}(x) \cdot \mathcal{FT}(h)$.

Solution

We have that

$$x(t) = \mathrm{tri}(t) = \Pi * \Pi(t)$$

where $\Pi(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$, is the rectangular impulse function.

Using the convolution theorem we get

$$\hat{x}(\omega) = \mathcal{FT}(\Pi)(\omega) \cdot \mathcal{FT}(\Pi)(\omega) = 4 \cdot \frac{\sin^2 \omega}{\omega^2}$$

The Discrete Fourier Transform

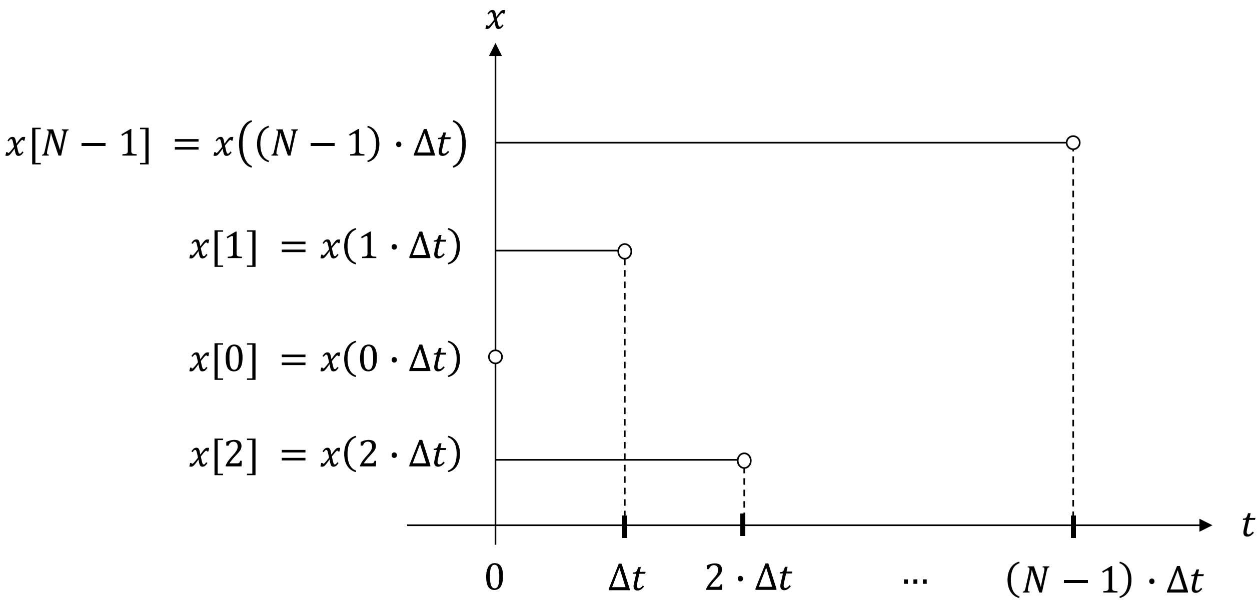
Let $T > 0$ be a fixed real number and N be a fixed positive integer and suppose that values

$$x[n] = x[n \times \Delta T], \quad n = 0,1,\dots,N-1$$

of signal x are provided by a sampling process.

The discrete Fourier transform of sampled signal $x[0],\dots,x[N-1]$ is

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}, \quad k = 0,1,\dots,N-1$$



Remark

Mathematically, both the input and the output of the discrete Fourier transform consist of N pure numbers.

If the sampling frequency is known, the 'discrete spectrum' can be determined from values $X[0], \ldots, X[N-1]$.

Consider

- T, the sampling time,
- N, the sample size (number of elements in the sample),
- ΔT, time between two measurements.
- $f_s = N/T = 1/\Delta T$, the sampling frequency.

Then the frequency resolution *is*

$$\Delta f = 1/T = f_s/N$$

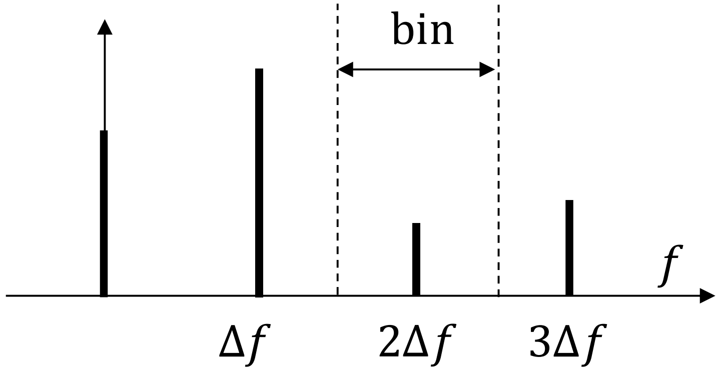
and the (possible) frequency values in the discrete spectrum are

$$k \times \Delta f, \quad k = 0,1, \ldots, N-1$$

Example

If the sampling frequency is $f_s = 1000\ Hz$, and the sample size is $N = 1024$, then the frequency resolution is

$$\Delta f = \frac{1000}{1024} = 0.9766\ \frac{Hz}{bin}$$



Example *(DFT provided by MS Excel)*

k	sampled signal $x[k]$	excel output	DFT $X[k]$	$\lvert X[k]\rvert$	frequency of components $k \cdot \Delta f$		amplitude spectrum $2 \cdot \lvert X[k]\rvert$
0	0,0000	0	0	0	constant		
1	-0,1306	-24 i	-1,5 i	**1,5**	$\boldsymbol{\Delta f}$	'useful'	**3**
2	5,4394	0	0	0	$\boldsymbol{2 \cdot \Delta f}$	frequencies	**0**
3	1,0374	-64 i	-4 i	**4**	$\boldsymbol{3 \cdot \Delta f}$		**8**
4	-8,0025	-40 i	-2,5 i	**2,5**	$\boldsymbol{4 \cdot \Delta f}$		**5**
5	6,8903	0	0	0	$\boldsymbol{5 \cdot \Delta f}$		**0**
6	-11,3740	0	0	0	$\boldsymbol{6 \cdot \Delta f}$		**0**
7	10,0187	0	0	0	$\boldsymbol{7 \cdot \Delta f}$		**0**
8	-1,5194	0	0	0	constant		
9	3,9285	0	0	0	$-7 \cdot \Delta f$		
10	-5,8108	0	0	0	$-6 \cdot \Delta f$		
11	5,0878	0	0	0	$-5 \cdot \Delta f$		
12	-13,3852	40 i	2,5 i	**2,5**	$-4 \cdot \Delta f$		
13	13,9040	64 i	4 i	**4**	$-3 \cdot \Delta f$		
14	-6,9681	0	0	0	$-2 \cdot \Delta f$		
15	7,2340	24 i	1,5 i	**1,5**	$-\Delta f$		

$\boldsymbol{\Delta f} = \frac{1}{T} = \frac{f_S}{N}$ is the frequency resolution, where T is the sampling time, f_S is the sampling frequency, N is the sample size.

For example, if the sampling frequency was $f_s = 200\ [Hz]$, then

$$\Delta \boldsymbol{f} = \frac{f_s}{N} = \frac{200\ [Hz]}{16} = 12{,}5\ [Hz]$$

(sample size is $N = 16$ in the example).

Thus, there are the following three frequencies in the spectrum:

$$12{,}5\ [Hz], \quad 37{,}5\ [Hz], \quad 50\ [Hz]$$

The discrete Fourier transform can also be calculated as a matrix multiplication.

Introducing the notation

$$W_N = e^{-i\cdot\frac{2\pi}{N}}$$

then

$$e^{-i\cdot k\cdot n\cdot\frac{2\pi}{N}} = W_N^{k\cdot n}$$

and the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2\cdot(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix}$$

If $N = 2$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

If $N = 4$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

If $N = 8$, the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & r & -i & -i\cdot r & -1 & -r & i & i\cdot r \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\cdot r & i & r & -1 & i\cdot r & -i & -r \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -r & -i & i\cdot r & -1 & r & i & -i\cdot r \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\cdot r & i & -r & -1 & -i\cdot r & -i & r \end{pmatrix}, \quad r = \frac{1}{\sqrt{2}}\cdot(1-i)$$

Calculation with the matrix:

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2\cdot(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

The inverse transformation is

$$x[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot e^{i\cdot k\cdot n\cdot\frac{2\pi}{N}} = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot W_N^{-k\cdot n} =$$

$$= \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot \left(W_N^{k\cdot n}\right)^*, \quad n = 0,1,\ldots,N-1$$

or in matrix form

$$\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{4} & \cdots & W_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2\cdot(N-1)} & \cdots & W_N^{-(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}$$

Fast Fourier Transform (FFT)

Formula of DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] \cdot W_N^{k \cdot n}, \quad k = 0,1,\ldots,N-1$$

where $W_N = e^{-i \cdot \frac{2\pi}{N}}$, or in matrix form

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2 \cdot (N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2 \cdot (N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}.$$

It is clear from the formulas that a DFT requires the evaluation of polynomial

$$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{N-1} \cdot x^{N-1}$$

where

$$a_0 = x[0], a_1 = x[1], \ldots, a_{N-1} = x[N-1]$$

on a special set

$$\{1, W_N, W_N^2, \ldots, W_N^{N-1}\}, \qquad W_N = e^{-i \cdot \frac{2\pi}{N}}, \quad (W^n = 1)$$

which is a so-called collapse set.

Remark

X is a collapse set if

$$|X^2| = \frac{1}{2} \cdot |X|$$

or $X = \{1\}$, where $|X|$ denotes the number of elements in X.)

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}.$$

$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{N-1} \cdot x^{N-1}$ is a polynomial of degree $N - 1$.

To reduce the computational time (number of steps) we use recursively that

$$A(x) = A_{\text{even}}(x^2) + x \cdot A_{\text{odd}}(x^2)$$

where A_{even} and A_{odd} are polynomials of degree $\frac{N}{2} - 1$

$$A_{\text{even}}(x) = a_0 + a_2 \cdot x + a_4 \cdot x^2 + \ldots + a_{N-2} \cdot x^{\frac{N}{2}-1} = \sum_{k=0}^{\frac{N}{2}-1} a_{2k} \cdot x^k$$

$$A_{\text{odd}}(x) = a_1 + a_3 \cdot x + a_5 \cdot x^2 + \ldots + a_{N-1} \cdot x^{\frac{N}{2}-1} = \sum_{k=0}^{\frac{N}{2}-1} a_{2k+1} \cdot x^k$$

Decimation in time

Here we suppose that N is a power of 2.

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{k \cdot n} = \sum_{n\ is\ even} x[n] \cdot W_N^{k \cdot n} + \sum_{n\ is\ odd} x[n] \cdot W_N^{k \cdot n} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_N^{2 \cdot r \cdot k} + \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_N^{(2 \cdot r + 1) \cdot k} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot (W_N^{2 \cdot})^{r \cdot k} + W_N^k \cdot \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot (W_N^{2 \cdot})^{r \cdot k} =$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_{\frac{N}{2}}^{r \cdot k} + W_N^k \cdot \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_{\frac{N}{2}}^{r \cdot k}$$

Since

$$G[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r] \cdot W_{\frac{N}{2}}^{r \cdot k} \quad \text{and} \quad H[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2 \cdot r + 1] \cdot W_{\frac{N}{2}}^{r \cdot k}$$

are $\frac{N}{2}$ point DFTs, we have that the calculation of an N point DFTs can be led back to the calculation of two $\frac{N}{2}$ point DFTs:

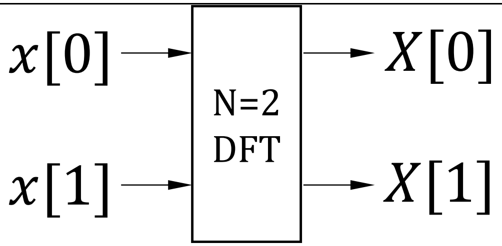
$$X[k] = G[k] + W_N^k \cdot H[k]$$

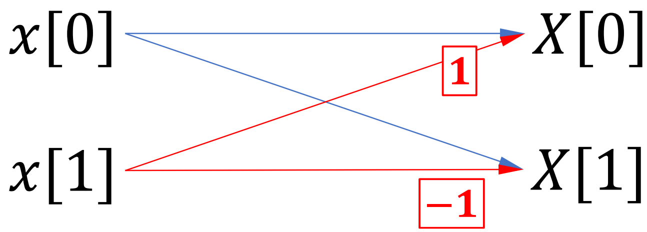
where $G[k]$ is calculated from values $X[0], X[2], X[4], \dots, X[N-2]$, while $H[k]$ is calculated from values $X[1], X[3], X[5], \dots, X[N-1]$.

2-point DFT

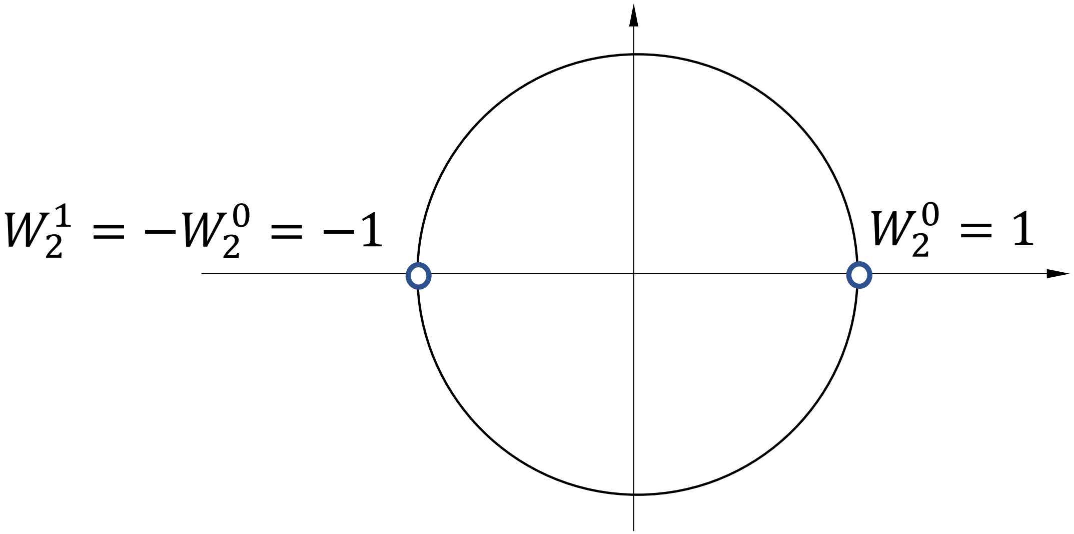
$$\begin{pmatrix} X[0] \\ X[1] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \end{pmatrix}$$

$$X[0] = x[0] + x[1]$$
$$X[1] = x[0] - x[1]$$





2nd roots of the unity



2-point inverse DFT

$$\begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \end{pmatrix}$$

$$x[0] = \frac{1}{2} \cdot (X[0] + X[1])$$

$$X[1] = \frac{1}{2} \cdot (X[0] - X[1])$$

4-point DFT

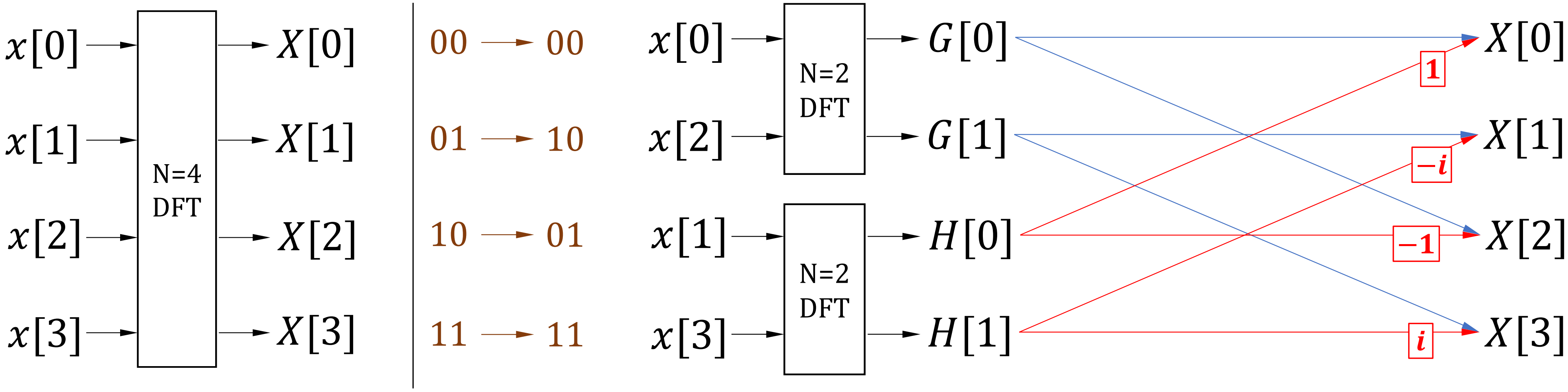
$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix}$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$



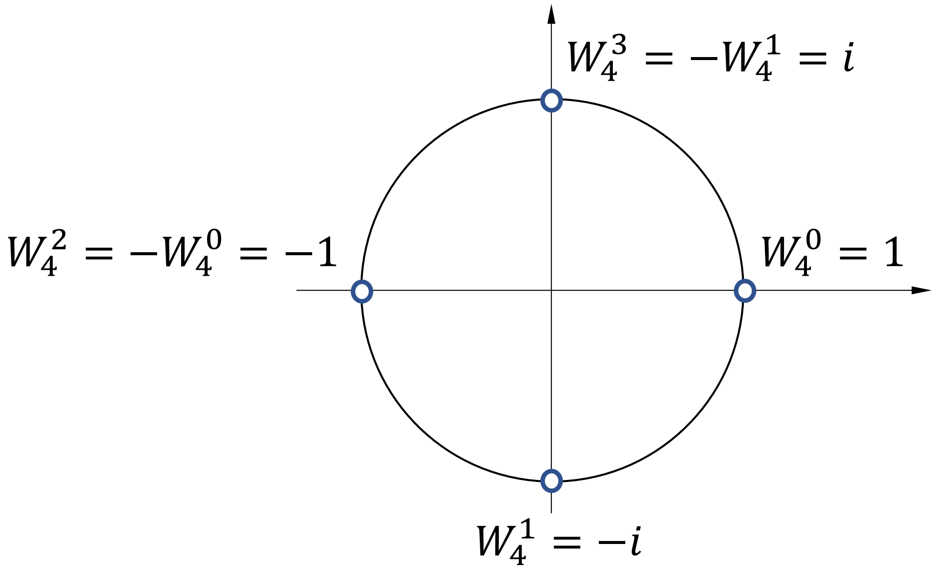
$$X[0] = \quad G[0] + 1 \cdot H[0] = \quad x[0] + x[2] + x[1] + x[3] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = \quad G[1] - i \cdot H[1] = \quad x[0] - x[2] - i \cdot (x[1] - x[3]) = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

$$X[2] = \quad G[0] - 1 \cdot H[0] = \quad x[0] + x[2] - (x[1] + x[3]) = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = \quad G[1] - i \cdot H[1] = \quad x[0] - x[2] + i \cdot (x[1] - x[3]) = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$

4th roots of the unity



4-point inverse DFT

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \cdot \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

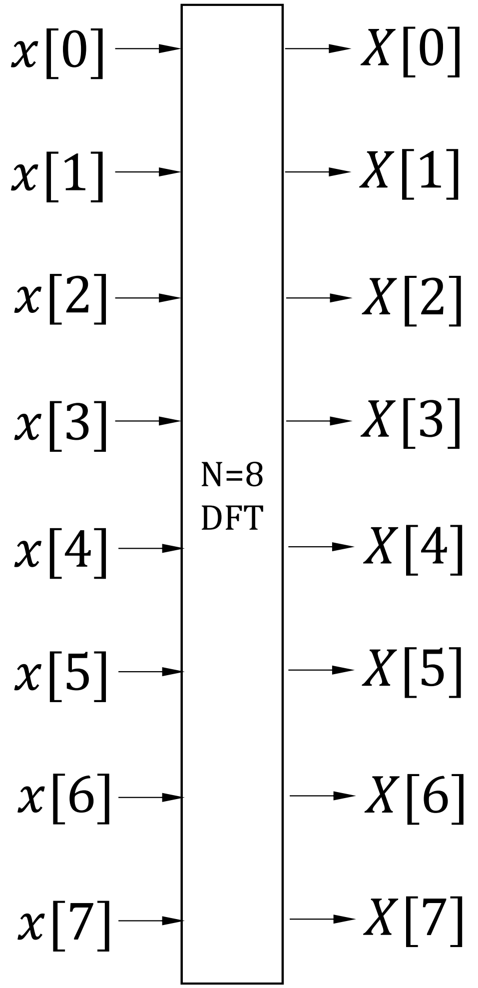
$$X[1] = x[0] + i \cdot x[1] - x[2] - i \cdot x[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

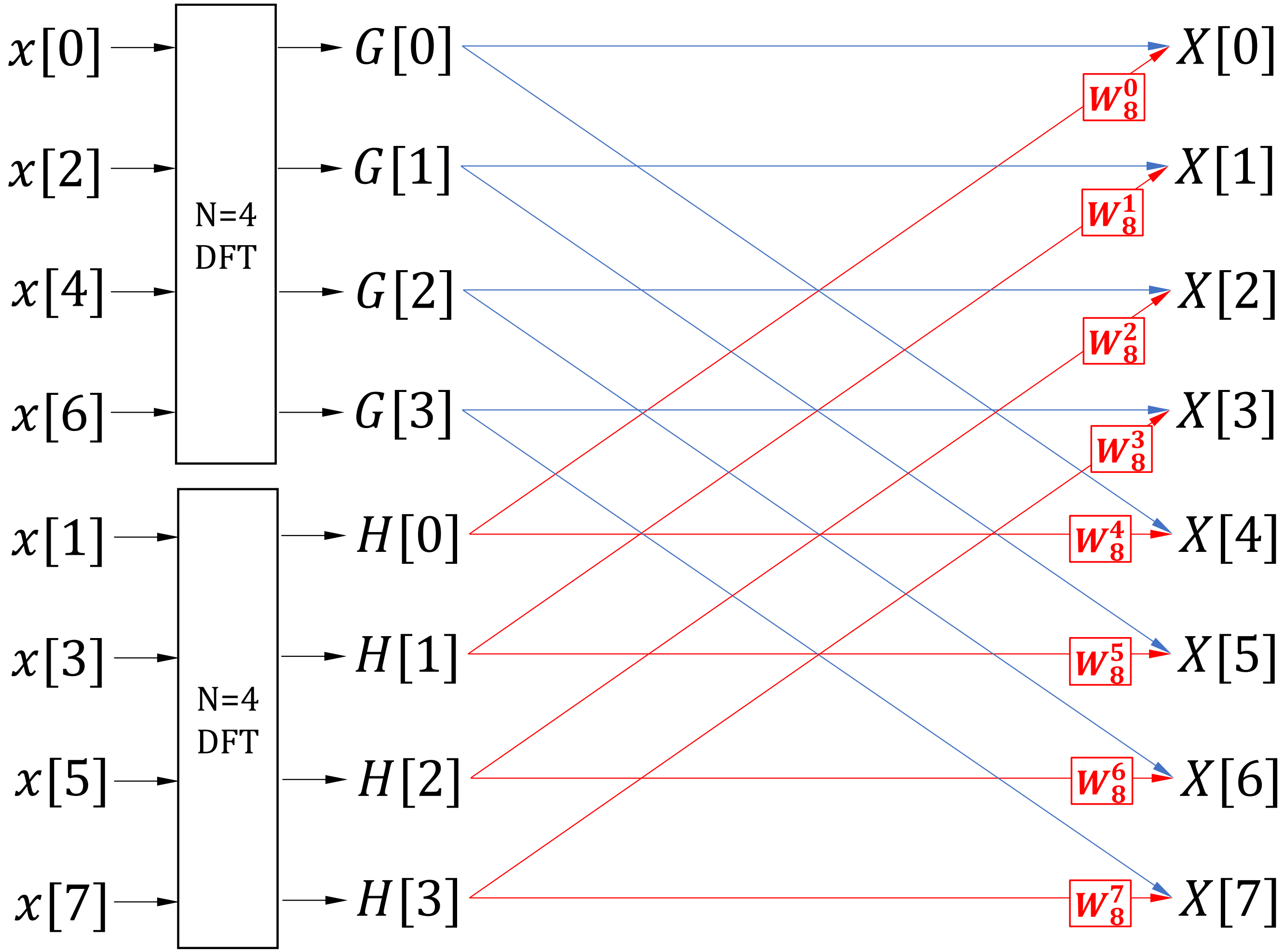
$$X[3] = x[0] - i \cdot x[1] - x[2] + i \cdot x[3]$$

8-point DFT

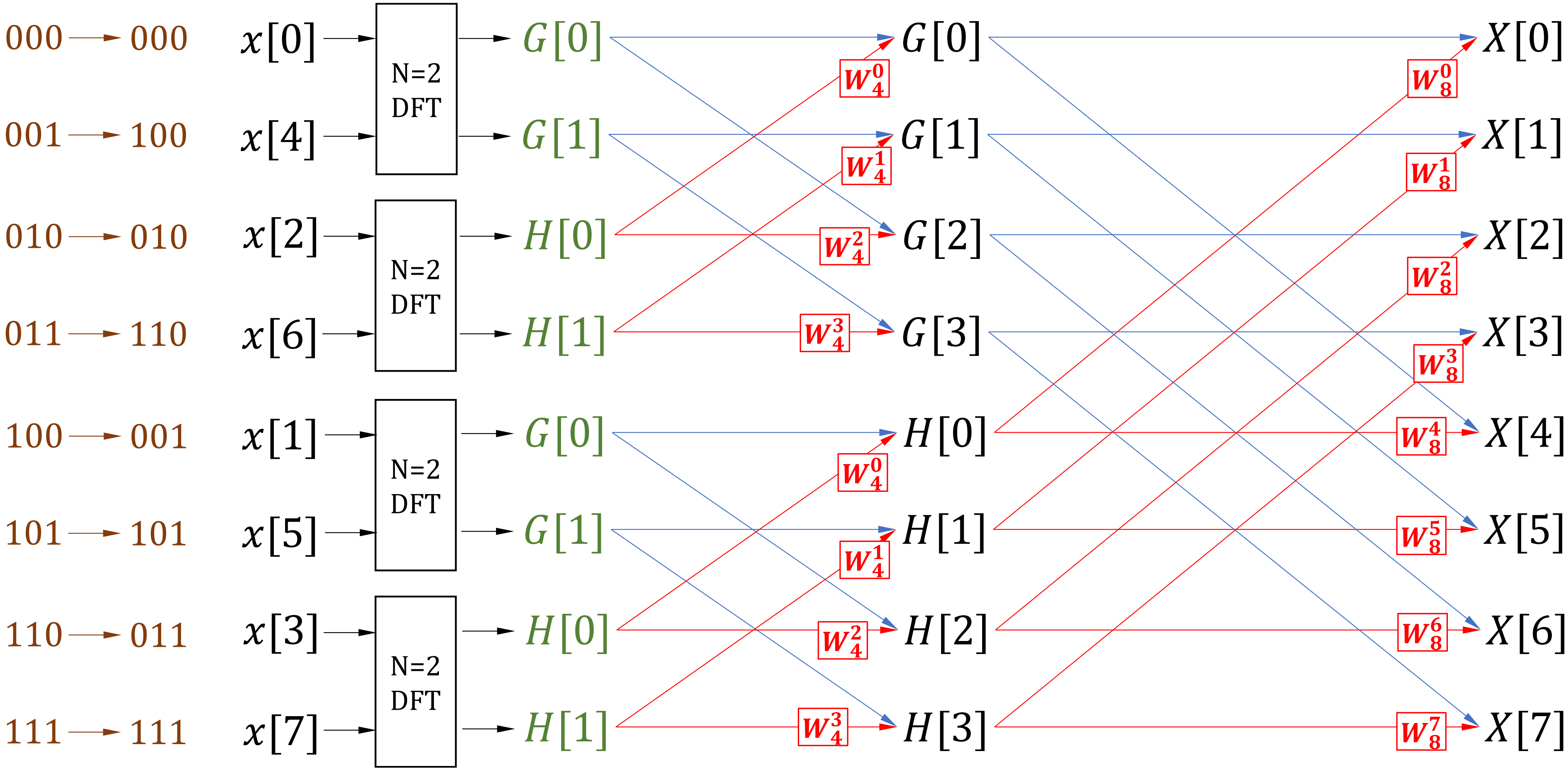
$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}}\cdot(1-i) & -i & \frac{1}{\sqrt{2}}\cdot(-1-i) & -1 & \frac{1}{\sqrt{2}}\cdot(-1+i) & i & \frac{1}{\sqrt{2}}\cdot(1+i) \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1}{\sqrt{2}}\cdot(-1-i) & i & \frac{1}{\sqrt{2}}\cdot(1-i) & -1 & \frac{1}{\sqrt{2}}\cdot(1+i) & -i & \frac{1}{\sqrt{2}}\cdot(-1+i) \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{1}{\sqrt{2}}\cdot(-1+i) & -i & \frac{1}{\sqrt{2}}\cdot(1+i) & -1 & \frac{1}{\sqrt{2}}\cdot(1-i) & i & \frac{1}{\sqrt{2}}\cdot(-1-i) \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1}{\sqrt{2}}\cdot(1+i) & i & \frac{1}{\sqrt{2}}\cdot(-1+i) & -1 & \frac{1}{\sqrt{2}}\cdot(-1-i) & -i & \frac{1}{\sqrt{2}}\cdot(1-i) \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$



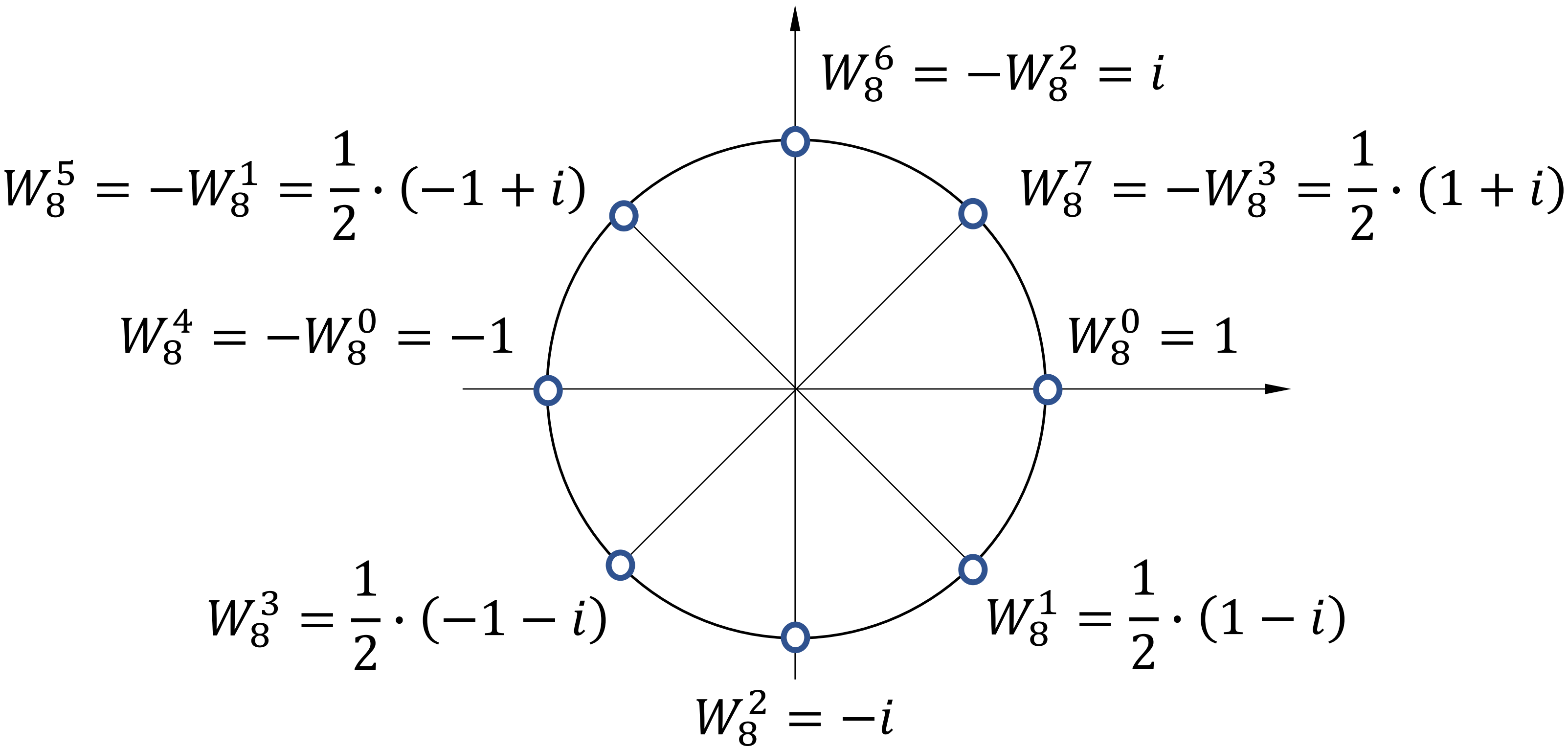
x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
N=8
DFT
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]



x[0]
x[2]
x[4]
x[6]
N=4
DFT
x[1]
x[3]
x[5]
x[7]
N=4
DFT
G[0]
G[1]
G[2]
G[3]
H[0]
H[1]
H[2]
H[3]
W_8^0
W_8^1
W_8^2
W_8^3
W_8^4
W_8^5
W_8^6
W_8^7
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]


000 → 000
001 → 100
010 → 010
011 → 110
100 → 001
101 → 101
110 → 011
111 → 111
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
N=2 DFT
N=2 DFT
N=2 DFT
N=2 DFT
G[0]
G[1]
H[0]
H[1]
G[0]
G[1]
H[0]
H[1]
W_4^0
W_4^1
W_4^2
W_4^3
W_4^0
W_4^1
W_4^2
W_4^3
G[0]
G[1]
G[2]
G[3]
H[0]
H[1]
H[2]
H[3]
W_8^0
W_8^1
W_8^2
W_8^3
W_8^4
W_8^5
W_8^6
W_8^7
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

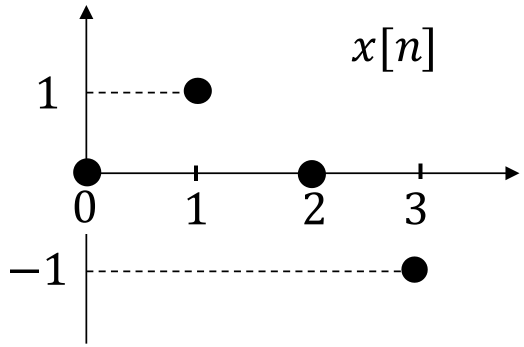
8th roots of the unity



Example

Determine the discrete Fourier transform of the sampled signal.

n	0	1	2	3
$x[n]$	0	1	0	-1



Plot the complex numbers in the complex plane appearing in the sums.

Solution

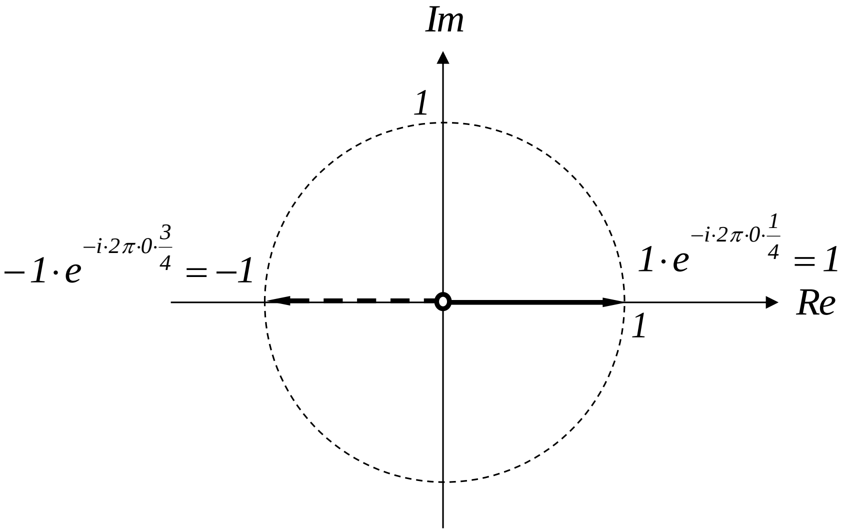
$$X[0] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 0 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] = 0 + 1 + 0 - 1 = 0$$

$$X[1] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 1 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \frac{\pi}{2}} =$$

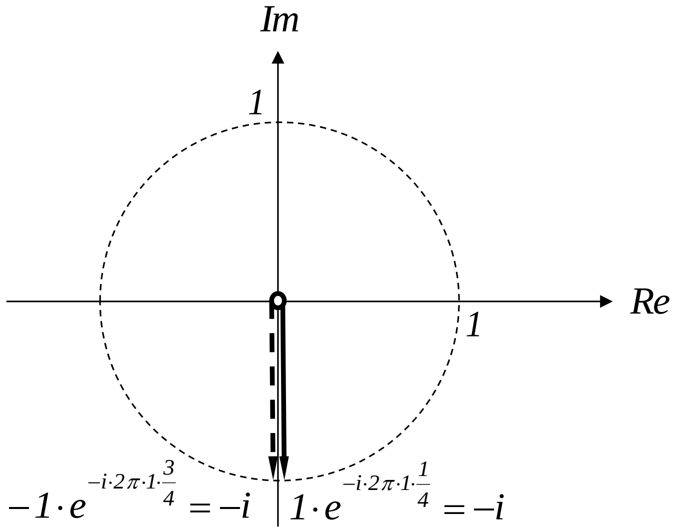
$$= 0 \cdot e^{-i \cdot 0 \cdot \frac{\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{\pi}{2}} = e^{-i \cdot \frac{\pi}{2}} - e^{-i \cdot 3 \cdot \frac{\pi}{2}} =$$

$$= \left(\cos\left(-\frac{\pi}{2}\right) + i \cdot \sin\left(-\frac{\pi}{2}\right)\right) - \left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right)\right) = 0 - i + 0 - i = -2 \cdot i$$

Values in the sum giving $X[0]$



Values in the sum giving $X[1]$



$$X[2] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 2 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \pi} =$$

$$= 0 \cdot e^{-i \cdot 0 \cdot \pi} + 1 \cdot e^{-i \cdot 1 \cdot \pi} + 0 \cdot e^{-i \cdot 2 \cdot \pi} - 1 \cdot e^{-i \cdot 3 \cdot \pi} = e^{-i \cdot \pi} - e^{-i \cdot 3 \cdot \pi} =$$

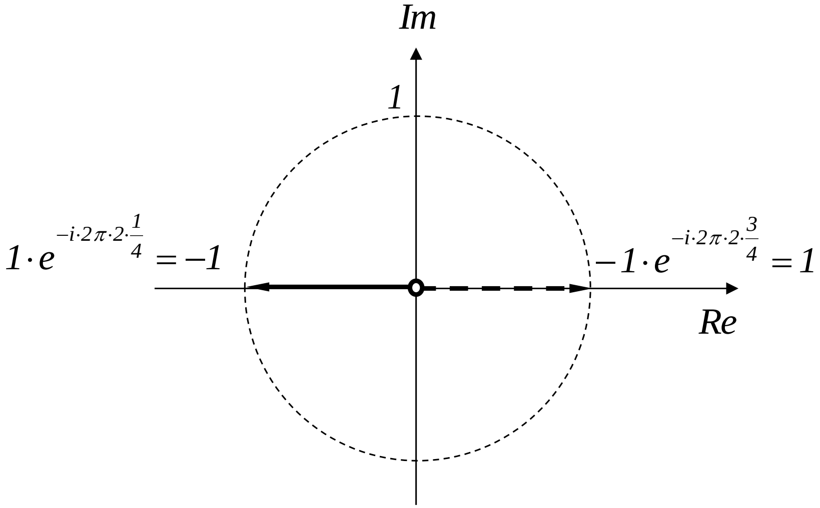
$$= \big(\cos(-\pi) + i \cdot \sin(-\pi)\big) - \big(\cos(-3\pi) + i \cdot \sin(-3\pi)\big) = 0 - 1 + 0 + 1 = 0$$

$$X[3] = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot 3 \cdot n \cdot \frac{2\pi}{4}} = \sum_{n=0}^{3} x[n] \cdot e^{-i \cdot n \cdot \frac{3\pi}{2}} =$$

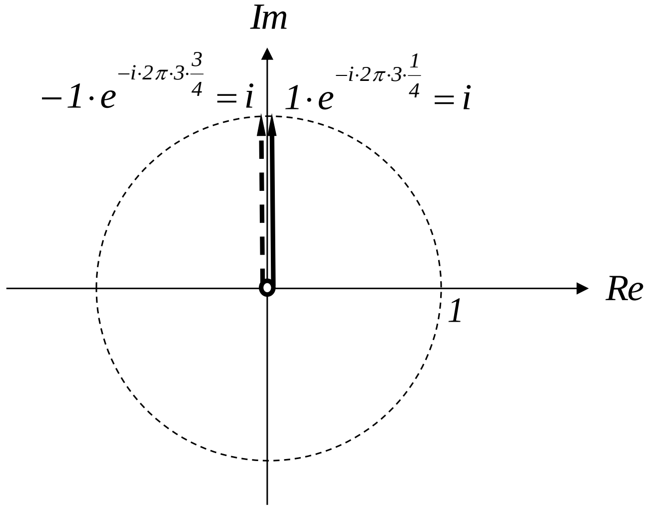
$$= 0 \cdot e^{-i \cdot 0 \cdot \frac{3\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{3\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{3\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{3\pi}{2}} = e^{-i \cdot \frac{3\pi}{2}} - e^{-i \cdot 3 \cdot \frac{3\pi}{2}} =$$

$$= \left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right)\right) - \left(\cos\left(-\frac{9\pi}{2}\right) + i \cdot \sin\left(-\frac{9\pi}{2}\right)\right) = 0 + i + 0 + i = 2 \cdot i$$

Values in the sum giving $X[2]$



Values in the sum giving $X[3]$



k	0	1	2	3
$X[k]$	0	$-2 \cdot i$	0	$2 \cdot i$
$\lvert X[k] \rvert$	0	2	0	2

Calculation with the transformation matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \cdot i \\ 0 \\ 2 \cdot i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

Example

Determine the discrete Fourier transform of the sampled signal

n	0	1	2	3
$x[n]$	8	4	8	0

using the transformation matrix.

Solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 4 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ -4 \cdot i \\ 12 \\ 4 \cdot i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$